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ON TRACES OF ANALYTIC Q_p TYPE SPACES, MIXED NORM SPACES AND HARMONIC BERGMAN CLASSES ON CERTAIN POLYDOMAINS

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Abstract

In this paper, we introduce new Q_p type spaces and mixed norm analytic function spaces on polyballs and describe completely their traces on unit ball. Complete descriptions of traces of harmonic Bergman classes on products of unit balls of $\mathbb{R}^n$ and products of $\mathbb{R}^{n+1}_+$ halfspaces will be also provided.

1 Introduction and preliminaries

The goal of this paper is to give complete descriptions of traces of certain Q_p type spaces and mixed norm spaces of analytic functions in polyballs. In recent years many papers were devoted to the study of Q_p type spaces on the unit disk and the unit ball (see, for example, [7], [11], [21], [22], [23] and the references there). The mixed norm classes in polyballs that we introduce and study in this paper have their origins in real analysis where they were investigated intensively for many years (see e.g. [3], [9], [10], [20]). They also can serve as an example of direct generalizations of well-studied analytic Bergman classes in the polydisk and in the unit ball at the same time (see [23], [7]). We also observe that for $n = 1$ the mentioned trace problem completely coincide with the well known problem of diagonal map which previously has been considered by many authors [5], [7], [8], [12], [13], [14]. Various applications of theorems of the diagonal map are well known [2], [7], [15]. This paper can be considered as a continuation of [16] and [17] where we considered and solved such a trace problem for classical Bergman classes in polyballs and some new function classes defined with the help of the Luzin area operator and the Bergman metric ball. Basic properties of the so called r-lattice in the Bergman metric ball can be found in [23] and are playing an important role in all our proofs. At the end

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of paper, the complete descriptions of traces of harmonic functions on the products of unit balls of $\mathbb{R}^n$ will be also provided.

Throughout the paper, we write C (sometimes with indexes) to denote a positive constant which might be different at each occurrence (even in a chain of inequalities) but is independent of the functions or variables being discussed.

Let $\mathbb{C}$ denote the set of complex numbers and let $\mathbb{C}^n = \mathbb{C} \times \cdots \times \mathbb{C}$ denote the Euclidean space of complex dimension n. The open unit ball in $\mathbb{C}^n$ is the set $B_n = \{z \in \mathbb{C}^n : |z| < 1\}$. We denote by $H(B_n)$ the space of holomorphic functions on the open unit ball in $\mathbb{C}^n$. Moreover, let $d\nu$ denote the Lebesgue measure on B_n normalized such that $\nu(B_n) = 1$ and for any $\alpha \in R$, let $d\nu_\alpha(z) = c_\alpha(1-|z|^2)^\alpha d\nu(z)$ for $z \in B_n$. Here, if $\alpha \le -1$, $c_\alpha = 1$ and if $\alpha > -1$, $c_\alpha = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)}$ is the normalizing constant so that ν_α has unit total mass. The Bergman metric on B_n is

$$\beta(z,w) = \frac{1}{2}\log\frac{1+|\varphi_z(w)|}{1-|\varphi_z(w)|},$$

where φ_z is the Möbius transformation of B_n that interchanges 0 and z. Let $D(a,r) = \{z \in B_n : \beta(z,a) < r\}$ denote the Bergman metric ball centered at $a \in B_n$ with radius $r > 0$.

The proofs of the following properties of the Bergman balls can be found in [23] (see lemmas 1.24, 2.20, 2.24 and 2.27 in [23]).

Lemma 1. (*a*) *There exists a positive number $N \ge 1$ such that for any $0 < r \le 1$ we can find a sequence $\{v_k\}_{k=1}^\infty$ in B_n to be r-lattice in the Bergman metric of B_n. This means that $B_n = \cup_{k=1}^\infty D(v_k, r)$, $D(v_l, r/4) \cap D(v_k, r/4) = \emptyset$ if $k \ne l$ and each $z \in B_n$ belongs to at most N of the sets $D(v_k, 2r)$.*
(*b*) *For any $r > 0$ there is a constant $C > 0$ so that $\frac{1}{C} \le |\frac{1-\langle z,w\rangle}{1-\langle z,v\rangle}| \le C$ for all $z \in B_n$ and all w, v with $\beta(w,v) < r$.*
(*c*) *For any $\alpha > -1$ and $r > 0$, $\int_{D(z,r)}(1-|w|^2)^\alpha d\nu(w)$ is comparable with $(1-|z|^2)^{n+1+\alpha}$ for all $z \in B_n$.*
(*d*) *Suppose $r > 0$ and $p > 0$ and $\alpha > -1$. Then there is a constant $C > 0$ such that*

$$|f(z)|^p \le \frac{C}{(1-|z|^2)^{n+1+\alpha}}\int_{D(z,r)}|f(w)|^p d\nu_\alpha(w),$$

for all $f \in H(B_n)$ and all $z \in B_n$.

The following estimate is well-known and will be used often in the paper. For a proof, see [23], Theorem 1.12. The conclusion about the behaviour of the constants as $t \to -1$ or $s \to 0$ follows from a careful examination of the above mentioned proof in [23].

Lemma 2. *Suppose that $c > 0$ and $t > -1$. Then there are positive constants C_1, C_2 such that*

$$C_1\frac{\Gamma(t+1)\Gamma(c)}{(1-|z|^2)^c} \le \int_{B_n}\frac{1-|w|^2)^t}{|1-\langle z,w\rangle|^{n+1+t+c}}d\nu(w) \le C_2\frac{\Gamma(t+1)\Gamma(c)}{(1-|z|^2)^c},$$

for all $z \in B_n$. The constants C_1 and C_2 depend on n, c and t and they are bounded as $t \to -1$ and $s \to 0$.

We also need the following estimate, a proof of which can be found in [11].

Lemma 3. *Suppose that $t > -1$, $s > 0$ and $0 \le r < n+1+t$. Then there is a constant $C > 0$ such that for all $w, v \in B_n$,*

$$\int_{B_n} \frac{(1-|z|)^t d\nu(z)}{|1-\langle w,z\rangle|^{n+1+t+s}\ |1-\langle v,z\rangle|^r} \le \frac{C}{(1-|w|^2)^s\ |1-\langle w,v\rangle|^r}.$$

For $\alpha > -1$ and $p > 0$ the weighted Bergman space A^p_α consists of holomorphic functions f in $L^p(B_n, d\nu_\alpha)$, that is, $A^p_\alpha = L^p(B_n, d\nu_\alpha) \cap H(B_n)$. It is well-known that A^p_α is a closed subspace of $L^p(B_n, d\nu_\alpha)$. See [23, Chapter 2] for more details.

Definition 1. Let X and Y be Banach analytic function spaces on the ball and the polyball so that $X \subset H(B_n)$ and $Y \subset H((B_n)^m)$. Then X is called the *trace* of Y, if for every function f, $f \in Y$, $f(z,\dots,z)$ is in X and the reverse is also true, for every function g, $g \in X$ there exists a function f in Y such that $f(z,\dots,z) = g(z)$ for all $z \in B_n$.

2 Traces of Q_p type spaces and mixed norm analytic function spaces on polyballs

From now on, we fix an integer $m \ge 1$. For any two m-tuples of real numbers $a = (a_1,\dots,a_m)$, and $b = (b_1,\dots,b_m)$, we define the integral operator

$$(S_{a,b}f)(z_1,\dots,z_m) = \prod_{j=1}^m (1-|z_j|^2)^{a_j} \int_{B_n} \frac{f(w)(1-|w|^2)^{-n-1+\sum_{j=1}^m b_j}}{\prod_{j=1}^m (1-\langle z_j,w\rangle)^{a_j+b_j}} d\nu(w),$$

where $z_1,\dots,z_m$ are in B_n and f is a function in $L^1(B_n, d\nu_{-n-1-\sum_{j=1}^m b_j})$. Note that for such f, the function $S_{a,b}f$ is defined on $(B_n)^m$, the product of m copies of B_n. The operator $S_{a,b}$ can be called an expanded Bergman projection in the unit ball (see [23, Chapter 2]).

