Starting from the study of the Shepard nonlinear operator of max-prod type in [6], [7], in the book [8], Open Problem 5.5.4, pp. 324-326, the Favard-Szász-Mirakjan max-prod type operator is introduced and the question of the approximation order by this operator is raised. In the recent paper [1], by using a pretty complicated method to this open question an answer is given by obtaining an upper pointwise estimate of the approximation error of the form $C\omega_1(f; \sqrt{n})$ (with an unexplicit absolute constant $C > 0$) and the question of improving the order of approximation $\omega_1(f; \sqrt{n})$ is raised. The first aim of this note is to obtain the same order of approximation but by a simpler method, which in addition presents, at least, two advantages: it produces an explicit constant in front of $\omega_1(f; \sqrt{n})$ and it can easily be extended to other max-prod operators of Bernstein type. Also, we prove by a counterexample that in some sense, in general this type of order of approximation with respect to $\omega_1(f; \cdot)$ cannot be improved. However, for some subclasses of functions, including for example the bounded, nondecreasing concave functions, the essentially better order $\omega_1(f; 1/n)$ is obtained. Finally, some shape preserving properties are obtained.

1 Introduction

Starting from the study of the Shepard nonlinear operator of max-prod type in [6], [7], by the Open Problem 5.5.4, pp. 324-326 in the recent monograph [8], the following nonlinear Favard-Szász-Mirakjan max-prod operator is introduced (here $\vee$ means maximum)

$$F_n^{(M)}(f)(x) = \vee_{k=0}^{\infty} \frac{(nx)^k}{k!} \frac{f \left( \frac{x}{n} \right)}{\vee_{k=0}^{\infty} \frac{(nx)^k}{k!}}.$$
for which by a pretty complicated method in [1], Theorem 8, the order of pointwise approximation \( \omega_1(f; \sqrt{x}/\sqrt{n}) \) is obtained. Also, by Remark 9, 2) in the same paper [1], the question if this order of approximation could be improved is raised.

The first aim of this note is to obtain the same order of approximation but by a simpler method, which in addition presents, at least, two advantages: it produces an explicit constant in front of \( \omega_1(f; \sqrt{x}/\sqrt{n}) \) and it can easily be extended to various max-prod operators of Bernstein type, see [2] – [5]. Also, one proves by a counterexample that in some sense, in general this type of order of approximation with respect to \( \omega_1(f; \cdot) \) cannot be improved, giving thus a negative answer to a question raised in [1] (see Remark 9, 2) there). However, for some subclasses of functions, including for example the bounded, nondecreasing concave functions, the essentially better order \( \omega_1(f; 1/n) \) is obtained. This allows us to put in evidence large classes of functions (e.g. bounded, nondecreasing concave polygonal lines on \([0, \infty)\)) for which the order of approximation given by the max-product Favard-Szász-Mirakjan operator, is essentially better than the order given by the linear Favard-Szász-Mirakjan operator. Finally, some shape preserving properties are presented.

Section 2 presents some general results on nonlinear operators, in Section 3 we prove several auxiliary lemmas, Section 4 contains the approximation results, while in Section 5 we present some shape preserving properties.

2 Preliminaries

For the proof of the main result we need some general considerations on the so-called nonlinear operators of max-prod kind. Over the set of positive reals, \( \mathbb{R}_+ \), we consider the operations \( \vee \) (maximum) and \( \cdot \), product. Then \((\mathbb{R}_+, \vee, \cdot)\) has a semiring structure and we call it as Max-Product algebra.

Let \( I \subset \mathbb{R} \) be a bounded or unbounded interval, and

\[
CB_+(I) = \{ f : I \to \mathbb{R}_+ ; f \text{ continuous and bounded on } I \}.
\]

The general form of \( L_n : CB_+(I) \to CB_+(I) \), (called here a discrete max-product type approximation operator) studied in the paper will be

\[
L_n(f)(x) = \bigvee_{i=0}^n K_n(x, x_i) \cdot f(x_i),
\]

or

\[
L_n(f)(x) = \bigvee_{i=0}^\infty K_n(x, x_i) \cdot f(x_i),
\]

where \( n \in \mathbb{N}, f \in CB_+(I), K_n(\cdot, x_i) \in CB_+(I) \) and \( x_i \in I, \) for all \( i. \) These operators are nonlinear, positive operators and moreover they satisfy a pseudo-linearity condition of the form

\[
L_n(\alpha \cdot f \vee \beta \cdot g)(x) = \alpha \cdot L_n(f)(x) \vee \beta \cdot L_n(g)(x), \forall \alpha, \beta \in \mathbb{R}_+, f, g : I \to \mathbb{R}_+.
\]
In this section we present some general results on these kinds of operators which will be useful later in the study of the Favard-Szász-Mirakjan max-product kind operator considered in Introduction.

**Lemma 2.1.** ([1]) Let \( I \subset \mathbb{R} \) be a bounded or unbounded interval, 
\[
CB_+ (I) = \{ f : I \rightarrow \mathbb{R}_+ ; f \text{ continuous and bounded on } I \},
\]
and \( L_n : CB_+(I) \rightarrow CB_+(I) \), \( n \in \mathbb{N} \) be a sequence of operators satisfying the following properties :

(i) if \( f, g \in CB_+(I) \) satisfy \( f \leq g \) then \( L_n(f) \leq L_n(g) \) for all \( n \in \mathbb{N} \); 
(ii) \( L_n(f + g) \leq L_n(f) + L_n(g) \) for all \( f, g \in CB_+(I) \).

Then for all \( f, g \in CB_+(I) \), \( n \in \mathbb{N} \) and \( x \in I \) we have 
\[
|L_n(f)(x) - L_n(g)(x)| \leq L_n(|f - g|(x)).
\]

**Proof.** Since is very simple, we reproduce here the proof in [1]. Let \( f, g \in CB_+(I) \).
We have \( f = f - g + g \leq |f - g| + g \), which by the conditions (i) - (ii) successively implies \( L_n(f)(x) \leq L_n(|f - g|)(x) + L_n(g)(x) \), that is \( L_n(f)(x) - L_n(g)(x) \leq L_n(|f - g|)(x) \).

Writing now \( g = g - f + f \leq |f - g| + f \) and applying the above reasonings, it follows \( L_n(g)(x) - L_n(f)(x) \leq L_n(|f - g|)(x) \), which combined with the above inequality gives \( |L_n(f)(x) - L_n(g)(x)| \leq L_n(|f - g|)(x) \).

**Remarks.** 1) It is easy to see that the Favard-Szász-Mirakjan max-product operator satisfy the conditions in Lemma 2.1, (i), (ii). In fact, instead of (i) it satisfies the stronger condition
\[
L_n(f \lor g)(x) = L_n(f)(x) \lor L_n(g)(x), \; f, g \in CB_+(I).
\]

Indeed, taking in the above equality \( f \leq g \), \( f, g \in CB_+(I) \), it easily follows \( L_n(f)(x) \leq L_n(g)(x) \).

2) In addition, it is immediate that the Favard-Szász-Mirakjan max-product operator is positive homogenous, that is \( L_n(\lambda f) = \lambda L_n(f) \) for all \( \lambda \geq 0 \).

