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SUBCLASS OF HARMONIC UNIVALENT FUNCTIONS ASSOCIATED WITH SALAGEAN DERIVATIVE

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ABSTRACT.
In the present paper a new subclass of Harmonic univalent functions is defined using Salegaon derivative operator and several interesting properties like coefficient bound, distortion theorem are obtained.

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1. Introduction

A continuous function f is said to be a complex-valued harmonic function in a simply connected domain D in complex plane $\mathbb{C}$ if both $\operatorname{Re}\{f\}$ and $\operatorname{Im}\{f\}$ are real harmonic in D. Such functions can be expressed as

$$f = h + \overline{g} \tag{1.1}$$

where h and g are analytic in D. We call h as analytic part and g as co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h'(z)| > |g'(z)|$ for all z in D, see[2].

Let S_H be the family of functions of the form (1.1) that are harmonic ,univalent and orientation preserving in the open unit disk $U = \{z : |z| < 1\}$, so that $f = h + \overline{g}$ is normalized by $f(0) = h(0) = f_z(0) - 1 = 0$. Further $f = h + \overline{g}$ can be uniquely determined by the coefficients of power series expansions

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad , \; g(z) = \sum_{n=1}^{\infty} b_n z^n, \;\; z \in U, \; |b_1| < 1, \tag{1.2}$$

where $a_n \in \mathbb{C}$ for $n = 2,3,4,....$ and $b_n \in \mathbb{C}$ for $n = 1,2,3,.....$

We note that this family S_H was investigated and studied by Clunie and Sheil-Small [2] and it reduces to the well-known family S, the class of all normalized analytic univalent functions h given in (1.2), whenever the co-analytic part g of f is identically zero.

Let $\overline{S_H}$ denote the subfamily of S_H consisting of harmonic functions of the form

$$f_m = h + \overline{g_m}$$

where

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad , \; g_m(z) = (-1)^m \sum_{n=1}^{\infty} b_n z^n, \; \; z \in \mathrm{U}, \; |b_1| < 1, \qquad (1.3) \; .$$

Recently, Jahangiri *et al* [5] defined the Salagean derivative of harmonic functions $f = h + \overline{g}$ in S_H by

$$D^m f(z) = D^m h(z) + (-1)^m \overline{D^m g(z)} \qquad (1.4)$$

$m \in N \cup \{0\}$, where Salagean derivative of power series $\phi(z) = \sum_{n=1}^{\infty} \phi_n z^n$ is given by

$$D^0 \phi(z) = \phi(z) \; , \; D^1 \phi(z) = z\phi'(z) \text{ and } D^m \phi(z) = D(D^{m-1}\phi(z)) = \sum_{n=1}^{\infty} n^m \phi_n z^n \; .$$

Definition. The function $f = h + \overline{g}$ defined by (1.2) is in the class $S_H(k, \beta; m)$ if

$$\mathrm{Re}\left\{ \frac{D^{m+1} f(z)}{D^m f(z)} \right\} \geq k \left| \frac{D^{m+1} f(z)}{D^m f(z)} - 1 \right| + \beta \qquad (1.5)$$

where $0 \leq k < \infty, 0 \leq \beta < 1, m \in N \cup \{0\}$.

Also let $\overline{S_H}(k, \beta; m) = S_H(k, \beta; m) \cap \overline{S_H}$ (1.6)

We note that by specializing the parameter, especially when $k = 0$, $S_H(k, \beta; m)$ reduces to well-known family of starlike harmonic functions of order β. In recent years many researchers have studied various subclasses of S_H for example [1],[3],[4],[7]and [8].

In the present paper we aim at systematic study of basic properties, in particular coefficient bound , distortion theorem and extreme points of aforementioned subclass of harmonic functions.

2. Main Results

Theorem1. *Let* $f = h + \overline{g}$ *be given by* (1.2). *If condition*

$$\sum_{n=1}^{\infty} \left[\frac{n^m \left[n(1+k) - k - \beta \right]}{(1-\beta)} |a_n| + \frac{n^m \left[n(1+k) + k + \beta \right]}{(1-\beta)} |b_n| \right] \leq 2 \qquad (2.1)$$

where $a_1 = 1, 0 \leq \beta < 1, 0 \leq k < \infty, m \in N \cup \{0\}$

then f *is sense-preserving harmonic univalent in* U *and* $f \in S_H(k, \beta; m)$.