The following two propositions study the boundedness of $S_{a,b}$ from certain L^p spaces on B_n into certain A^p spaces of $(B_n)^m$. Note that for $m = 1$ and $r_j = 0$, $j = 1,\dots,m$ these assertions are well known (see, for example, [23]).

Proposition 1. *Let $1 < p < \infty$. Suppose $s_1,\dots,s_m > -1$ and $r_1,\dots,r_m \ge 0$ are such that for each $j = 1,\dots,m$, we have $-pa_j < min\{s_j+1, s_j+1+n-r_j\}$ and $ms_j + 1 < p(mb_j - n) - (m-1)(n+1)$. Denote $t = (m-1)(n+1) + \sum_{j=1}^m s_j$. Then there is a constant $C > 0$ such that*

$$\int_{B_n} \cdots \int_{B_n} |(S_{a,b}f)(z_1,\dots,z_m)|^p \prod_{j=1}^m \frac{(1-|z_j|^2)^{s_j}}{|1-\langle u_j,z_j\rangle|^{r_j}} d\nu(z_1)\cdots d\nu(z_m)$$

$$\leq C\int_{B_n}|f(w)|^p\frac{|1-|w|^2|^t}{\prod_{j=1}^m|1-\langle u_j,w\rangle|^{r_j}}d\nu(w),$$

for all $f\in L^p(B_n,d\nu_t)$ *and* $u_1,\dots,u_m\in B_n$.

Proof. Let q denote the exponential conjugate of p, that is $\frac{1}{p}+\frac{1}{q}=1$. Choose a positive number γ such that $p\gamma<\min\{p(mb_j-n)-(m-1)(n-1)-ms_j-1: j=1,\dots,m\}$. Put $\alpha=\frac{1}{m}(\gamma-\frac{1}{q})$ and $\beta=-n-1+\sum_{j=1}^m b_j-m\alpha=-n-1+\sum_{j=1}^m b_j-\gamma+\frac{1}{q}$. For each j, choose e_j such that

$$\frac{n+1}{mq}+\alpha<e_j<\frac{n+1}{mq}+\alpha+min\{\frac{pa_j+s_j+1}{p},\frac{pa_j+s_j+1+npr_j}{p}\}.$$

It is possible to choose such an e_j since $pa_j+s_j+1>0$. Further, let $d_j=a_j+b_j-e_j$. For any measurable function f on B_n and $z_1,\dots,z_m\in B_n$, using Hölder's inequality, we have

$$\int_{B_n}|f(w)|\frac{(1-|w|^2)^{-n-1+\sum_{j=1}^m b_j}}{\prod_{j=1}^m(1-\langle z_j,w\rangle)^{a_j+b_j}}d\nu(w)$$

$$=\int_{B_n}\frac{|f(w)|(1-|w|^2)^{\beta}}{\prod_{j=1}^m|1-\langle z_j,w\rangle|^{d_j}}\prod_{j=1}^m\frac{(1-|w|^2)^{\alpha}}{(1-\langle z_j,w\rangle)^{e_j}}d\nu(w)$$

$$\leq\left(\int_{B_n}\frac{|f(w)|^p(1-|w|^2)^{p\beta}}{\prod_{j=1}^m|1-\langle z_j,w\rangle|^{pd_j}}d\nu(w)\right)^{\frac{1}{p}}\prod_{j=1}^m\left(\int_{B_n}\frac{(1-|w|^2)^{mq\alpha}}{(1-\langle z_j,w\rangle)^{mqe_j}}d\nu(w)\right)^{\frac{1}{mq}}.$$

For each j, since $mq\alpha=q\mu-1>-1$ and $mqe_j>n+1+mq\alpha$, Lemma 2 shows that

$$\int_{B_n}\frac{(1-|w|^2)^{mq\alpha}}{(1-\langle z_j,w\rangle)^{mqe_j}}d\nu(w)\leq C(1-|z_j|^2)^{n+1+mq\alpha-mqe_j},$$

where C is independent of $z_1,\dots,z_m$. This implies that

$$|(S_{a,b}f)(z_1,\dots,z_m)|^p\leq$$
$$\leq C\int_{B_n}\frac{|f(w)|^p(1-|w|^2)^{p\beta}}{\prod_{j=1}^m|1-\langle z_j,w\rangle|^{pd_j}}d\nu(w)\prod_{j=1}^m(|1-|z_j|^2)^{\frac{p(n+1)}{mq}+p(\alpha-e_j+a_j)}.$$

Now by Fubini's theorem

$$\int_{B_n}\cdots\int_{B_n}|(S_{a,b}f)(z_1,\dots,z_m)|^p\prod_{j=1}^m\frac{(1-|z_j|^2)^{s_j}}{|1-\langle u_j,z_j\rangle|^{r_j}}d\nu(z_1)\cdots d\nu(z_m)\qquad(1)$$

$$\leq C\int_{B_n}\{\prod_{j=1}^m\int_{B_n}\frac{(1-|z_j|^2)^{\frac{p(n+1)}{mq}+p(\alpha-e_j+a_j)+s_j}}{|1-\langle z_j,w\rangle|^{pd_j}|1-\langle u_j,z_j\rangle|^{r_j}}d\nu(z_j)\}|f(w)|^p(1-|w|^2)^{p\beta}d\nu(w).$$

For each j, by the choice of e_j and γ, we have $\frac{p(n+1)}{mq} + p(\alpha - e_j + a_j) + s_j > -1$ and $r_j < n + 1 + \frac{p(n+1)}{mq} + p(\alpha - e_j + a_j) + s_j < pd_j$. Applying Lemma 3, we have

$$\int_{B_n} \frac{(1-|z_j|^2)^{\frac{p(n+1)}{mq}+p(\alpha-e_j+a_j)+s_j}}{|1-\langle z_j, w\rangle|^{pd_j}|1-\langle u_j, z_j\rangle|^{r_j}} d\nu(z_j)$$

$$\leq C\frac{(1-|w|^2)^{n+1+\frac{p(n+1)}{mq}+p(\alpha-e_j+a_j)+s_j-pd_j}}{|1-\langle u_j, w\rangle|^{r_j}} \tag{2}$$

$$= C\frac{(1-|w|^2)^{\frac{p\gamma-p(mb_j-n)+(m-1)(n+1)+(ms_j+1)}{m}}}{|1-\langle u_j, w\rangle|^{r_j}},$$

where C is independent of w and u_j. From (1) and (2) and the fact that

$$\sum_{j=1}^{m} \frac{p\gamma - p(mb_j - n) + (m-1)(n+1) + (ms_j+1)}{m}$$

$$= (m-1)(n+1)\sum_{j=1}^{m} s_j - p(\sum_{j=1}^{m} b_j - \mu - n) + 1 = (m-1)(n+1) + \sum_{j=1}^{m} s_j - p\beta,$$

the conclusion of the proposition follows. □

For the case $0 < p \leq 1$, we have the following result.

Proposition 2. *Let $0 < p \leq 1$. Suppose that $s_1, \ldots, s_m > -1$ and $r_1, \ldots, r_m \geq 0$ are such that for each $j = 1, \ldots, m$, we have $-pa_j < min\{s_j + 1, s_j + 1 + n - r_j\}$ and $s_j + 1 < pb_j - n)$. Denote $t = (m-1)(n+1) + \sum_{j=1}^{m} s_j$. Then there is a constant $C > 0$ such that*

$$\int_{B_n} \cdots \int_{B_n} |(S_{a,b}f)(z_1, \ldots, z_m)|^p \prod_{j=1}^{m} \frac{(1-|z_j|^2)^{s_j}}{|1-\langle u_j, z_j\rangle|^{r_j}} d\nu(z_1)\cdots d\nu(z_m)$$

$$\leq C\int_{B_n} |f(w)|^p \frac{(1-|w|^2)^t}{\prod_{j=1}^{m}|1-\langle u_j, w\rangle|^{r_j}} d\nu(w),$$

for all $f \in A^p(B_n, d\nu_t)$ and $u_1, \ldots, u_m \in B_n$.