**Corollary 2.2.** ([1]) Let \( L_n : CB_+(I) \rightarrow CB_+(I) \), \( n \in \mathbb{N} \) be a sequence of operators satisfying the conditions (i)-(ii) in Lemma 1 and in addition being positive homogenous. Then for all \( f \in CB_+(I) \), \( n \in \mathbb{N} \) and \( x \in I \) we have
\[
|f(x) - L_n(f)(x)| \leq \left[ \frac{1}{\delta} L_n(\varphi_x)(x) + L_n(e_0)(x) \right] \omega_1(f; \delta) f + f(x) \cdot |L_n(e_0)(x) - 1|,
\]
where \( \delta > 0 \), \( e_0(t) = 1 \) for all \( t \in I \), \( \varphi_x(t) = |t - x| \) for all \( t \in I \), \( x \in I \), \( \omega_1(f; \delta) = \max\{|f(x) - f(y)| ; x, y \in I, |x - y| \leq \delta\} \) and if \( I \) is unbounded then we suppose that there exists \( L_n(\varphi_x)(x) \in \mathbb{R}_+ \cup \{+\infty\} \), for any \( x \in I \), \( n \in \mathbb{N} \).

**Proof.** The proof is identical with that for positive linear operators and because of its simplicity we reproduce it below. Indeed, from the identity
\[
L_n(f)(x) - f(x) = [L_n(f)(x) - f(x) \cdot L_n(e_0)(x)] + f(x)[L_n(e_0)(x) - 1],
\]
it follows (by the positive homogeneity and by Lemma 2.1)
\[ |f(x) - L_n(f(x))| \leq |L_n(f(x))(x) - L_n(f(t))(x)| + |f(x)| \cdot |L_n(e_0)(x) - 1| \leq L_n(|f(t) - f(x)|)(x) + |f(x)| \cdot |L_n(e_0)(x) - 1|. \]

Now, since for all \( t, x \in I \) we have
\[ |f(t) - f(x)| \leq \omega_1(f; |t - x|)_I \leq \left[ \frac{1}{\delta} |t - x| + 1 \right] \omega_1(f; \delta)_I, \]
replacing above we immediately obtain the estimate in the statement. □

An immediate consequence of Corollary 2.2 is the following.

**Corollary 2.3.** ([1]) Suppose that in addition to the conditions in Corollary 2.2, the sequence \( \{L_n\}_n \) satisfies \( L_n(e_0) = e_0 \), for all \( n \in N \). Then for all \( f \in CB^+(I) \), \( n \in N \) and \( x \in I \) we have
\[ |f(x) - L_n(f(x))| \leq \left[ 1 + \frac{1}{\delta} L_n(\varphi_x)(x) \right] \omega_1(f; \delta)_I. \]

3 Auxiliary Results

Since it is easy to check that \( F_n^{(M)}(f)(0) - f(0) = 0 \) for all \( n \), notice that in the notations, proofs and statements of the all approximation results, that is in Lemmas 3.1-3.3, Theorem 4.1, Lemma 4.2, Corollary 4.4, Corollary 4.5, in fact we always may suppose that \( x > 0 \).

For each \( k, j \in \{0, 1, 2, \ldots, \} \) and \( x \in \left[ j/n, j/n + 1 \right] \), let us denote \( s_{n,k}(x) = \frac{(nx)^k}{k!} \),
\[ M_{k,n,j}(x) = \frac{s_{n,k}(x) \{ k/n - x \}}{s_{n,j}(x)}, m_{k,n,j}(x) = \frac{s_{n,k}(x)}{s_{n,j}(x)}. \]

It is clear that if \( k \geq j + 1 \) then
\[ M_{k,n,j}(x) = \frac{s_{n,k}(x) \{ k/n - x \}}{s_{n,j}(x)} \]
and if \( k \leq j - 1 \) then
\[ M_{k,n,j}(x) = \frac{s_{n,k}(x) \{ x - k/n \}}{s_{n,j}(x)}. \]

**Lemma 3.1.** For all \( k, j \in \{0, 1, 2, \ldots, \} \) and \( x \in \left[ j/n, j/n + 1 \right] \) we have
\[ m_{k,n,j}(x) \leq 1. \]

**Proof.** We have two cases: 1) \( k \geq j \) and 2) \( k \leq j \).
Case 1. Since clearly the function $h(x) = \frac{1}{x}$ is nonincreasing on $[j/n, (j+1)/n]$, it follows
\[
\frac{m_{k,n,j}(x)}{m_{k+1,n,j}(x)} = \frac{k+1}{n} \cdot \frac{1}{x} \geq \frac{k+1}{n} \cdot \frac{n}{j+1} = \frac{k+1}{j+1} \geq 1,
\]
which implies $m_{j,n,j}(x) \geq m_{j+1,n,j}(x) \geq m_{j+2,n,j}(x) \geq \ldots$.

Case 2). We get
\[
\frac{m_{k,n,j}(x)}{m_{k-1,n,j}(x)} = \frac{nx}{k} \geq \frac{n}{k} = \frac{j}{k} \geq 1,
\]
which immediately implies
\[
m_{j,n,j}(x) \geq m_{j-1,n,j}(x) \geq m_{j-2,n,j}(x) \geq \ldots \geq m_{0,n,j}(x).
\]
Since $m_{j,n,j}(x) = 1$, the conclusion of the lemma is immediate. \hfill \Box

Lemma 3.2. Let $x \in [\frac{j}{n}, \frac{j+1}{n}]$.

(i) If $k \in \{j+1, j+3, \ldots, \}$ is such that $k - \sqrt{k+1} \geq j$, then $M_{k,n,j}(x) \geq M_{k+1,n,j}(x)$.

(ii) If $k \in \{1, 2, \ldots, j-1\}$ is such that $k + \sqrt{k} \leq j$, then $M_{k,n,j}(x) \geq M_{k-1,n,j}(x)$.

**Proof.** (i) We observe that
\[
\frac{M_{k,n,j}(x)}{M_{k+1,n,j}(x)} = \frac{k+1}{n} \cdot \frac{1}{x} \cdot \frac{k-n}{x} \geq \frac{k-n}{x}.
\]
Since the function $g(x) = \frac{1}{x} \cdot \frac{k-x}{x}$ is clearly nonincreasing, it follows that $g(x) \geq g(\frac{j+1}{n}) = \frac{n}{j+1} \cdot \frac{k-j-1}{k-j}$ for all $x \in [\frac{j}{n}, \frac{j+1}{n}]$.

Then, since the condition $k - \sqrt{k+1} \geq j$ implies $(k+1)(k-j-1) \geq (j+1)(k-j)$, we obtain
\[
\frac{M_{k,n,j}(x)}{M_{k+1,n,j}(x)} \geq \frac{k+1}{j+1} \cdot \frac{k-j-1}{k-j} \geq 1.
\]

(ii) We observe that
\[
\frac{M_{k,n,j}(x)}{M_{k-1,n,j}(x)} = \frac{n}{k} \cdot \frac{x}{x} \cdot \frac{x-k}{x} = \frac{k-n}{x}.
\]
Since the function $h(x) = x \cdot \frac{x-k}{x}$ is nondecreasing, it follows that $h(x) \geq h(\frac{j}{n}) = \frac{j-k}{n}$ for all $x \in [\frac{j}{n}, \frac{j+1}{n}]$.