Proof. If the inequality (2.1) holds for coefficients of $f = h + \overline{g}$ then by (1.2), f is orientation preserving and harmonic univalent in U. Now it remains to show that

$f \in S_H(k,\beta;m)$. According to (1.4) and (1.5) we have

$$\operatorname{Re}\left\{\frac{D^{m+1}f(z)}{D^m f(z)}\right\} \geq k\left|\frac{D^{m+1}f(z)}{D^m f(z)}-1\right|+\beta$$

which is equivalent to $\operatorname{Re}\left(A(z)/B(z)\right) > \beta$ where

$A(z)=(1+k)D^{m+1}f(z)-kD^m f(z)$ and $B(z)=D^m f(z)$

Using the fact that, $\operatorname{Re}(w)>\beta$ if $|1-\beta+w| \geq |1+\beta-w|$ it suffices to show that

$|A(z)+(1-\beta)B(z)| \geq |A(z)-(1+\beta)B(z)|$

substituting values of $A(z)$ and $B(z)$ and with simple calculations we led to

$$\left|(2-\beta)z+\sum_{n=2}^{\infty} n^m\left[(1+k)n-k+(1-\beta)\right]a_n z^n-(-1)^m\sum_{n=1}^{\infty} n^m\left[(1+k)n-k-(1-\beta)\right]\overline{b_n}\,\overline{z^n}\right|$$

$$-\left|\beta z+\sum_{n=2}^{\infty} n^m\left[(1+k)n-k+(1-\beta)\right]a_n z^n+(-1)^m\sum_{n=1}^{\infty} n^m\left[(1+k)n-k-(1-\beta)\right]\overline{b_n}\,\overline{z^n}\right|$$

$$\geq 2(1-\beta)|z|-\sum_{n=2}^{\infty} n^m\left[2(1+k)n-2k-2\beta\right]|a_n||z|^n-(-1)^m\sum_{n=1}^{\infty} n^m\left[2(1+k)n+2k+2\beta\right]|\overline{b}_n||\overline{z}|^n$$

$$\geq 2(1-\beta)|z|\left\{1-\sum_{n=2}^{\infty} n^m\left[\frac{(1+k)n-k-\beta}{(1-\beta)}\right]|a_n||z|^{n-1}-(-1)^m\sum_{n=1}^{\infty} n^m\left[\frac{(1+k)n+k+\beta}{(1-\beta)}\right]|\overline{b}_n||\overline{z}|^{n-1}\right\}$$

$$\geq 0$$

by assumption. Hence proof is completed.
The functions

$$f(z)=z+\sum_{n=2}^{\infty}\frac{(1-\beta)}{n^m\left[(1+k)n-k-\beta\right]}x_n z^n+\sum_{n=1}^{\infty}\frac{(1-\beta)}{n^m\left[(1+k)n+k+\beta\right]}\overline{y_n}\,\overline{z^n}$$

where $\sum_{n=2}^{\infty}|x_n|+\sum_{n=1}^{\infty}|y_n|=1,$ (2.3)

shows that the coefficient bound given (2.1) is sharp.

Theorem2. *Let* $f_m=h+\overline{g_m}$ *be so that* h and g_m *are given by* (1.6). *Then* $f_m \in \overline{S_H}(k,\beta,m)$ *if and only if*

$$\sum_{n=1}^{\infty}\left[\frac{n^k\left[(1+k)n-\beta-k\right]}{(1-\beta)}|a_n|+\frac{n^k\left[(1+k)n+\beta+k\right]}{(1-\beta)}|b_n|\right] \leq 2 \qquad (2.4)$$

where $a_1=1, 0\leq\beta<1, 0\leq k<\infty, m\in N\cup\{0\}$.

Proof. The 'if' part follows form Theorem 1 with the fact that $\overline{S_H}(k,\beta,m)\subset S_H(k,\beta,m)$.

For only if part, we show that $f_m \notin \overline{S_H}(k,\beta,m)$ if the condition (2.4) is not satisfied.

Note that necessary and sufficient condition for $f_m = h + \overline{g_m}$ given by (1.6) to be in $\overline{S_H}(k,\beta,m)$ is that

$$\operatorname{Re}\left\{\frac{D^{m+1} f_m(z)}{D^m f_m(z)}\right\} \ge k\left|\frac{D^{m+1} f_m(z)}{D^m f_m(z)} - 1\right| + \beta$$

which is equivalent to

$$\operatorname{Re}\left\{\frac{(1+k)D^{m+1} f_m(z) + (k-\beta)D^m f_m(z)}{D^m f_m(z)}\right\}$$

$$=\operatorname{Re}\left\{\frac{(1-\beta)z - \sum_{n=2}^{\infty} n^m\left[(1+k)n - k - \beta\right]a_n z^n - (-1)^{2k}\sum_{n=1}^{\infty} n^m\left[(1+k)n + k + \beta\right]b_n \overline{z}^n}{z - \sum_{n=2}^{\infty} n^m a_n z^n + (-1)^{2k}\sum_{n=1}^{\infty} n^m b_n \overline{z^n}}\right\}$$