Proof. Fix $0 < r \leq 1$ and choose $\{v_k\}_{k=1}^{\infty}$ to be an r-lattice in the Bergman metric of B_n. This means that $B_n = \cup_{k=1}^{\infty} D(v_k, r)$ and $D(v_l, r/4) \cap D(v_k, r/4) = \emptyset$ if $k \neq l$ and there is an integer $N \geq 1$ such that each $z \in B_n$ belongs to at most N of the sets $D(v_k, 2r)$ (see Lemma 1). For any function $f \in L^1(B_n, d\nu_n)$ and any $z_1, \ldots, z_m \in B_n$ we have

$$|(S_{a,b}f)(z_1, \ldots, z_m)| \leq \prod_{j=1}^{m}(1-|z_j|^2)^{a_j} \sum_{k=1}^{\infty} \int_{D(v_k,r)} \frac{|f(w)|^p(1-|w|^2)^{-n-1+\sum_{j=1}^{m} b_j}}{\prod_{j=1}^{m}|1-\langle z_j, w\rangle|^{a_j+b_j}} d\nu(w).$$

By Lemma 1 (b) there is a constant $C > 0$ so that for each $j = 1, \dots, m$ and $k \geq 1$, $\frac{1}{C} \leq |\frac{1-\langle z_j, w\rangle}{1-\langle z_j, v_k\rangle}| \leq C$ for all $w \in D(v_k, r)$. Also by Lemma 1 (c), $\int_{D(v_k,r)}(1 - |w|^2)^{-n-1+\sum_{j=1}^m b_j} d\nu(w)$ is comparable with $(1 - |v_k|^2)^{\sum_{j=1}^m b_j}$. Thus we obtain

$$|(S_{a,b}f)(z_1, \dots, z_m)| \leq$$
$$\leq C \sum_{k=1}^{\infty} \prod_{j=1}^{m} \frac{(1 - |z_j|^2)^{a_j}}{|1 - \langle z_j, v_k\rangle|^{a_j+b_j}} \int\limits_{D(v_k,r)} |f(w)|^p (1 - |w|^2)^{-n-1+\sum_{j=1}^m b_j} d\nu(w)$$
$$\leq C \sum_{k=1}^{\infty} \prod_{j=1}^{m} \frac{(1 - |z_j|^2)^{a_j} (1 - |v_k|^2)^{\sum_{j=1}^m b_j}}{|1 - \langle z_j, v_k\rangle|^{a_j+b_j}} sup\{|f(w)|^p : w \in D(v_k, r)\}.$$

Now since $0 < p \leq 1$, using the inequality $(x_1 + x_2 + \dots)^p \leq x_1^p + x_2^p + \dots$, which is valid for non-negative numbers $x_1, x_2, \dots$, we get

$$|(S_{a,b}f)(z_1, \dots, z_m)|^p \prod_{j=1}^{m} \frac{1}{|1 - \langle u_j, z_j\rangle|^{r_j}}$$

$$\leq C \sum_{k=1}^{\infty} \prod_{j=1}^{m} \frac{(1 - |z_j|^2)^{pa_j} (1 - |v_k|^2)^{p\sum_{j=1}^m b_j}}{|1 - \langle z_j, v_k\rangle|^{pa_j+pb_j} |1 - \langle u_j, z_j\rangle|^{r_j}} \sup\{|f(w)|^p : w \in D(v_k, r)\}.$$

Integrating with respect to $d\nu_{s_1}(z_1) \cdots d\nu_{s_m}(z_m)$ and using Lemma 3 (note that by assumption, $pa_j + s_j > -1$ and $pa_j + pb_j > n + 1 + pa_j + s_j > r_j$), we obtain

$$\int_{B_n} \cdots \int_{B_n} |(S_{a,b}f)(z_1, \dots, z_m)|^p \prod_{j=1}^{m} \frac{(1 - |z_j|^2)^{s_j}}{|1 - \langle u_j, z_j\rangle|^{r_j}} d\nu(z_1) \cdots d\nu(z_m)$$

$$\leq C \sum_{k=1}^{\infty} (\prod_{j=1}^{m} \frac{(1 - |v_k|^2)^{n+1+s_j-pb_j}}{|1 - \langle u_j, v_k\rangle|^{r_j}}) (1 - |v_k|^2)^{p\sum_{j=1}^m b_j} sup\{|f(w)|^p : w \in D(v_k, r)\}$$

$$\leq C \sum_{k=1}^{\infty} \frac{(1 - |v_k|^2)^{m(n+1)+\sum_{k=1}^m s_j}}{\prod_{j=1}^m |1 - \langle u_j, v_k\rangle|^{r_j}} \sup\{|f(w)|^p : w \in D(v_k, r)\}.$$

By Lemma 1 (b), $1 - |v_k|^2$ is comparable with $1 - |w|^2$ and $|1 - \langle u_j, v_k\rangle|$ when $w \in D(v_k, r)$. This together with Lemma 1 (d) implies that, if $f \in H(B_n)$, then

$$\int_{B_n} \cdots \int_{B_n} |(S_{a,b}f)(z_1, \dots, z_m)|^p \prod_{j=1}^{m} \frac{(1 - |z_j|^2)^{s_j}}{|1 - \langle u_j, z_j\rangle|^{r_j}} d\nu(z_1) \cdots d\nu(z_m)$$

$$\leq C \sum_{k=1}^{\infty} \sup\{|f(w)|^p \frac{(1 - |w|^2)^{m(n+1)+\sum_{k=1}^m s_j}}{\prod_{j=1}^m |1 - \langle u_j, w\rangle|^{r_j}} : w \in D(v_k, r)\}$$
$$\leq C \sum_{k=1}^{\infty} \int_{D(v_k,2r)} |f(w)|^p \frac{(1 - |w|^2)^{(m-1)(n+1)+\sum_{k=1}^m s_j}}{\prod_{j=1}^m |1 - \langle u_j, w\rangle|^{r_j}} d\nu(w)$$

$$\leq C\int_{B_n}|f(w)|^p\frac{(1-|w|^2)^{(m-1)(n+1)+\sum_{k=1}^m s_j}}{\prod_{j=1}^m|1-\langle u_j,w\rangle|^{r_j}}d\nu(w).$$

To derive the last inequality we have used the fact that each $z\in B_n$ belongs to at most N of the sets $\{D(v_k,2r):k=1,2,\ldots\}$. □

Remark 1. The condition $ms_j+1<p(mb_j-n)-(m-1)(n+1)-n(1-p)$ is equivalent to $s_j+1<mb_j-n$. This shows that there is an extra summand $n(1-p)$ in the condition on the s_j's compared to that in Proposition 1. This extra summand vanishes when $p=1$. Also note that in Proposition 2 we require that f to be holomorphic, which is not needed in Proposition 1.

Lemma 4. *Let $0<p<\infty$. Suppose that $s_1,\ldots,s_m>-1$ are real numbers and let $t=(m-1)(n+1)+\sum_{j=1}^m s_j$. Then there exists a constant $C>0$ such that for all $f\in H((B_n)^m)$,*

$$\int_{B_n}|f(z,\ldots,z)|^p d\nu_t(z)\leq C\int_{B_n}\cdots\int_{B_n}|f(z_1,\ldots,z_m)|^p d\nu_{s_1}(z_1)\cdots d\nu_{s_m}(z_m).$$

Proof. Let $0<r<1$ and choose $\{u_k\}_{k=1}^\infty$ to be an r-lattice in the Bergman metric of B_n as in the proof of Proposition 2. For any integer $k\geq 1$, $\nu_t(D(u_k,r))$ is comparable with $(1-|u_k|^2)^{n+1+t}$, which is in turn comparable with $(1-|w|^2)^{n+1+t}$ for any $w\in D(u_k,r)$ (see Lemma 1). Also, $B_n=\cup_{k=1}^\infty D(u_k,r)$. Hence we have

$$\int_{B_n}|f(z,\ldots,z)|^p d\nu_t(z)\leq\sum_{k=1}^\infty\int_{D(u_k,r)}|f(z,\ldots,z)|^p d\nu_t(z)$$

$$\leq C\sum_{k=1}^\infty(1-|u_k|^2)^{n+1+t}\sup\{|f(z,\ldots,z)|^p:z\in D(u_k,r)\}$$

$$\leq C\sum_{k=1}^\infty\sup\{\prod_{j=1}^m(1-|z_j|^2)^{n+1+s_j}|f(z_1,\ldots,z_m)|^p:z_1,\ldots,z_m\in D(u_k,r)\}.$$