Then, since the condition $k + \sqrt{k} \leq j$ implies $j(j-k) \geq k(j-k+1)$, we obtain
\[
\frac{M_{k,n,j}(x)}{M_{k-1,n,j}(x)} \geq \frac{n}{k} \cdot \frac{j}{j-k+1} \geq 1,
\]
which proves the lemma. \hfill \Box

Also, a key result in the proof of the main result is the following.
Lemma 3.3. Denoting \( s_{n,k}(x) = \frac{(nx)^k}{k!} \), we have

\[
\sum_{k=0}^{\infty} \frac{(nx)^k}{k!} = s_{n,j}(x), \quad \text{for all } x \in \left[ \frac{j}{n}, \frac{j+1}{n} \right], \quad j = 0, 1, \ldots.
\]

Proof. First we show that for fixed \( n \in \mathbb{N} \) and \( 0 \leq k \) we have

\[
0 \leq s_{n,k+1}(x) \leq s_{n,k}(x), \quad \text{if and only if } x \in [0, (k+1)/n].
\]

Indeed, the inequality one reduces to

\[
0 \leq \frac{(nx)^{k+1}}{(k+1)!} \leq \frac{(nx)^k}{k!},
\]

which after simplifications is obviously equivalent to

\[
0 \leq x \leq \frac{k+1}{n}.
\]

By taking \( k = 0, 1, \ldots \), in the inequality just proved above, we get

\[
s_{n,1}(x) \leq s_{n,0}(x), \quad \text{if and only if } x \in [0, 1/n],
\]

\[
s_{n,2}(x) \leq s_{n,1}(x), \quad \text{if and only if } x \in [0, 2/n],
\]

\[
s_{n,3}(x) \leq s_{n,2}(x), \quad \text{if and only if } x \in [0, 3/n],
\]

so on,

\[
s_{n,k+1}(x) \leq s_{n,k}(x), \quad \text{if and only if } x \in [0, (k+1)/n],
\]

and so on.

From all these inequalities, reasoning by recurrence we easily obtain :

if \( x \in [0, 1/n] \) then \( s_{n,k}(x) \leq s_{n,0}(x) \), for all \( k = 0, 1, \ldots \),

if \( x \in [1/n, 2/n] \) then \( s_{n,k}(x) \leq s_{n,1}(x) \), for all \( k = 0, 1, \ldots \),

if \( x \in [2/n, 3/n] \) then \( s_{n,k}(x) \leq s_{n,2}(x) \), for all \( k = 0, 1, \ldots \),

and so on, in general

if \( x \in [j/n, (j+1)/n] \) then \( s_{n,k}(x) \leq s_{n,j}(x) \), for all \( k = 0, 1, \ldots \),

which proves the lemma. \( \square \)
4 Approximation Results

If $F_n^M(f)(x)$ represents the nonlinear Favard-Szász-Mirakjan operator of max-product kind defined in Introduction, then the main result is the following.

**Theorem 4.1.** Let $f : [0, \infty) \to \mathbb{R}_+$ be bounded and continuous on $[0, \infty)$. Then we have the estimate

$$|F_n^M(f)(x) - f(x)| \leq 8\omega_1 \left( f, \frac{\sqrt{x}}{\sqrt{n}} \right), \text{ for all } n \in \mathbb{N}, x \in [0, \infty),$$

where

$$\omega_1(f, \delta) = \sup\{|f(x) - f(y)| ; x, y \in [0, \infty), |x - y| \leq \delta\}.$$

**Proof.** It is easy to check that the max-product Favard-Szász-Mirakjan operators fulfil the conditions in Corollary 2.3 and we have

$$|F_n^M(f)(x) - f(x)| \leq \left( 1 + \frac{1}{\delta_n} F_n^M(\varphi_x)(x) \right) \omega_1(f, \delta_n),$$

where $\varphi_x(t) = |t - x|$. So, it is enough to estimate

$$E_n(x) := F_n^M(\varphi_x)(x) = \frac{\infty}{k=0} \frac{(nx)^k |x_n - x|}{k!}, x \in [0, \infty).$$

Let $x \in [j/n, (j + 1)/n]$, where $j \in \{0, 1, \ldots, \}$ is fixed, arbitrary. By Lemma 3.3 we easily obtain

$$E_n(x) = \max_{k=0,1,\ldots} \{ M_{k,n,j}(x) \}, x \in [j/n, (j + 1)/n].$$

In all what follows we may suppose that $j \in \{1, 2, \ldots, \}$, because for $j = 0$ we get $E_n(x) \leq \frac{\sqrt{x}}{\sqrt{n}}$, for all $x \in [0, 1/n]$. Indeed, in this case we obtain $M_{k,n,0}(x) = \frac{(nx)^k |x - x_n|}{k!}$, which for $k = 0$ gives $M_{k,n,0}(x) = x = \sqrt{x} \cdot \sqrt{x} \leq \sqrt{x} \cdot \frac{1}{\sqrt{n}}$. Also, for any $k \geq 1$ we have $\frac{k^k}{n^k} \leq \frac{k}{n^k} \leq \frac{k}{n^k}$ and we obtain

$$M_{k,n,0}(x) \leq \frac{(nx)^k}{k!} \cdot \frac{k}{n} = \sqrt{x} \cdot \frac{n^{k-1}x^{k-1/2}}{(k-1)!} \leq \sqrt{x} \cdot \frac{n^{k-1}}{(k-1)!} \cdot \frac{1}{\sqrt{n}}.$$

So it remains to obtain an upper estimate for each $M_{k,n,j}(x)$ when $j = 1, 2, \ldots$, is fixed, $x \in [j/n, (j + 1)/n]$ and $k = 0, 1, \ldots$. In fact we will prove that

$$M_{k,n,j}(x) \leq \frac{4\sqrt{x}}{\sqrt{n}}, \text{ for all } x \in [j/n, (j + 1)/n], k = 0, 1, \ldots,$$

which immediately will imply that

$$E_n(x) \leq \frac{4\sqrt{x}}{\sqrt{n}} \text{ for all } x \in [0, \infty), n \in \mathbb{N},$$
and taking $\delta_n = \frac{4\sqrt{x}}{\sqrt{n}}$ in (1) we immediately obtain the estimate in the statement.

In order to prove (2) we distinguish the following cases:

1) $k = j$; 2) $k \geq j + 1$ and 3) $k \leq j - 1$.

Case 1). If $k = j$ then $M_{j,n,j}(x) = \lfloor \frac{j}{n} - x \rfloor$. Since $x \in [\frac{j}{n}, \frac{j+1}{n})$, it easily follows that $M_{j,n,j}(x) \leq \frac{1}{n}$. Now, since $j \geq 1$ we get $x \geq \frac{1}{n}$, which implies $\frac{k}{n} = \frac{1}{\sqrt{n}} \cdot \frac{1}{\sqrt{n}} \leq \sqrt{x} \cdot \frac{1}{\sqrt{n}}$.

Case 2). Subcase a). Suppose first that $k - \sqrt{k + 1} < j$. We get

$$M_{k,n,j}(x) = m_{k,n,j}(x)(\frac{k}{n} - x) \leq \frac{k}{n} - x \leq \frac{k}{n} - \frac{j}{n} \leq \frac{k - \sqrt{k + 1}}{n} = \sqrt{k + 1}.$$

But we necessarily have $k \leq 3j$. Indeed, if we suppose that $k > 3j$, then because $g(x) = x - \sqrt{x + 1}$ is nondecreasing, it follows $j > k - \sqrt{k + 1} \geq 3j - \sqrt{3j + 1}$, which implies the obvious contradiction $j > 3j - \sqrt{3j + 1}$.