>0

The above conditions must hold for all values of z , $|z| = r < 1$. Choosing z on positive axis where $0 \le |z| = r < 1$. we have

$$\frac{(1-\beta)z - \sum_{n=2}^{\infty} n^m\left[(1+k)n - k - \beta\right]a_n r^{n-1} - (-1)^{2k}\sum_{n=1}^{\infty} n^m\left[(1+k)n + k + \beta\right]b_n \overline{r}^{n-1}}{z - \sum_{n=2}^{\infty} n^m a_n r^{n-1} + (-1)^{2k}\sum_{n=1}^{\infty} n^m b_n \overline{r^{n-1}}} \ge 0$$

(2.5)

or equivalently if the condition (2.4) dose not hold then the numerator in (2.5) is negative for r sufficiently close to 1.

Thus there exists $z_0 = r_0$ in (0,1) for which the quotient in (2.5) is negative .This contradicts that required condition for $f_m \in \overline{S_H}(k,\beta,m)$ and hence proof is completed.

Theorem3. *Let* f_m *be given by* (1.6). *Then* $f_m \in \overline{S_H}(k,\beta,m)$ *if and only if*

$$f_m(z) = \sum_{n=1}^{\infty}\left(x_n h_n(z) + y_n g_{m_n}(z)\right)$$

where $$h_1(z)=z, h_n(z)=z-\frac{1-\beta}{n^m\left[(1+k)n-k-\beta\right]}z^n \qquad , n=2,3,...$$

$$g_{m_n}(z)=z+(-1)^{m-1}\frac{1-\beta}{n^m\left[(1+k)n+k+\beta\right]}\overline{z}^n \qquad , n=1,2,3,...$$

and

$$x_m\geq 0, y_m\geq 0, x_1=1-\sum_{n=2}^{\infty}(x_n+y_n)\geq 0.$$

In particular, the extreme points of $\overline{S_H}(k,\beta,m)$ *are* $\{h_m\}$ *and* $\{g_{m_n}\}$.

Proof. Let

$$f_m(z)=\sum_{n=1}^{\infty}\left(x_nh_n(z)+y_ng_{m_n}(z)\right)$$

$$=\sum_{n=1}^{\infty}(x_n+y_n)z-\sum_{n=2}^{\infty}\frac{1-\beta}{n^m\left[(1+k)n-k-\beta\right]}x_nz^n+(-1)^{m-1}\sum\frac{1-\beta}{n^m\left[(1+k)n+k+\beta\right]}y_n\overline{z}^{n} \quad .$$

Then

$$\sum_{n=2}^{\infty}\frac{n^m\left[(1+k)n-k-\beta\right]}{1-\beta}|a_n|+\sum_{n=1}^{\infty}\frac{n^m\left[(1+k)n+k+\beta\right]}{1-\beta}|b_n|$$

$$=\sum_{n=2}^{\infty}x_n+\sum_{n=1}^{\infty}y_n=1-x_1\leq 1$$

and so $f_m\in\overline{S_H}(k,\beta,m)$.

Conversely, suppose that $f_m\in\overline{S_H}(k,\beta,m)$.

Setting

$$x_n=\frac{n^m\left[(1+k)n-k-\beta\right]}{1-\beta}a_n, \qquad n=2,3,...$$

$$y_n=\frac{n^m\left[(1+k)n+k+\beta\right]}{1-\beta}b_n, \qquad n=1,2,...$$

where $\sum_{n=1}^{\infty}\left(x_n+y_n\right)=1,$ we obtain $f_m(z)=\sum_{n=1}^{\infty}\left(x_nh_n(z)+y_ng_{m_n}(z)\right)$

as required.