Using Lemma 1 (d), we see that, for $z_1,\ldots,z_m\in D(u_k,r)$,

$$\prod_{j=1}^m(1-|z_j|^2)^{n+1+s_j}|f(z_1,\ldots,z_m)|^p\leq$$

$$\leq C\int\limits_{D(z_j,r)}\cdots\int\limits_{D(z_m,r)}|f(z_1,\ldots,z_m)|^p d\nu_{s_1}(z_1)\cdots d\nu_{s_m}(z_m)$$

$$\leq C\int_{D(u_k,2r)}\cdots\int_{D(u_k,2r)}|f(z_1,\ldots,z_m)|^p d\nu_{s_1}(z_1)\cdots d\nu_{s_m}(z_m).$$

Therefore $\int_{B_n}|f(z_1,\ldots,z_m)|^p d\nu_t(z)$

$$\leq C\sum_{k=1}^\infty\int_{D(u_k,2r)}\cdots\int_{D(u_k,2r)}|f(z_1,\ldots,z_m)|^p d\nu_{s_1}(z_1)\cdots d\nu_{s_m}(z_m)$$

$$\leq C\sum_{k=1}^{\infty}\int_{B_n}\cdots\int_{B_n}|f(z_1,\ldots,z_m)|^p d\nu_{s_1}(z_1)\cdots d\nu_{s_m}(z_m),$$

where the last inequality follows from the fact that each $w \in B_n$ belongs to at most N of the sets $\{D(v_k, 2r) : k = 1, 2, \ldots\}$. □

The following result follows directly from Lemma 4 and Propositions 1 and 2 and was obtained by us in [17].

Theorem 1. *Let $0 < p < \infty$. Suppose that $s_1, \ldots, s_m > -1$ are real numbers and let $t = (m-1)(n+1) + \sum_{j=1}^{m} s_j$. Then $A^p(B_n, d\nu_t)$ is the trace of $A^p((B_n)^m), d\nu_{s_1}\cdots d\nu_{s_m})$.*

Let $m \geq 1$ be an integer. Suppose μ is a positive Borel measure on $(B_n)^m$ and $(r_1, \ldots, r_m)$ is a m-tuple of positive real numbers. We say that μ is a *bounded* $(r_1, \ldots, r_m)$*-Carleson measure* if

$$\sup\{\frac{\mu(E(a_1)\times\cdots\times E(a_m))}{(1-|a_1|)^{r_1}\cdots(1-|a_m|)^{r_m}} : a_1, \ldots, a_m \in B_n\} < \infty, \tag{3}$$

where $E(0) = B_n$ and for $a \in B_n \setminus \{0\}$, $E(a) = \{z \in B_n : |1 - \langle z, \frac{a}{|a|}\rangle| < 1 - |a|\}$. The following result is proved in [23] for the case $m = 1$ but the same proof also works for $m \geq 1$.

Theorem 2. *Let $0 < \tau_1, \ldots, \tau_m < \infty$ and $0 < r_1, \ldots, r_m < \infty$. A positive Borel measure μ on $(B_n)^m$ is a bounded $(r_1, \ldots, r_m)$-Carleson measure if and only if*

$$\sup\{\int_{B_n}\cdots\int_{B_n}\frac{(1-|a_j|)^{\tau_j}}{|1-\langle a_j, z_j\rangle|^{\tau_j+r_j}}d\mu(z_1,\ldots,z_m) : a_1,\ldots,a_m \in B_n\} < \infty. \tag{4}$$

Furthermore, the two suprema in (3) and (4) are equivalent.

We now introduce Q_p type spaces in polyballs. Let $0 < p < \infty$. For the real numbers $s_1, \ldots, s_m > -1$ and $r_1, \ldots, r_m > 0$, we define $M^p_{r_1,\ldots,r_k}((B_n)^m, d\nu_{s_1}\cdots d\nu_{s_m})$ to be the space of all measurable functions f on $(B_n)^m$ for which the measure $d\mu_f = |f|^p d\nu_{s_1}\cdots d\nu_{s_m}$ is a bounded $(r_1, \ldots, r_m)$-Carleson measure. For any $f \in M^p_{r_1,\ldots,r_k}((B_n)^m, d\nu_{s_1}\cdots d\nu_{s_m})$, we define

$$|||f|||^p = \sup\{\frac{\mu_f(E(a_1)\times\cdots\times E(a_m))}{(1-|a_1|)^{r_1}\cdots(1-|a_m|)^{r_m}} : a_1,\ldots,a_m \in B_n\} < \infty.$$

It can be checked that when $1 \leq p < \infty$, the space $M^p_{r_1,\ldots,r_k}((B_n)^m, d\nu_{s_1}\cdots d\nu_{s_m})$ is a Banach space with the above norm. For $0 < p < 1$, it is a complete metric given by $d(f,g) = |||f-g|||^p$. The space $HM^p_{r_1,\ldots,r_k}((B_n)^m, d\nu_{s_1}\cdots d\nu_{s_m})$ is the intersection of $H((B_n)^m)$ with $M^p_{r_1,\ldots,r_k}((B_n)^m, d\nu_{s_1}\cdots d\nu_{s_m})$.

Proposition 3. *Let $0 < p < \infty$. Suppose that $s_1, \ldots, s_m \geq -1$ and $r_1, \ldots, r_m \geq 0$ are such that for each $j = 1, \ldots, m$, we have $-pa_j < min\{s_j + 1, s_j + 1 + n - r_j\}$ and $ms_j + 1 < p(mb_j - n) - (m-1)(n+1)$ when $1 \leq p < \infty$ and $-pa_j <$*

$min\{s_j+1, s_j+1+n-r_j\}$ and $s_j+1 < pb_j - n$ when $0 < p < 1$. Let $t = (m-1)(n+1)+\sum_{j=1}^m s_j$ and $r = \sum_{j=1}^m r_j$. Then $S_{a,b}$ is a bounded operator from $HM_r^p(B_n, d\nu_t)$ into $M_{r_1,\dots,r_k}^p((B_n)^m, d\nu_{s_1}\cdots d\nu_{s_m})$.

Proof. For $j = 1,\dots,m$, since $-pa_j < s_j+1+n-r_j$, we can choose $\tau_j > 0$ such that $-pa_j < s_j+1+n-(r_j+\tau_j)$. By Propositions 1 and 2, there is a constant $C > 0$ such that for any $f \in HM^p(B_n, d\nu_t)$ and $u_1,\dots,u_2 \in B_n$,

$$\int_{B_n}\cdots\int_{B_n} |(S_{a,b}f)(z_1,\dots,z_m)|^p \prod_{j=1}^m \frac{(1-|z_j|^2)^{s_j}}{|1-\langle u_j, z_j\rangle|^{\tau_j+r_j}} d\nu(z_1)\cdots d\nu(z_m)$$

$$\leq C\int_{B_n} |f(w)|^p \frac{(1-|w|^2)^t}{\prod_{j=1}^m |1-\langle u_j, w\rangle|^{\tau_j+r_j}} d\nu(w).$$

Multiplying both sides with $\prod_{j=1}^m (1-|u_j|^2)^{\tau_j}$ we obtain

$$\int_{B_n}\cdots\int_{B_n} |(S_{a,b}f)(z_1,\dots,z_m)|^p \prod_{j=1}^m \frac{(1-|z_j|^2)^{s_j}(1-|u_j|^2)^{\tau_j}}{|1-\langle u_j, z_j\rangle|^{\tau_j+r_j}} d\nu(z_1)\cdots d\nu(z_m)$$

$$\leq C\int_{B_n} |f(w)|^p \prod_{j=1}^m \frac{(1-|u_j|^2)^{\tau_j}}{|1-\langle u_j, w\rangle|^{\tau_j+r_j}} (1-|w|)^t d\nu(w). \tag{5}$$