In conclusion, we obtain

$$M_{k,n,j}(x) \leq \frac{\sqrt{k + 1}}{n} \leq \frac{\sqrt{3j + 1}}{n} \leq 2\frac{\sqrt{j}}{n} \leq 2\frac{\sqrt{x}}{\sqrt{n}},$$

taking into account that $\sqrt{x} \geq \frac{\sqrt{j}}{\sqrt{n}}$.

Subcase b). Suppose now that $k - \sqrt{k + 1} \geq j$. Since the function $g(x) = x - \sqrt{x + 1}$ is nondecreasing on the interval $[0, \infty)$ it follows that there exists $k \in \{1, 2, ..., \}$, of maximum value, such that $k - \sqrt{k + 1} < j$. Then for $k_1 = k + 1$ we get $k_1 - \sqrt{k_1 + 1} \geq j$ and

$$M_{k_1,n,j}(x) = m_{k_1,n,j}(x)(\frac{k_1 - \sqrt{k_1 + 1}}{n} - x) \leq \frac{k_1 + 1}{n} - x \leq \frac{k_1 + 1}{n} - \frac{j}{n} \leq \frac{k_1 + 1}{n} - \frac{\sqrt{k_1 + 1}}{n} \leq 3\frac{\sqrt{x}}{\sqrt{n}}.$$

The last above inequality follows from the fact that $k - \sqrt{k + 1} < j$ necessarily implies $k \leq 3j$ (see the similar reasonings in in the above subcase a). Also, we have $k_1 \geq j + 1$. Indeed, this is a consequence of the fact that $g$ is nondecreasing and because is easy to see that $g(j) < j$.

By Lemma 3.2, (i) it follows that $M_{k_1,n,j}(x) \geq M_{k_1+2,n,j}(x) \geq ...$. We thus obtain $M_{k,n,j}(x) \leq 3\frac{\sqrt{x}}{\sqrt{n}}$ for any $k \in \{k + 1, k + 2, ..., \}$.

Case 3). Subcase a). Suppose first that $k + \sqrt{k} > j$. Then we obtain

$$M_{k,n,j}(x) = m_{k,n,j}(x)(x - \frac{k}{n}) \leq \frac{j + 1}{n} - \frac{k}{n} \leq \frac{k + \sqrt{k} + 1}{n} - \frac{k}{n} \leq \frac{\sqrt{k} + 1}{n} \leq \frac{\sqrt{j - 2} + 1}{n} \leq \frac{1}{\sqrt{n}} \cdot \frac{\sqrt{j - 2} + 1}{\sqrt{n}} \leq 2\frac{\sqrt{x}}{\sqrt{n}}.$$
taking into account that \( \sqrt{\frac{k+1}{n}} \leq 2 \sqrt{\frac{k}{n}} \leq 2 \sqrt{\frac{k}{n}}. \)

Subcase b). Suppose now that \( k + \sqrt{k} \leq j. \) Let \( \tilde{k} \in \{0, 1, ...\} \) be the minimum value such that \( \tilde{k} + \sqrt{k} > j. \) Then \( k_2 = \tilde{k} - 1 \) satisfies \( k_2 + \sqrt{k_2} \leq j \) and

\[
M_{k_2,n,j}(x) = m_{k_2,n,j}(x)(x - \frac{\tilde{k} - 1}{n}) \leq \frac{j + 1 - \tilde{k} - 1}{n} \leq \frac{\sqrt{k + 1} + 1}{n} = \frac{\sqrt{k + 2}}{n} \leq 4 \frac{\sqrt{\frac{k}{n}}}{\sqrt{n}}.
\]

For the last inequality we used the obvious relationship \( \tilde{k} - 1 = k_2 \leq k_2 + \sqrt{k_2} \leq j, \) which implies \( \tilde{k} \leq j + 1 \) and \( \sqrt{k + 2} \leq \sqrt{j + 1 + 2} \leq 4 \sqrt{\frac{j}{n}}. \) Also, because \( j \geq 1 \) it is immediate that \( k_2 \leq j - 1. \)

By Lemma 3.2, (ii) it follows that \( M_{k_2,n,j}(x) \geq M_{k_2-1,n,j}(x) \geq ... \geq M_{0,n,j}(x). \)

We thus obtain \( M_{k,n,j}(x) \leq 4 \frac{\sqrt{\frac{k}{n}}}{\sqrt{n}} \) for any \( k \leq j - 1 \) and \( x \in \left[ \frac{j}{n}, \frac{j+1}{n} \right]. \)

Collecting all the above estimates we get (2), which completes the proof. □

**Remark.** It is clear that on each compact subinterval \([0, a]\), with arbitrary \( a > 0, \) the order of approximation in Theorem 4.1 is \( O(1/\sqrt{n}). \) In what follows, we will prove that this order cannot be improved. In this sense, first we observe that

\[
M_{k,n,j}(x) = (nx)^{k-j} \frac{j!}{k!} \left| \frac{k}{n} - x \right| = (nx)^{k-j} \frac{1}{(j+1)(j+2)...k} \left| \frac{k}{n} - x \right|
\]

for any \( k > j. \)

Now, for \( n \in \mathbb{N} \) and \( a > 0, \) let us denote \( j_n = [na], k_n = [na] + [\sqrt{n}], x_n = \frac{[na]}{n}. \) Then

\[
M_{k_n,n,j_n}(x_n) \geq \left( \frac{[na]}{[na] + [\sqrt{n}]} \right)^{\sqrt{n}} \frac{\sqrt{n}}{n} \geq \left( \frac{na - 1}{na + \sqrt{n}} \right)^{\sqrt{n}} \frac{\sqrt{n}}{n} - 1
\]

for any \( n \geq \max\{4, 1/a\}. \) Because \( \lim_{n \to \infty} \left( \frac{na - 1}{na + \sqrt{n}} \right)^{\sqrt{n}} = e^{-1/a} \) it follows that there exists \( n_0 \in \mathbb{N}, \) \( n_0 \geq \max\{4, 1/a\}, \) such that

\[
\left( \frac{na - 1}{na + \sqrt{n}} \right)^{\sqrt{n}} \geq e^{-1-1/a},
\]

for any \( n \geq n_0. \) Then we get

\[
M_{k_n,n,j_n}(x_n) \geq \frac{1}{2} e^{-1-1/a} \frac{1}{\sqrt{n}}.
\]
Since \( x_n \leq a \) and \( \lim_{n \to \infty} x_n = a \), we get \( x_n \in [0, a] \) for any \( n \in \mathbb{N} \), and combining that with the relationship (2) in the proof of Theorem 4.1, it easily implies that \( \frac{1}{\sqrt{n}} \), the order of \( \max_{x \in [0, a]} \{ E_n(x) \} \), cannot be made smaller. Finally, this implies that the order of approximation \( \omega_1(f; 1/\sqrt{n}) \) on \([0, a]\) obtained by the statement of Theorem 4.1, cannot be improved.

In what follows we will prove that for some subclasses of functions \( f \), the order of approximation \( \omega_1(f; \sqrt{x}/\sqrt{n}) \) in Theorem 4.1 can essentially be improved to \( \omega_1(f; 1/n) \).