Theorem 4. *Let* $f_m\in\overline{S_H}(k,\beta,m)$ *then for* $|z|=r<1$

we have

$$\left|f_m(z)\right|\leq\left(1+|b_1|\right)r+\frac{1}{2^m}\left(\frac{1-\beta}{2(k+1)-k-\beta}-\frac{2(k+1)+k+\beta}{2(k+1)-k-\beta}|b_1|\right)r^2$$

and

$$\left|f_m(z)\right| \geq \left(1-|b_1|\right)r - \frac{1}{2^m}\left(\frac{1-\beta}{2(k+1)-k-\beta} - \frac{2(k+1)+k+\beta}{2(k+1)-k-\beta}|b_1|\right)r^2 .$$

Proof. Let $f_m \in \overline{S_H}(k,\beta,m)$ .Taking absolute value of f_m we obtain

$$\left|f_m(z)\right| \leq (1+|b_1|)r + \sum_{n=2}^{\infty}(|a_n|+|b_n|)r^n$$

$$\leq (1+|b_1|)r + \sum_{n=2}^{\infty}(|a_n|+|b_n|)r^2$$

$$\leq (1+|b_1|)r + \frac{1-\beta}{2^m(2(k+1)-k-\beta)}\sum_{n=2}^{\infty}\left(\frac{2^m(2(k+1)-k-\beta)}{1-\beta}(|a_n|+|b_n|)\right)r^2$$

$$\leq (1+|b_1|)r + \frac{1-\beta}{2^m(2(k+1)-k-\beta)}\sum_{n=2}^{\infty}\left(\frac{2^m(2(k+1)-k-\beta)}{1-\beta}|a_n| + \frac{2^m(2(k+1)+k+\beta)}{1-\beta}|b_n|\right)r^2$$

$$\leq (1+|b_1|)r + \frac{1-\beta}{2^m(2(k+1)-k-\beta)}\sum_{n=2}^{\infty}\left(1-\frac{(2(k+1)+k+\beta)}{1-\beta}|b_1|\right)r^2$$

$$\leq (1+|b_1|)r + \frac{1}{2^m}\left(\frac{1-\beta}{(2(k+1)-k-\beta)} - \frac{2(k+1)+k+\beta}{2(k+1)-k-\beta}|b_1|\right)r^2$$

The forthcoming result follows from left hand inequality in Theorem 2.4.

Corollary. *Let* f_m *of the form* (1.3) *be so that if* $f_m \in \overline{S_H}(k,\beta,m)$,
then

$$\left\{\omega;|\omega| < \frac{2^m(k+2)-1-\left(2^m-1\right)\beta}{2^m(k+2-\beta)} - \frac{\left(2^m-1\right)(k+2)-k-\left(2^m+1\right)\beta}{2^m(k+2-\beta)}|b_1|\right\} \subset f_m(U).$$

Theorem 5. *The class* $\overline{S_H}(k,\beta,m)$ *is closed under convex combination*.

Proof. For $i=1,2,\ldots$ suppose $f_{m_i}(z) \in \overline{S_H}(k,\beta,m)$ where

$$f_{m_i}(z) = z - \sum_{n=2}^{\infty}|a_{in}|z^n + (-1)^m\sum_{n=1}^{\infty}|b_{in}|\overline{z}^n$$

then by Theorem 2

$$\sum_{n=2}^{\infty}\frac{n^m\left((k+1)n-k-\beta\right)}{(1-\beta)}|a_{in}| + \sum_{n=2}^{\infty}\frac{n^m\left((k+1)n+k+\beta\right)}{(1-\beta)}|b_{in}| \leq 1 \quad . \qquad (2.6)$$

For $\sum_{n=1}^{\infty} t_i = 1, 0 \le t_i \le 1,$ the convex combination of f_{m_i} may be written as

$$\sum_{i=1}^{\infty} t_i f_{m_i}(z) = z - \sum_{n=2}^{\infty}\left(\sum_{i=1}^{\infty} t_i |a_{in}|\right) z^n + (-1)^m \sum_{n=1}^{\infty}\left(\sum_{i=1}^{\infty} t_i |b_{in}|\right)\overline{z}^n$$

hence by (2.6)

$$\sum_{n=2}^{\infty} \frac{n^m\left((k+1)n-k-\beta\right)}{1-\beta}\left(\sum_{i=1}^{\infty} t_i |a_{in}|\right) + \sum_{n=1}^{\infty} \frac{n^m\left((k+1)n+k+\beta\right)}{1-\beta}\left(\sum_{i=1}^{\infty} t_i |b_{in}|\right)$$

$$= \sum_{i=1}^{\infty} t_i \left[\sum_{n=2}^{\infty} \frac{n^m\left((k+1)n-k-\beta\right)}{1-\beta}|a_{in}| + \sum_{n=1}^{\infty} \frac{n^m\left((k+1)n+k+\beta\right)}{1-\beta}|b_{in}|\right] \le \sum_{i=1}^{\infty} t_i \le 1$$

and therefore $\sum_{i=1}^{\infty} t_i f_{m_i}(z) \in \overline{S_H}(k, \beta, m)$.

This completes the proof.

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