Since $\sum_{j=1}^m \frac{r_j}{r} = 1$, applying Hölder's inequality and Theorem 2, we get

$$\int_{B_n} |f(w)|^p \prod_{j=1}^m \frac{(1-|u_j|^2)^{\tau_j}(1-|w|)^t}{|1-\langle u_j, w\rangle|^{\tau_j+r_j}} d\nu(w)$$

$$\leq \prod_{j=1}^m \{\int_{B_n} |f(w)|^p (1-|w|^2)^t \frac{(1-|u_j|^2)^{(\tau_j r)/r_j}}{|1-\langle u_j, w\rangle|^{(\tau_j r)/r_j+r_j}} d\nu(w)\}^{r_j/r}$$

$\leq C\prod_{j=1}^m (|||f|||^p)^{r_j/r} = C\ |||f|||^p$, where C is independent of f and $u_1,\dots,u_m$. From this and (5), we get

$$\int_{B_n}\cdots\int_{B_n} |(S_{a,b}f)(z_1,\dots,z_m)|^p \prod_{j=1}^m \frac{(1-|z_j|^2)^{s_j}(1-|u_j|^2)^{\tau_j}}{|1-\langle u_j, z\rangle|^{\tau_j+r_j}} d\nu_{s_1}(z_1)\cdots d\nu_{s_m}(z_m)$$
$$\leq C|||f|||^p$$

for all $u_1,\dots,u_m \in B_n$. The statement, then follows from Theorem 2. □

Theorem 3. *Let $0 < p < \infty$. Suppose that $s_1,\dots,s_m > -1$ and $r_1,\dots,r_m > 0$ are such that $r_j < n+1+s_j$ for $j = 1,\dots,m$. Put $t = (m-1)(n+1)+\sum_{j=1}^m s_j$ and $r = \sum_{j=1}^m r_j$. Then $Trace(HM_{r_1,\dots,r_m}^p((B_n)^m, d\nu_{s_1}\cdots d\nu_{s_m})) = HM_r^p(B_n, d\nu_t)$.*

Proof. For any $a \in B_n$ and $f \in HM^p_{r_1,\dots,r_m}((B_n)^m, d\nu_{s_1}\cdots d\nu_{s_m})$, define

$$\tilde{f}(z_1,\dots,z_m) = f(z_1,\dots,z_m)\frac{(1-|a|^2)^m}{\prod_{j=1}^m (1-\langle z_j,a\rangle)^{1+r_j/p}}.$$

Applying Lemma 4 to the function $\tilde{f}$ we get

$$\int_{B_n} |f(z,\dots,z)|^p \frac{(1-|a|^2)^{mp}}{|1-\langle z,a\rangle|^{mp+r}} d\nu_t(z)$$

$$\leq C\int\limits_{B_n}\cdots\int\limits_{B_n} |f(z_1,\dots,z_m)|^p \prod_{j=1}^m \frac{(1-|a|^2)^p}{|1-\langle z_j,a\rangle|^{p+r_j}} d\nu_{s_1}(z_1)\cdots d\nu_{s_m}(z_m) \leq C|||f|||^p.$$

The last inequality follows from Theorem 2. This shows that the function $z \mapsto f(z,\dots,z)$ is in $HM^p_r(B_n, d\nu_t)$.

On the other hand, for $j=1,\dots,m$, let $a_j = 0$ and b_j be large enough so that the hypotheses of Propositions 1 and 2 are satisfied. Let $a=(0,\dots,0)$ and $b=(b_1,\dots,b_m)$. Put $\alpha = -n-1+\sum_{j=1}^m b_j$. Then Proposition 3 together with the fact that the image of $S_{a,b}$ is contained $H((B_n)^m)$ show that $S = c_\alpha S_{a,b}$ maps $HM^p_r(B_n, d\nu_t)$ boundedly into $HM^p_{r_1,\dots,r_m}((B_n)^m, d\nu_{s_1}\cdots d\nu_{s_m})$. In addition, for $f \in HM^p_r(B_n, d\nu_t)$ and $z \in B_n$,

$$(Sf)(z,\dots,z) = c_\alpha \int_{B_n} \frac{f(w)(1-|w|^2)^{-n-1-\sum_{j=1}^m b_j}}{(1-\langle z,w\rangle)^{\sum_{j=1}^m b_j}} d\nu(w)$$

$$= \int_{B_n} \frac{f(w)d\nu_\alpha(w)}{(1-\langle z,w\rangle)^{n+1+\alpha}} = f(z).$$

The last equality follows from [23, Theorem 2.2]. So the conclusion of the theorem follows. □

We now introduce the mixed norm classes in polyballs

$$A^{p_1,\dots,p_m}_{\alpha_1,\dots,\alpha_m}(B^m_n) = \{f \in H(B^m_n) : \|f\|_{A^{p_1,\dots,p_m}_{\alpha_1,\dots,\alpha_m}} := (\int\limits_{B_n} (1-|z_m|)^{\alpha_m} (\int\limits_{B_n} (1-|z_{m-1}|)^{\alpha_{m-1}}$$

$$\cdots \int\limits_{B_n} |f(z_1,\dots,z_m)|^{p_1}(1-|z_1|)^{\alpha_1} d\nu(z_1))^{\frac{p_2}{p_1}} \cdots d\nu(z_{m-1}))^{\frac{p_m}{p_{m-1}}} d\nu(z_m))^{\frac{1}{p_m}} < \infty\},$$

where $0 < p_i < \infty$, $\alpha_i > -1$, $i=1,\dots,m$. Note that for $n=1$ these classes were studied in [14]. For $m=1$ we have the classical Bergman spaces on the unit ball. Formally replacing B_n by $\mathbb{R}^n$ we arrive at well studied function classes in $\mathbb{R}^n$ (see [3], [4], [10]).

Let $L^{p_1,\dots,p_m}_{\alpha_1,\dots,\alpha_m}(B^m_n)$ denote the space of all measurable functions $f : B^m_n \to \mathbb{C}$ such that $\|f\|_{A^{p_1,\dots,p_m}_{\alpha_1,\dots,\alpha_m}} < \infty$. It is not difficult to show that $A^{p_1,\dots,p_m}_{\alpha_1,\dots,\alpha_m}(B^m_n)$ is a

Banach space for $1 \leq p_i < \infty$, $i = 1, \dots, m$. Moreover, it can be shown that $A^{p_1,\dots,p_m}_{\alpha_1,\dots,\alpha_m}(B^m_n)$ is a complete metric space for $0 < p_i < 1$, $i = 1, \dots, m$.

For $f \in A^{p_1,\dots,p_m}_{\alpha_1,\dots,\alpha_m}(B^m_n)$, we have the following estimate

$$|f(z_1,\dots,z_m)| \leq C \frac{\|f\|_{A^{p_1,\dots,p_m}_{\alpha_1,\dots,\alpha_m}}}{\prod_{k=1}^m (1-|z_k|)^{\frac{\alpha_k}{p_k}+\frac{n+1}{p_k}}}, \tag{6}$$

where $z_j \in B_n$, $j = 1, \dots, m$. The proof of (6) can be obtained by modification of standard arguments from [23].

Our intention is to describe the traces of these function spaces. This result also generalizes our previous mentioned description of the trace of $A^p_\alpha(B^m_n)$ (see [16], [17]) and for $n = 1$ it coincides with a theorem from [14].

Theorem 4. *Let* $\gamma = \alpha_m + \sum_{j=1}^{m-1}(n+1+\alpha_j)\frac{p_m}{p_j}$, $\alpha_j > -1$ *and* $p_j > 1$, $j = 1, \dots, m$. *If* $f \in A^{p_1,\dots,p_m}_{\alpha_1,\dots,\alpha_m}(B^m_n)$, *then*

$$\int_{B_n} |f(z,\dots,z)|^{p_m} d\nu_\gamma(z) \leq C\ \|f\|_{A^{p_1,\dots,p_m}_{\alpha_1,\dots,\alpha_m}}.$$

Moreover, $Trace(A^{p_1,\dots,p_m}_{\alpha_1,\dots,\alpha_m}(B^m_n)) = A^{p_m}_\gamma(B_n)$, *for* $1 < p_j < \infty$, $j = 1, \dots, m$.

Proof. Note the first assertion of theorem is a direct mixed norm generalization of Lemma 4 and we will provide only sketch of the proof.