For this purpose, for any \( k, j \in \{0, 1, \ldots, \} \), let us define the functions \( f_{k,n,j} : [\frac{j}{n}, \frac{j+1}{n}] \to \mathbb{R} \),

\[
f_{k,n,j}(x) = m_{k,n,j}(x)f\left(\frac{k}{n}\right) = \frac{s_{n,k}(x)}{s_{n,j}(x)}f\left(\frac{k}{n}\right) = \frac{j!}{k!} \cdot (nx)^{k-j} f\left(\frac{k}{n}\right).
\]

Then it is clear that for any \( j \in \{0, 1, \ldots, \} \) and \( x \in [\frac{j}{n}, \frac{j+1}{n}] \) we can write

\[
F_n^{(M)}(f)(x) = \bigvee_{k=0}^{\infty} f_{k,n,j}(x).
\]

Also, we need the following auxiliary lemmas.

**Lemma 4.2.** Let \( f : [0, \infty) \to [0, \infty) \) be bounded and such that

\[
F_n^{(M)}(f)(x) = \max\{f_{j,n,j}(x), f_{j+1,n,j}(x)\} \quad \text{for all } x \in [j/n, (j+1)/n].
\]

Then

\[
\left| F_n^{(M)}(f)(x) - f(x) \right| \leq \omega_1\left(f; \frac{1}{n}\right), \quad \text{for all } x \in [j/n, (j+1)/n],
\]

where \( \omega_1(f; \delta) = \max\{|f(x) - f(y)|; x, y \in [0, \infty), |x - y| \leq \delta\} < \infty. \)

**Proof.** We distinguish two cases:

Case (i). Let \( x \in [j/n, (j+1)/n] \) be fixed such that \( F_n^{(M)}(f)(x) = f_{j,n,j}(x) \). Because by simple calculation we have \( 0 \leq x - \frac{k}{n} \leq \frac{1}{n} \) and \( f_{j,n,j}(x) = f\left(\frac{j}{n}\right) \), it follows that

\[
\left| F_n^{(M)}(f)(x) - f(x) \right| \leq \omega_1\left(f; \frac{1}{n}\right).
\]

Case (ii). Let \( x \in [j/n, (j+1)/n] \) be such that \( F_n^{(M)}(f)(x) = f_{j+1,n,j}(x) \). We have two subcases:

(\(ii_a\)) \( F_n^{(M)}(f)(x) \leq f(x) \), when evidently \( f_{j,n,j}(x) \leq f_{j+1,n,j}(x) \leq f(x) \) and we immediately get

\[
\left| F_n^{(M)}(f)(x) - f(x) \right| = \left| f_{j+1,n,j}(x) - f(x) \right|
\]

\[
= f(x) - f_{j+1,n,j}(x) \leq f(x) - f(j/n) \leq \omega_1\left(f; \frac{1}{n}\right).
\]

(iii_b) \( F_n^{(M)}(f)(x) > f(x) \), when

\[
\left| F_n^{(M)}(f)(x) - f(x) \right| = f_{j+1,n,j}(x) - f(x) = m_{j+1,n,j}(x)f\left(\frac{j+1}{n}\right) - f(x)
\]
$f\left(\frac{j+1}{n}\right) - f(x)$.

Because $0 \leq \frac{j+1}{n} - x \leq \frac{1}{n}$ it follows $f\left(\frac{j+1}{n}\right) - f(x) \leq \omega_1 \left(f; \frac{1}{n}\right)$, which proves the lemma.

**Lemma 4.3.** If the function $f : [0, \infty) \rightarrow [0, \infty)$ is concave, then the function $g : (0, \infty) \rightarrow (0, \infty)$, $g(x) = \frac{f(x)}{x}$ is nonincreasing.

**Proof.** Let $x, y \in (0, \infty)$ be with $x \leq y$. Then

$$f(x) = f\left(\frac{x}{y} + \frac{y-x}{y}0\right) \geq \frac{x}{y}f(y) + \frac{y-x}{y}f(0) \geq \frac{x}{y}f(y),$$

which implies $\frac{f(x)}{x} \geq \frac{f(y)}{y}$. □

**Corollary 4.4.** If $f : [0, \infty) \rightarrow [0, \infty)$ is bounded, nondecreasing and such that the function $g : (0, \infty) \rightarrow (0, \infty)$, $g(x) = \frac{f(x)}{x}$ is nonincreasing, then

$$|F_n^{(M)}(f)(x) - f(x)| \leq \omega_1 \left(f; \frac{1}{n}\right), \text{ for all } x \in [0, \infty).$$

**Proof.** Since $f$ is nondecreasing it follows (see the proof of Theorem 5.4 in the next section)

$$F_n^{(M)}(f)(x) = \bigvee_{k \geq j} f_{k,n,j}(x), \text{ for all } x \in [j/n, (j+1)/n].$$

Let $x \in [0, \infty)$ and $j \in \{0, 1, \ldots\}$ such that $x \in \left[\frac{j}{n}, \frac{j+1}{n}\right]$. Let $k \in \{0, 1, \ldots, n\}$ be with $k \geq j$. Then

$$f_{k+1,n,j}(x) = \frac{j!}{(k+1)!}(nx)^{k+1-j}f\left(\frac{k+1}{n}\right) = \frac{(nx)^j}{(k+1)!}(nx)^{k-j}f\left(\frac{k+1}{n}\right).$$

Since $g(x)$ is nonincreasing we get $\frac{f(k+1)}{x} \leq \frac{f(k)}{x}$ that is $f\left(\frac{k+1}{n}\right) \leq \frac{k+1}{k}f\left(\frac{k}{n}\right)$. From $x \leq \frac{j+1}{n}$ it follows

$$f_{k+1,n,j}(x) \leq \frac{(j+1)!}{(k+1)!}(nx)^{k-j} \cdot \frac{k+1}{k}f\left(\frac{k}{n}\right) = f_{k,n,j}(x)\frac{j+1}{k}.$$

It is immediate that for $k \geq j+1$ we have $f_{k,n,j}(x) \geq f_{k+1,n,j}(x)$. Thus we obtain

$$f_{j+1,n,j}(x) \geq f_{j+2,n,j}(x) \geq \cdots \geq f_{n,n,j}(x) \geq \cdots$$

that is

$$F_n^{(M)}(f)(x) = \max\{f_{j,n,j}(x), f_{j+1,n,j}(x)\}, \text{ for all } x \in [j/n, (j+1)/n],$$

and from Lemma 4.2 we obtain

$$\left|F_n^{(M)}(f)(x) - f(x)\right| \leq \omega_1 \left(f; \frac{1}{n}\right).$$
Corollary 4.5. Let \( f : [0, \infty) \to [0, \infty) \) be a bounded, nondecreasing concave function. Then

\[
\left| F_n^{(M)}(f)(x) - f(x) \right| \leq \omega_1 \left( f; \frac{1}{n} \right), \quad \text{for all } x \in [0, \infty).
\]

Proof. The proof is immediate by Lemma 4.3 and Corollary 4.4.

Remarks. 1) If we suppose, for example, that in addition to the hypothesis in Corollary 4.5, \( f : [0, \infty) \to [0, \infty) \) is a Lipschitz function, that is there exists \( M > 0 \) such that \( |f(x) - f(y)| \leq M|x - y| \), for all \( x, y \in [0, \infty) \), then it follows that the order of uniform approximation on \([0, \infty)\) by \( F_n^{(M)}(f)(x) \) is \( \frac{1}{n} \), which is essentially better than the order \( \frac{1}{\sqrt{n}} \) obtained from Theorem 4.1 on each compact subinterval \([0, a]\) for \( f \) Lipschitz function on \([0, \infty)\).

2) It is known that for the linear Favard-Szász-Mirakjan operator given by

\[
F_n(f)(x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f(k/n),
\]

the best possible uniform approximation result is given by the equivalence (see [10]),

\[
\|F_n(f) - f\| \sim \omega_2^\varphi(f; 1/\sqrt{n}),
\]

where \( \|f\| = \sup\{|f(x)|; x \in [0, \infty)\} \) and \( \omega_2^\varphi(f; \delta) \) is the Ditzian-Totik second order modulus of smoothness on \([0, \infty)\) given by

\[
\omega_2^\varphi(f; \delta) = \sup \{ \sup\{|f(x + h\varphi(x)) - 2f(x) + f(x - h\varphi(x))|; x \in [h^2, \infty)\}; h \in [0, \delta]\},
\]

with \( \varphi(x) = \sqrt{x}, \ \delta \leq 1 \).

Now, if \( f \) is, for example, a nondecreasing concave polygonal line on \([0, \infty)\), constant on an interval \([a, \infty)\), then by simple reasonings we get that \( \omega_2^\varphi(f; \delta) \sim \delta \) for \( \delta \leq 1 \), which shows that the order of approximation obtained in this case by the linear Favard-Szász-Mirakjan operator is exactly \( \frac{1}{\sqrt{n}} \). On the other hand, since such of function \( f \) obviously is a Lipschitz function on \([0, \infty)\) (as having bounded all the derivative numbers) by Corollary 4.5 we get that the order of approximation by the max-product Favard-Szász-Mirakjan operator is less than \( \frac{1}{n} \), which is essentially better than \( \frac{1}{\sqrt{n}} \). In a similar manner, by Corollary 4.4 we can produce many subclasses of functions for which the order of approximation given by the max-product Favard-Szász-Mirakjan operator is essentially better than the order of approximation given by the linear Favard-Szász-Mirakjan operator. Intuitively, the max-product Favard-Szász-Mirakjan operator has better approximation properties than its linear counterpart, for non-differentiable functions in a finite number of points (with the graphs having some “corners”), as for example for functions defined as a maximum of a finite number of continuous functions on \([0, \infty)\).

3) Since it is clear that a bounded nonincreasing concave function on \([0, \infty)\) necessarily one reduces to a constant function, the approximation of such functions is not of interest.
5 Shape Preserving Properties

In this section we will present some shape preserving properties. First we have the following simple result.

**Lemma 5.1.** For any arbitrary bounded function \( f : [0, \infty) \to \mathbb{R}_+ \), the max-product operator \( F_n^{(M)}(f)(x) \) is positive, bounded, continuous on \([0, \infty)\) and satisfies \( F_n^{(M)}(f)(0) = f(0) \).

**Proof.** The positivity of \( F_n^{(M)}(f)(x) \) is immediate. Also, if \( f(x) \leq K \) for all \( x \in [0, \infty) \) it is immediate that \( F_n^{(M)}(f)(x) \leq K \), for all \( x \in [0, \infty) \).

From Lemma 3.3, taking into account that \( s_{n,j}((j+1)/n) = s_{n,j+1}((j+1)/n) \), we immediately obtain that the denominator is a continuous function on \([0, \infty)\). Also, since \( s_{n,k}(x) > 0 \) for all \( x \in (0, \infty) \), \( n \in \mathbb{N} \), \( k \in \{0, 1, ..., \} \), it follows that the denominator \( \bigvee_{k=0}^{\infty} s_{n,k}(x) > 0 \) for all \( x \in (0, \infty) \) and \( n \in \mathbb{N} \).

To prove the continuity on \([0, \infty)\) of the numerator, let us denote \( h(x) = \bigvee_{k=0}^{\infty} s_{n,k}(x)f(k/n) \), and for each \( m \in \mathbb{N} \), \( h_m(x) = \bigvee_{k=0}^{m} s_{n,k}(x)f(k/n) \). It is clear that for each \( m \in \mathbb{N} \), the function \( h_m(x) \) is continuous on \([0, \infty)\), as a maximum of finite number of continuous functions. Also, fix \( a > 0 \) arbitrary and consider \( x \in [0, a] \). First, since

\[
0 \leq h(x) = \max \left\{ \bigvee_{k=0}^{m} s_{n,k}(x)f(k/n), \bigvee_{k=m+1}^{\infty} s_{n,k}(x)f(k/n) \right\} \\
\leq \bigvee_{k=0}^{m} s_{n,k}(x)f(k/n) + \bigvee_{k=m+1}^{\infty} s_{n,k}(x)f(k/n),
\]

it follows that for all \( m \in \mathbb{N} \) we have

\[
0 \leq h(x) - h_m(x) \leq \bigvee_{k=m+1}^{\infty} s_{n,k}(x)f(k/n) \leq \bigvee_{k=m+1}^{\infty} \frac{(na)^k}{k!} K, \text{ for all } x \in [0, a],
\]

where \( 0 \leq f_m(x) \leq K \) for all \( x \in [0, \infty) \).

Now, fix \( \varepsilon > 0 \). Since \( s_{n,k+1}(a) = s_{n,k}(a) \) there exists an index \( k_0 > 0 \) (independent of \( x \)), such that \( \frac{na}{k+1} < \varepsilon, \) for all \( k \geq k_0 \). Choose now \( m = k_0 \). It is immediate that

\[
\bigvee_{k=m+1}^{\infty} \frac{(na)^k}{k!} K < \varepsilon \cdot \frac{K(na)^{k_0}}{k_0!}, \text{ which implies that}
\]

\[
0 \leq h(x) - h_m(x) < \varepsilon \cdot \frac{K(na)^{k_0}}{k_0!}, \text{ for all } x \in [0, a] \text{ and } m \geq k_0.
\]

This implies that the numerator \( h(x) \) is the uniform limit (as \( m \to \infty \)) of a sequence of continuous functions on \([0, a]\), \( h_m(x), m \in \mathbb{N} \), which implies the continuity of \( h(x) \) on \([0, a]\). Because \( a > 0 \) was chosen arbitrary, it follows the continuity of \( h(x) \) on \([0, \infty)\).
As a first conclusion, we get the continuity of $F_n^{(M)}(f)(x)$ on $(0, \infty)$.

To prove now the continuity of $F_n^{(M)}(f)(x)$ at $x = 0$, we observe that $s_{n,k}(0) = 0$ for all $k \in \{1, 2, \ldots \}$ and $s_{n,k}(0) = 1$ for $k = 0$, which implies that $\bigvee_{k=0}^{\infty} s_{n,k}(x) = 1$ if $x = 0$. Thus the fact that $F_n^{(M)}(f)(x)$ coincides with $f(x)$ at $x = 0$ immediately follows from the above considerations, proving the theorem.

**Remark.** Note that because of the continuity of $F_n^{(M)}(f)(x)$ on $[0, \infty)$, it suffices to prove the shape properties of $F_n^{(M)}(f)(x)$ on $(0, \infty)$ only. As a consequence, in the notations and proofs below we always may suppose that $x > 0$.