The proof of the first part is based on the following estimate which can be obtained from Lemma 1 applied $(n-2)$ times separately by each variable and is based also on the fact that

$$\left(\int_{B_n} |f(z_1,\dots,z_m)|^{p_i}(1-|z_i|)^{\alpha_i} d\nu(z_i)\right)^{\frac{p_{i+1}}{p_i}}$$

is subharmonic by z_{i+1} when other variables are fixed.

$$\Big(\sum_{k_m=1}^{\infty} \sup_{z_m \in D(u_{k_m},2r)} \cdots \Big(\sum_{k_2=1}^{\infty} \sup_{z_2 \in D(u_{k_2},2r)} \Big(\sum_{k_1=1}^{\infty} \sup_{z_1 \in D(u_{k_1},2r)} |f(z_1,\dots,z_m)|^{p_1}$$

$$(1-|u_{k_1}|^2)^{\alpha_1+n+1}\Big)^{\frac{p_2}{p_1}}(1-|u_{k_2}|^2)^{\alpha_2+n+1}\Big)^{\frac{p_3}{p_2}}\cdots\Big)^{\frac{p_m}{p_{m-1}}}(1-|u_{k_m}|^2)^{\alpha_m+n+1} \leq$$

$$\leq C\ \|f\|^{p_m}_{A^{p_1,\dots,p_m}_{\alpha_1,\dots,\alpha_m}} < \infty,$$

where $0 < p_i < \infty$, $\alpha_i > -1$, $i = 1, \dots, m$. Taking only those members of this multidimensional sum that have the same indexes, that is, $k_1 = \cdots = k_m$, we arrive at estimates as follows

$$\sum_{k=1}^{\infty} \sup_{z \in D(u_k,2r)} |f(z,\dots,z)|^{p_m}(1-|z|)^{\widetilde{\gamma}} \leq C\ \|f\|^{p_m}_{A^{p_1,\dots,p_m}_{\alpha_1,\dots,\alpha_m}},\ \widetilde{\gamma} = \sum_{j=1}^{m}(\alpha_j+n+1)\frac{p_m}{p_j},$$

and hence again applying Lemma 1 we finally have

$$\|f(z,\dots,z)\|^{p_m}_{A^{p_m}_{\gamma}(B_n)} \le C\sum_{k=1}^{\infty}\sup_{z\in D(u_k,2r)}|f(z,\dots,z)|^{p_m}(1-|z|)^{\widetilde{\gamma}} \le C\|f\|^{p_m}_{A^{p_1,\dots,p_m}_{\alpha_1,\dots,\alpha_m}}.$$

To prove the reverse estimate we restrict ourselves to the case of the biball, that is, $B_n\times B_n$. The general case can be considered similarly. The expansion is done again with the help of the expanded Bergman projection.

For every $s>-1$, we have as above $f(z,z)=g(z)$, where

$$f(z_1,z_2)=C(n,\beta)\int_{B_n}\frac{g(w)(1-|w|)^{2(s+n+1)-(n+1)}d\nu(w)}{\prod_{j=1}^{2}(1-\langle z_j,\overline{w}\rangle)^{s+n+1}}$$

by Bergman representation formula (see [23, Theorem 2.11]). Using duality, we show that $f\in A^{p_1,p_2}_{\alpha_1,\alpha_2}$ if $g\in A^{p_2}_{\gamma}(B_n)$. Let $\Psi\in L^{q_1,q_2}_{\alpha_1,\alpha_2}$, $\frac{1}{p_i}+\frac{1}{q_i}=1,\ i=1,2$. Then

$$\left|\int_{B_n}\int_{B_n}f(w_1,w_2)\Psi(w_1,w_2)(1-|w_1|)^{\alpha_1}(1-|w_2|)^{\alpha_2}d\nu(w_1,w_2)\right| \tag{7}$$

$$\le C(n)\int_{B_n}\int_{B_n}|\Psi(w_1,w_2)|(1-|w_1|)^{\alpha_1}(1-|w_2|)^{\alpha_2}$$

$$\int_{B_n}\frac{|g(w)|(1-|w|)^{2(s+n+1)-(n+1)}d\nu(w)}{\prod_{j=1}^{2}|1-\langle w_j,\overline{w}\rangle|^{s+n+1}}$$

$$\le C(s,n)\int_{B_n}|g(w)|(1-|w|)^{2(s+n+1)-(n+1)}$$

$$\int_{B_n}\int_{B_n}\frac{\Psi(w_1,w_2)(1-|w_1|)^{\alpha_1}(1-|w_2|)^{\alpha_2}d\nu(w_1,w_2)}{|1-\langle\overline{w_1},w\rangle|^{s+n+1}|1-\langle\overline{w_2},w\rangle|^{s+n+1}}d\nu(w).$$

Let $F(w_1,w_2)=\int_{B_n}\int_{B_n}\frac{\Psi(w_1,w_2)(1-|w_1|)^{\alpha_1}(1-|w_2|)^{\alpha_2}d\nu(w_1,w_2)}{(1-\langle\overline{w_1},w\rangle)^{s+n+1}(1-\langle\overline{w_2},w\rangle)^{s+n+1}}$.

Then by first part of theorem

$$F(w,w)\in A^{q_2}_{\widetilde{\alpha}_2+(n+1+\widetilde{\alpha}_1)\frac{q_2}{q_1}}, \tag{8}$$

where $\widetilde{\alpha}_1=\alpha_1-q_1\alpha_1+sq_1,\ \widetilde{\alpha}_2=\alpha_2-q_2\alpha_2+sq_2$. Since it can be easily shown that if $\Psi\in L^{q_1,q_2}_{\alpha_1,\alpha_2}$ then $F(w_1,w_2)\in A^{q_1,q_2}_{\widetilde{\alpha}_1,\widetilde{\alpha}_2}$.

Using the fact that $\left(L^{q_1,q_2}_{\alpha_1,\alpha_2}\right)^*=L^{p_1,p_2}_{\alpha_1,\alpha_2}$ (see [3]), it remains to use in (7) Hölder's inequality with q_2 and p_2 and use (8) choosing appropriate s to get what we need, $f\in A^{p_1,p_2}_{\alpha_1,\alpha_2}$. We omit calculations. The theorem is proved. □

A complete analogue of the last theorem is also valid for $p_j\le 1,\ j=1,\dots,m$ case.

Theorem 5. *Let* $\gamma = \alpha_m + \sum_{j=1}^{m-1}(n+1+\alpha_j)\frac{p_m}{p_j}$, $\alpha_j > -1$ *and* $p_j \leq 1$, $j = 1, \dots, m$. *If* $f \in A^{p_1,\dots,p_m}_{\alpha_1,\dots,\alpha_m}(B_n^m)$, *then*

$$\int_{B_n} |f(z, \dots, z)|^{p_m} d\nu_\gamma(z) \leq C \ ||f||_{A^{p_1,\dots,p_m}_{\alpha_1,\dots,\alpha_m}}.$$

Moreover, $Trace(A^{p_1,\dots,p_m}_{\alpha_1,\dots,\alpha_m}(B_n^m)) = A^{p_m}_\gamma(B_n)$, *for* $p_j \leq 1$, $j = 1, \dots, m$.

The proof of this assertion is based on representations of bounded linear functionals for analytic mixed norm spaces in polyballs we introduced above for $p_j \leq 1$, $j = 1, \dots, m$ case. We provided the formulation for completeness of our exposition. The complete proof of this last will be presented in a separate paper.

We can easily notice that generally sharp trace theorems for analytic spaces we considered above will be valid for any G domain and any $G \times \cdots \times G$ polydomain for which an appropriate substitution of r-lattice we had above in the ball and appropriate Bergman type integral representations can be found. By this we mean substitutions of properties of mentioned r-lattices in the ball we enlisted in text above that were used by us during the proof of trace theorems in $B_n \times \cdots \times B_n$ polyballs.

3 Traces of the harmonic Bergman function spaces in the unit ball of $\mathbb{R}^n$ and $\mathbb{R}^{n+1}$

In this section we will provide descriptions of traces of products of harmonic Bergman classes in the unit ball of $\mathbb{R}^n$ and in $\mathbb{R}^{n+1}$. Since these results are very similar to those we obtained in holomorphic case we mostly restrict ourselves to formulations and short sketches of proofs.