As in Section 4, for any $k, j \in \{0, 1, \ldots \}$, let us consider the functions $f_{k,n,j} : [\frac{j}{n}, \frac{j+1}{n}] \to \mathbb{R}$,

$$f_{k,n,j}(x) = m_{k,n,j}(x)f\left(\frac{k}{n}\right) = \frac{s_{n,k}(x)}{s_{n,j}(x)} f\left(\frac{k}{n}\right) = \frac{j!}{k!} \cdot (nx)^{k-j} f\left(\frac{k}{n}\right).$$

For any $j \in \{0, 1, \ldots \}$ and $x \in [\frac{j}{n}, \frac{j+1}{n}]$ we can write

$$F_n^{(M)}(f)(x) = \bigvee_{k=0}^{\infty} f_{k,n,j}(x).$$

**Lemma 5.2.** If $f : [0, \infty) \to \mathbb{R}_+$ is a nondecreasing function then for any $k, j \in \{0, 1, \ldots \}$ with $k \leq j$ and $x \in [\frac{j}{n}, \frac{j+1}{n}]$ we have $f_{k,n,j}(x) \geq f_{k-1,n,j}(x)$.

**Proof.** Because $k \leq j$, by the proof of Lemma 3.1, case 2), it follows that $m_{k,n,j}(x) \geq m_{k-1,n,j}(x)$. From the monotonicity of $f$ we get $f\left(\frac{k}{n}\right) \geq f\left(\frac{k-1}{n}\right)$. Thus we obtain

$$m_{k,n,j}(x)f\left(\frac{k}{n}\right) \geq m_{k-1,n,j}(x)f\left(\frac{k-1}{n}\right),$$

which proves the lemma. 

**Corollary 5.3.** If $f : [0, \infty) \to \mathbb{R}_+$ is nonincreasing then $f_{k,n,j}(x) \geq f_{k+1,n,j}(x)$ for any $k, j \in \{0, 1, \ldots \}$ with $k \geq j$ and $x \in [\frac{j}{n}, \frac{j+1}{n}]$.

**Proof.** Because $k \geq j$, by the proof of Lemma 3.1, case 1), it follows that $m_{k,n,j}(x) \geq m_{k+1,n,j}(x)$. From the monotonicity of $f$ we get $f\left(\frac{k}{n}\right) \geq f\left(\frac{k+1}{n}\right)$. Thus we obtain

$$m_{k,n,j}(x)f\left(\frac{k}{n}\right) \geq m_{k+1,n,j}(x)f\left(\frac{k+1}{n}\right),$$

which proves the corollary.

**Theorem 5.4.** If $f : [0, \infty) \to \mathbb{R}_+$ is nondecreasing and bounded on $[0, \infty)$ then $F_n^{(M)}(f)$ is nondecreasing (and bounded).

**Proof.** Because $F_n^{(M)}(f)$ is continuous (and bounded) on $[0, \infty)$, it suffices to prove that on each subinterval of the form $[\frac{j}{n}, \frac{j+1}{n}]$, with $j \in \{0, 1, \ldots \}$, $F_n^{(M)}(f)$ is nondecreasing.
So let \( j \in \{0, 1, \ldots \} \) and \( x \in \left[\frac{j}{n}, \frac{j+1}{n}\right] \). Because \( f \) is nondecreasing, from Lemma 5.2 it follows that
\[
f_{j,n,j}(x) \geq f_{j-1,n,j}(x) \geq f_{j-2,n,j}(x) \geq \ldots \geq f_{0,n,j}(x).
\]
But then it is immediate that
\[
F_n^{(M)}(f)(x) = \bigvee_{k \geq j} f_{k,n,j}(x),
\]
for all \( x \in \left[\frac{j}{n}, \frac{j+1}{n}\right] \). Clearly that for \( k \geq j \) the function \( f_{k,n,j} \) is nondecreasing and since \( F_n^{(M)}(f) \) is defined as supremum of nondecreasing functions, it follows that it is nondecreasing. \( \square \)

**Corollary 5.5.** If \( f : [0, \infty) \to \mathbb{R}_+ \) is nonincreasing then \( F_n^{(M)}(f) \) is nonincreasing.

**Proof.** By hypothesis, \( f \) implicitly is bounded on \([0, \infty)\). Because \( F_n^{(M)}(f) \) is continuous and bounded on \([0, \infty)\), it suffices to prove that on each subinterval of the form \([\frac{j}{n}, \frac{j+1}{n}]\), with \( j \in \{0, 1, \ldots \} \), \( F_n^{(M)}(f) \) is nonincreasing.

So let \( j \in \{0, 1, \ldots \} \) and \( x \in \left[\frac{j}{n}, \frac{j+1}{n}\right] \). Because \( f \) is nonincreasing, from Corollary 5.3 it follows that
\[
f_{j,n,j}(x) \geq f_{j+1,n,j}(x) \geq f_{j+2,n,j}(x) \geq \ldots
\]
But then it is immediate that
\[
F_n^{(M)}(f)(x) = \bigvee_{k \geq 0} f_{k,n,j}(x),
\]
for all \( x \in \left[\frac{j}{n}, \frac{j+1}{n}\right] \). Clearly that for \( k \leq j \) the function \( f_{k,n,j} \) is nonincreasing and since \( F_n^{(M)}(f) \) is defined as the maximum of nonincreasing functions, it follows that it is nonincreasing. \( \square \)

In what follows, let us consider the following concept generalizing the monotonicity and convexity.

**Definition 5.6.** Let \( f : [0, \infty) \to \mathbb{R} \) be continuous on \([0, \infty)\). One says that \( f \) is quasi-convex on \([0, \infty)\) if it satisfies the inequality
\[
f(\lambda x + (1 - \lambda)y) \leq \max\{f(x), f(y)\}, \text{ for all } x, y \in [0, \infty) \text{ and } \lambda \in [0, 1].
\]
(see e.g. the book [8], p. 4, (iv)).

**Remark.** By [9], the continuous function \( f \) is quasi-convex on the bounded interval \([0, a]\), equivalently means that there exists a point \( c \in [0, a] \) such that \( f \) is nonincreasing on \([0, c]\) and nondecreasing on \([c, a]\). But this property easily can be extended to continuous quasiconvex functions on \([0, \infty)\), in the sense that there exists \( c \in [0, \infty) \) (\( c = \infty \) by convention for nonincreasing functions on \([0, \infty)\)) such that \( f \) is nonincreasing on \([0, c]\) and nondecreasing on \([c, \infty)\). This easily follows
from the fact that the quasiconvexity of \( f \) on \([0, \infty)\) means the quasiconvexity of \( f \) on any bounded interval \([0, a]\), with arbitrary large \( a > 0\).

The class of quasi-convex functions includes the both classes of nondecreasing functions and of nonincreasing functions (obtained from the class of quasi-convex functions by taking \( c = 0 \) and \( c = \infty \), respectively). Also, it obviously includes the class of convex functions on \([0, \infty)\).

**Corollary 5.7.** If \( f : [0, \infty) \to \mathbb{R}_+ \) is continuous, bounded and quasi-convex on \([0, \infty)\) then for all \( n \in \mathbb{N} \), \( F_n^{(M)}(f) \) is quasi-convex on \([0, \infty)\).