Let B be the unit ball in $\mathbb{R}^n$, that is $B = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : |x| = (\sum_{i=1}^n |x_i|^2)^{\frac{1}{2}} \leq 1\}$, and $S^{n-1} = \partial B = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : |x| = 1\}$, and $x = rx', r = |x| \in (0, 1)$, $x' \in S^{n-1}$. We consider as usual the normalized Lebesgue measure on B as $dx = dx_1 \cdots dx_n$ or in the sphere coordinates as $r^{n-1}drdx'$, so that

$$\int_B dx = \int_0^1 \int_{S^{n-1}} r^{n-1} dr dx' = 1.$$

Let $h(B)$ be the space of all harmonic functions on B. We consider Banach Bergman spaces in B (see [6], [7]),

$$A^p_\alpha(B) = \{f \in h(B) : \|f\|_{p,\alpha} := \left(\int_0^1 \int_{S^{n-1}} |f(rx')|^p (1-r)^\alpha r^{n-1} dr dx'\right)^{\frac{1}{p}} < \infty\},$$

when $1 \leq p < \infty$ and $0 \leq \alpha < \infty$ and for $p = \infty$ and $0 \leq \alpha < \infty$ we have

$$\|f\|_{\infty,\alpha} := \sup_{x \in B} |f(x)|(1 - |x|)^\alpha < \infty.$$

We now introduce a new $Q_\alpha(x,y)$ function that serve as a nice substitution of Bergman kernel for harmonic functions in the unit ball of $\mathbb{R}^n$ (see [7]),

$$Q_\alpha(x,y) = 2\sum_{k=0}^{\infty} \frac{\Gamma(\alpha+1+k+n/2)}{\Gamma(\alpha+1)\Gamma(k+n/2)} r^k \rho^k z_{x'}^{(k)}(y').$$

Note that $x = rx'$ and $y = \rho y'$, $Q_\alpha(x,y) = Q_\alpha(y,x)$ and $\|Q_\alpha\|_{A^p_\alpha} = 1$, where Γ is the Gamma function of Euler and $z_{x'}^{(k)}(y')$ is a zonal harmonic of order k, (see [7]).

$$z_{x'}^{(k)}(y') = \sum_{j=1}^{\alpha_k} y_j^{(k)}(x')\overline{y_j^{(k)}(y')},$$

where $y^{(k)}$ and $y^{(l)}$ are so called spherical harmonics, see [7]. Various nice properties of spherical harmonics can be found in [18] and [19].

From properties of these spherical harmonics it follows that every harmonic function in B has an expansion

$$f(x) = f(rx') = \sum_{k=0}^{\infty} r^k c_k y^{(k)}(x'),$$

where $c_k y^{(k)} = \sum_{j=1}^{\alpha_k} c_k^j y_j^{(k)}$ and $\cup_{k=0}^{\infty}\{y_1^{(k)}, \ldots, y_{\alpha_k}^{(k)}\}$ form a orthonormal basis in $L^2(S^{n-1})$ (by dx' measure), see [19].

We need the following result from [7].

Theorem A. *Let* $1 \le p \le \infty$ *and* $0 \le \alpha < \infty$. *If* $f \in A^p_\alpha(B)$, *then*

$$f(x) = \int_0^1 \int_{S^{n-1}} (1-\rho^2)^\alpha Q_\alpha(x,y) f(\rho y')\rho^{n-1} d\rho dy'.$$

Theorem A provides Bergman type representation for harmonic Bergman spaces in the unit ball.

In order to get a sharp trace theorem for this case we introduce a modified Q_α function to form an analogue of the expanded Bergman projection we considered above, that is, $Q_\alpha(\frac{x_1+\cdots+x_m}{m}, y)$. Let $\alpha_j > 0$ for $j = 1, ..., m$ and denote by $A^\infty_{\alpha_1,\ldots,\alpha_m}(B\times\cdots\times B)$ the space of all functions f harmonic by each variable in B such that

$$\sup_{x_i\in B, i=1,\ldots,m} |f(x_1,\ldots,x_m)| \prod_{i=1}^{m} (1-|x_i|^2)^{\alpha_i} < \infty.$$

Theorem 6. *Let* $\alpha_j > 0$ *for* $j = 1, \ldots, m$. *Then* $Trace(A^\infty_{\alpha_1,\ldots,\alpha_m}(B^m)) = A^\infty_{\sum_{j=1}^m \alpha_j}(B)$.

Proof. One implication is obvious. We show the reverse by using Theorem A. Indeed, for $g \in A^\infty_{\sum_{j=1}^m \alpha_j}(B)$ we have $f(x,\ldots,x) = g(x)$ since we put

$$f(x_1,\ldots,x_m) = \int_0^1 \int_{S^n} Q_\beta(\frac{x_1+\cdots+x_m}{m}, y)(1-\rho^2)^\beta \rho^{n-1} g(\rho y') d\rho dy',$$

where β can be large enough. Hence we get obviously

$$\|f\|_{A^\infty_{\alpha_1,\dots,\alpha_m}(B^m)} \le c \sup_{x_i\in B, i=1,\dots,m} \int_0^1 \int_{S^n} Q_\beta(\frac{x_1+\cdots+x_m}{m}, y)(1-\rho^2)^{\beta-\sum_{j=1}^m \alpha_j} \prod_{k=1}^m (1-|x_k|)^{\alpha_k} d\rho dy'.$$

We need an estimate for the expanded Q_β kernel. For that reason we apply the following inequality can be found in [7].

$$Q_\beta(x,y) \le \frac{c}{|r\rho x'-y'|^{n+\beta}} + \frac{c(1-\rho r)^{-\{\beta\}}}{|r\rho x'-y'|^{n+[\beta]}} + \frac{c_1}{(1-\rho r)^{1+\beta}}, \tag{9}$$

where $x = rx',\ y = \rho y',\ \beta > 0$.

We will need the following two classical estimates which can be obtained by direct calculations,

$$\int_{S^{n-1}} \frac{dx'}{|r\rho x'-y'|^\gamma} < \frac{c}{(1-\rho r)^{\gamma-n+1}}, \quad \gamma > n-1, \rho, r \in (0,1) \tag{10}$$

and

$$\int_0^1 (1-\rho r)^{-\alpha}(1-\rho)^\beta d\rho \le c(1-r)^{-\alpha+\beta+1}, \quad \beta > -1, \alpha > \beta+1, \rho, r \in (0,1). \tag{11}$$

For $\widetilde{x} := \frac{x_1+\cdots+x_m}{m}$, where $x_i \in B,\ |x_i| < 1$, using (9) we have

$$\int_0^1 \int_{S^n} Q_\beta(\widetilde{x}, y)(1-\rho^2)^{\beta-\sum_{j=1}^m \alpha_j} d\rho dy \le I_1 + I_2 + I_3.$$

We estimate I_2 and I_3. The estimate for I_1 can be obtained similarly. We have

$$I_3 \le c\int_0^1 \frac{(1-\rho^2)^{\beta-\sum_{j=1}^m \alpha_j}}{(1-|\tilde{x}|\rho)^{1+\beta}} d\rho \le c\int_0^1 \frac{(1-\rho^2)^{\beta-\sum_{j=1}^m \alpha_j} d\rho}{(1-\frac{\sum_{j=1}^m |x_j|\rho}{n})^{1+\beta}}$$

$$\le c\int_0^1 \frac{(1-\rho^2)^{\beta-\sum_{j=1}^m \alpha_j} d\rho}{(\frac{\sum_{j=1}^m (1-|x_j|\rho)}{m})^{1+\beta}} \le c\int_0^1 \frac{(1-\rho)^{\beta-\sum_{j=1}^m \alpha_j} d\rho}{(\prod_{j=1}^m \sqrt[m]{1-|x_j|\rho})^{1+\beta}} \le \frac{c_1}{\prod_{j=1}^m (1-|x_j|)^{\alpha_j}}.$$

$$I_2 \le C\int_0^1 \int_{S^{n-1}} \frac{(1-\rho)^{\beta-\sum_{j=1}^m \alpha_j} d\rho dy'}{|\widetilde{r}\rho\widetilde{x}-y'|^{n+\beta}} \le C\int_0^1 \frac{(1-\rho)^{\beta-\sum_{j=1}^m \alpha_j} d\rho}{(1-\rho\widetilde{r})^{\beta+1}},$$

$\widetilde{r} = \frac{\sum_{k=1}^m |x_k|}{m}$, and we continue as for I_3.