**Proof.** If \( f \) is nonincreasing (or nondecreasing) on \([0, \infty)\) (that is the point \( c = \infty \) (or \( c = 0 \)) in the above Remark) then by the Corollary 5.5 (or Theorem 5.4, respectively) it follows that for all \( n \in \mathbb{N} \), \( F_n^{(M)}(f) \) is nonincreasing (or nondecreasing) on \([0, \infty)\).

Suppose now that there exists \( c \in (0, \infty) \), such that \( f \) is nonincreasing on \([0, c]\) and nondecreasing on \([c, \infty)\). Define the functions \( F, G : [0, \infty) \to \mathbb{R}_+ \) by \( F(x) = f(x) \) for all \( x \in [0, c]\), \( F(x) = f(c) \) for all \( x \in [c, \infty) \) and \( G(x) = f(c) \) for all \( x \in [0, c] \), \( G(x) = f(x) \) for all \( x \in [c, \infty) \).

It is clear that \( F \) is nonincreasing and continuous on \([0, \infty)\), \( G \) is nondecreasing and continuous on \([0, \infty)\) and that \( f(x) = \max\{F(x), G(x)\} \), for all \( x \in [0, \infty) \).

But it is easy to show (see also Remark 1 after the proof of Lemma 2.1) that
\[
F_n^{(M)}(f)(x) = \max\{F_n^{(M)}(F)(x), F_n^{(M)}(G)(x)\}, \text{ for all } x \in [0, \infty),
\]
where by the Corollary 5.5 and Theorem 5.4 , \( F_n^{(M)}(F)(x) \) is nonincreasing and continuous on \([0, \infty)\) and \( F_n^{(M)}(G)(x) \) is nondecreasing and continuous on \([0, \infty)\).

We have two cases: 1) \( F_n^{(M)}(F)(x) \) and \( F_n^{(M)}(G)(x) \) do not intersect each other ; 2) \( F_n^{(M)}(F)(x) \) and \( F_n^{(M)}(G)(x) \) intersect each other.\footnote{This is clear that \( F_n^{(M)}(f)(x) \) is quasi-convex on \([0, \infty)\).

**Case 1.** We have \( \max\{F_n^{(M)}(F)(x), F_n^{(M)}(G)(x)\} = F_n^{(M)}(f)(x) \) for all \( x \in [0, \infty) \) or \( \max\{F_n^{(M)}(F)(x), F_n^{(M)}(G)(x)\} = F_n^{(M)}(f)(x) \) for all \( x \in [0, \infty) \), which obviously proves that \( F_n^{(M)}(f)(x) \) is quasi-convex on \([0, \infty)\).

**Case 2.** In this case it is clear that there exists a point \( c' \in [0, \infty) \) such that \( F_n^{(M)}(f)(x) \) is nonincreasing on \([0, c']\) and nondecreasing on \([c', \infty)\), which by the considerations in the above Remark implies that \( F_n^{(M)}(f)(x) \) is quasiconvex on \([0, \infty)\) and proves the corollary.\footnote{It is of interest to exactly calculate \( F_n^{(M)}(f)(x) = e_0(x) = 1 \) and for \( e_1(x) = x \). In this sense we can state the following.}

**Lemma 5.8.** For all \( x \in [0, \infty) \) and \( n \in \mathbb{N} \) we have \( F_n^{(M)}(e_0)(x) = 1 \) and \( F_n^{(M)}(e_1)(x) = x \).

**Proof.** The formula \( F_n(e_0)(x) = 1 \) is immediate by the definition of \( F_n^{(M)}(f)(x) \). To find the formula for \( F_n^{(M)}(e_1)(x) \), we observe that
\[
\lim_{k \to \infty} s_{n,k}(x) = \lim_{k \to \infty} s_{n,k}(x) = x \cdot \lim_{k \to \infty} s_{n,k-1}(x) = x \lim_{j \to \infty} s_{n,j}(x),
\]
which implies
\[ F_n^{(M)}(e_1)(x) = x \cdot \frac{\sum_{j=0}^{\infty} s_{n,j}(x)}{\sum_{k=0}^{\infty} s_{n,k}(x)} = x. \]

Also, we can prove the interesting property that for any arbitrary function \( f \), the max-product Bernstein operator \( F_n^{(M)}(f) \) is piecewise convex on \([0, \infty)\). In this sense the following result holds.

**Theorem 5.9.** For any function \( f : [0, \infty) \to [0, \infty) \), \( F_n^{(M)}(f) \) is convex on any interval of the form \([\frac{j}{n}, \frac{j+1}{n}]\), \( j = 0, 1, \ldots \).

**Proof.** For any \( k, j \in \{0, 1, \ldots \} \) let us consider the functions \( f_{k,n,j} : [\frac{j}{n}, \frac{j+1}{n}] \to \mathbb{R} \),
\[ f_{k,n,j}(x) = m_{k,n,j}(x)f(\frac{k}{n}) = \frac{j!(nx)^{k-j}}{k!}f(\frac{k}{n}). \]
Clearly we have
\[ F_n^{(M)}(f)(x) = \sup_{k=0}^{\infty} f_{k,n,j}(x), \]
for any \( j \in \{0, 1, \ldots \} \) and \( x \in [\frac{j}{n}, \frac{j+1}{n}] \).

We will prove that for any fixed \( j \), each function \( f_{k,n,j}(x) \) is convex on \([\frac{j}{n}, \frac{j+1}{n}]\), which will imply that \( F_n^{(M)}(f) \) can be written as a supremum of some convex functions on \([\frac{j}{n}, \frac{j+1}{n}]\).

Since \( f \geq 0 \) and \( f_{k,n,j}(x) = \frac{j!(nx)^{k-j}}{k!} \cdot x^{k-j} \cdot f(k/n) \), it suffices to prove that the functions \( g_{k,j} : [0, 1] \to \mathbb{R}_+ \), \( g_{k,j}(x) = x^{k-j} \) are convex on \([\frac{j}{n}, \frac{j+1}{n}]\).

For \( k = j \), \( g_{j,j} \) is constant so is convex.

For \( k = j + 1 \) we get \( g_{j+1,j}(x) = x \) for any \( x \in [\frac{j}{n}, \frac{j+1}{n}] \), which obviously is convex.

For \( k = j - 1 \) it follows \( g_{j-1,j}(x) = \frac{1}{x} \) for any \( x \in [\frac{j}{n}, \frac{j+1}{n}] \). Then \( g_{j-1,j}(x) = \frac{2}{x^2} > 0 \) for any \( x \in [\frac{j}{n}, \frac{j+1}{n}] \).

If \( k \geq j + 2 \) then \( g''_{k,j}(x) = (k-j)(k-j-1)x^{k-j-2} > 0 \) for any \( x \in [\frac{j}{n}, \frac{j+1}{n}] \).

If \( k \leq j - 2 \) then \( g''_{k,j}(x) = (k-j)(k-j-1)x^{k-j-2} > 0 \) for any \( x \in [\frac{j}{n}, \frac{j+1}{n}] \).

Since all the functions \( g_{k,j} \) are convex on \([\frac{j}{n}, \frac{j+1}{n}]\), we get that \( F_n^{(M)}(f) \) is convex on \([\frac{j}{n}, \frac{j+1}{n}]\) as maximum of these functions, proving the theorem. □

**References**


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