Note above we used an elementary estimate

$$\int_0^1 \frac{(1-r)^t dr}{\prod_{k=1}^m (1-r\rho_k)^{\alpha_k}} \le \frac{c}{\prod_{k=1}^m (1-\rho_k)^{\alpha_k-\beta_k(t+1)}}, \tag{12}$$

$t > -1,\ \sum_{k=1}^m \beta_k = 1,\ \alpha_k > (t+1)b_k,\ \rho_k \in (0,1),\ k = 1,\dots,m$, which follows directly from Hölder's inequality for m functions. Note we used (12) for $\beta_k = \frac{\frac{1+\beta}{m}-\alpha_k}{\beta-\sum_{k=1}^m \alpha_k+1}$, $t = \beta-\sum_{k=1}^m \alpha_k,\ \alpha_k = \frac{1+\beta}{n}, k = 1,\dots,m$. The theorem is proved. □

Let $hA^p_{\overrightarrow{\alpha}}(B\times\cdots\times B)=\{f\in h(B^m):$

$$\int_B\int_B\cdots\int_B\int_B|f(x_1,\dots,x_m)|^p$$

$$\prod_{k=1}^m(1-|x_k|)^{\alpha_k}(|x_1|\cdots|x_m|)^{n-1}d|x_1|\cdots d|x_m|dx'_1\cdots dx'_m<\infty\}.$$

Traces of $hA^p_{\overrightarrow{\alpha}}(B\times\cdots\times B)$ classes for $p<\infty$ can be described in a following manner.

Theorem 7. *Let $p\geq 1,\ m\in\mathbb{N},\ \alpha_j>-1,\ j=1,\dots,m$. Then $Trace(hA^p_{\overrightarrow{\alpha}}(B^m))=hA^p_{\sum_{k=1}^m\alpha_k+\beta}(B)$, where $\beta=(n+1)(m-1)$.*

The proof follows directly from arguments we provided above for holomorphic case. $\{D(a_k,r)\}$ families from there should be replaced by dyadic cubes $\{\Delta_k\}_{k=1}^\infty$, $\bigcup_{k=0}^\infty\Delta_k=B$ such a decomposition can be found in [18], [19] and we should use the estimate

$$|u(a)|^p\leq\frac{C(p,\alpha)}{r^\alpha}\int_{|x-a|<r}|u(x)|^p dx,\ u\in h(B),\ a\in\mathbb{R}^n,\ 0<p,r<\infty\ \text{(see [1])}.$$

As usual, denote by $\mathbb{R}^{n+1}_+$ the upper half-space in $\mathbb{R}^{n+1}$, $\mathbb{R}^{n+1}_+=\{x=(x_1,\dots,x_n,x_{n+1})\in\mathbb{R}^{n+1},\ x_{n+1}>0\}$, the Lebesque measure in $\mathbb{R}^{n+1}$ will be denoted by $dx=dx'dx_{n+1}$, $dx'=dx_1\cdots dx_n$.

Let $Q^k(x,y)=c_k\frac{\partial^{k+1}}{\partial^{k+1}_{n+1}}P(x'-y',x_{n+1}+y_{n+1}),\ k\geq 0$ which is defined on $\mathbb{R}^{n+1}_+\times\cdots\times\mathbb{R}^{n+1}_+$, $P(x',x_{n+1})=\frac{x_{n+1}}{(|x'|^2+x^2_{n+1})^{\frac{n+1}{2}}}$ is a usual Poisson kernel on $\mathbb{R}^{n+1}_+$, (see [7]).

We introduce harmonic Bergman classes on $\mathbb{R}^{n+1}$ and on product spaces as follows. Let $0<p<\infty,\ \alpha>-1$,

$$A^p_\alpha(\mathbb{R}^{n+1}_+)=\{f\ \text{is harmonic in}\ \mathbb{R}^{n+1}_+:$$

$$\|f\|^p_{p,\alpha}=\int_{\mathbb{R}^n}\int_0^\infty|f(x',x_{n+1})|^p x^\alpha_{n+1}dx'dx_{n+1}<\infty\},$$

and $A^p_\alpha((\mathbb{R}^{n+1}_+\times\cdots\times\mathbb{R}^{n+1}_+))=\{f$ is harmonic in $\mathbb{R}^{n+1}_+\times\cdots\times\mathbb{R}^{n+1}_+:$

$$\int_{\mathbb{R}^n}\cdots\int_{\mathbb{R}^n}\int_0^\infty\cdots\int_0^\infty|f(x'_1,\dots,x'_m,x^1_{n+1},\dots,x^m_{n+1})|^p\times$$

$$\times\prod_{k=1}^m(x^k_{n+1})^{\alpha_k}dx'_1\cdots dx'_m dx^1_{n+1}\cdots dx^m_{n+1}<\infty\}\ \text{(see [7])}.$$

Theorem B. *(see [7]) Let $f\in A^p_\alpha(\mathbb{R}^{n+1}_+),\ \alpha>-1,\ k\geq\frac{\alpha+n+1}{p}-(n+1),\ 0<p\leq 1$ or $k>\frac{1+\alpha}{p}-1,\ 1<p<\infty$. Then the following integral representation holds*

$$f(y)=\int_{\mathbb{R}^{n+1}_+}f(x)Q^k(y,x)x^k_{n+1}dx.$$

Theorem B provides Bergman type representation for harmonic Bergman spaces in $\mathbb{R}_+^{n+1}$.

Using this assertion we will get description of traces Bergman type classes on $\mathbb{R}_+^{n+1} \times \cdots \times \mathbb{R}_+^{n+1}$. We will use expanded projections (integral operators). Expanded $Q^k(y, x)$ kernel in $\mathbb{R}^{n+1}$ will have the following form

$$Q^k(y_1, \dots, y_m, x) = c_k \frac{\partial^{k+1}}{\partial x_{n+1}^{k+1}} \frac{\prod_{j=1}^m (x_{n+1} + y_{n+1}^j)^{\frac{1}{m}}}{\prod_{j=1}^m (|x' - y_1^j|^2 + (x_{n+1} + y_{n+1}^j)^2)^{\frac{n+1}{2m}}}, y_j \in \mathbb{R}^{n+1},$$

$$j = 1, \dots, m.$$

Obviously $Q^k(y, \dots, y, x) = Q^k(x, y)$, $x, y \in \mathbb{R}_+^{n+1}$, (see [7]).

The expanded Bergman projection type operator has the form

$$(Tf)(y_1, \dots, y_m) = \int_{\mathbb{R}_+^{n+1}} f(x) Q^k(y_1, \dots, y_m, x) x_{n+1}^k dx' dx_{n+1}, y_j \in \mathbb{R}^{n+1},$$

$$j = 1, \dots, m.$$

We will provide now descriptions of traces of harmonic Bergman classes on products of $\mathbb{R}^{n+1}$.

Theorem 8. *Let* $0 < p < \infty$, $m \in \mathbb{N}$, $\alpha_j > -1$, $j = 1, \dots, m$. *Then*

$$Trace(A^p_{\alpha_1, \dots, \alpha_m}(\mathbb{R}_+^{n+1} \times \cdots \times \mathbb{R}_+^{n+1})) = A^p_{\sum_{k=1}^m \alpha_k + (n+1)(m-1)}(\mathbb{R}_+^{n+1}).$$

Remark 2. We would like to mention finally that sharp trace theorems in harmonic Bergman spaces we formulated at the second part of our paper can be at least partially extended to mixed norm classes of harmonic functions which can be defined in appropriate manner as we did in holomorphic spaces case above. The formulations of these assertions and arguments that are needed for proofs of these assertions are very close or even repeat those we used in holomorphic case above and we would like to omit details leaving them to readers.

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