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A CHARACTERIZATION THEOREM FOR LEVELWISE STATISTICAL CONVERGENCE

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Abstract

In the present paper, we prove a characterization theorem which gives a necessary and sufficient condition for a sequence of fuzzy numbers to be levelwise statistically convergent in the space of fuzzy numbers. As an application of this theorem we utilize the idea of statistical equi-continuity in order to obtain a condition which guarantees the set of levelwise statistical cluster points of a statistically bounded sequence to be nonempty and a levelwise statistically Cauchy sequence to be levelwise statistically convergent.

1 Introduction

Several types of convergence in the space of fuzzy numbers have been introduced so far (see [8, 15, 18, 24]). Among these types, the ones using different metrics on fuzzy number spaces (see [8, 15, 18]), levelwise convergence, levelwise convergence almost everywhere on $[0, 1]$ (see [16, 21, 24]) and the statistical convergence (see [1, 2, 3, 4, 5, 17, 19, 22]) are well known. Fang and Huang [9] have recently presented a necessary and sufficient condition for a sequence of fuzzy numbers to be levelwise convergent and established certain fundamental theorems in a fuzzy number space; e.g. the levelwise monotone convergence theorem, the nested intervals theorem and the Cauchy criterion for levelwise convergence.

In [6], we introduced the statistical analogue of the notion of levelwise convergence, and investigated the relations between levelwise statistical convergence of a sequence of fuzzy numbers and pointwise statistical convergence of the sequence of corresponding $\alpha-$cuts. In [7], we introduced the concepts of levelwise statistical limit and levelwise statistical cluster points of a sequence of fuzzy numbers, and examined the relations between the sets of ordinary limit points, levelwise limit points, levelwise statistical limit points, statistical limit points, levelwise statistical cluster points and statistical cluster points of a sequence of fuzzy numbers. Finally,

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as an example we have shown that Bolzano-Weierstrass Theorem is not valid in the setting of levelwise statistical convergence.

In this work we introduce the notions of statistical equi-left and right-continuity of a sequence of functions, and present a necessary and sufficient condition which guarantees that given a sequence $\{X_n\}$ of fuzzy numbers satisfying $st-\lim \underline{X}_n^{\alpha} = f(\alpha)$ and $st-\lim \overline{X}_n^{\alpha} = g(\alpha)$ for each $\alpha \in [0,1]$, the pair of functions $[f(\alpha), g(\alpha)]$ determine a fuzzy number. To prove the above assertion we utilize the techniques used in Lemma 4.1 and Theorem 4.1 of [9]. In the third section we apply the notion of statistical equi-continuity of a sequence of functions in order to obtain a condition which guarantees the set of levelwise statistical cluster points of a statistically bounded sequence to be non-empty and a levelwise statistically Cauchy sequence to be levelwise statistically convergent.

2 Preliminaries, Background and Notations

Statistical convergence ([10, 20]) is a generalization of the usual notion of convergence that parallels the theory of ordinary convergence, and is defined in terms of the natural density of a set of positive integers.

If K is a subset of the set of all positive integers $\mathbb{N}$, let us denote the set $\{k \in K : k \leq n\}$ as K_n. The natural density of K is defined by $\delta(K) := \lim\limits_{n\to\infty} \frac{|K_n|}{n}$, where $|K_n|$ denotes the number of elements in K_n. The upper density of the set K is defined by $\overline{\delta}(K) := \limsup\limits_{n\to\infty} \frac{|K_n|}{n}$ (see [11]).

A sequence (x_n) of (real or complex) numbers is said to be statistically convergent to some number l if for each $\varepsilon > 0$ the set $\{n \in \mathbb{N} : |x_n - l| \geq \varepsilon\}$ has natural density zero; in this case we write $st-\lim\limits_n x_n := l$ (see [12]).

If $(x_{n(j)})$ is a subsequence of (x_n) and $K := \{n(j) \in \mathbb{N} : j \in \mathbb{N}\}$ then we abbreviate $(x_{n(j)})$ by $\{x\}_K$. If $\delta(K) = 0$, then $\{x\}_K$ is called a subsequence of density zero or a thin subsequence. We say that $\{x\}_K$ is a non-thin subsequence of (x_n) if K does not have density zero [13].

A real number γ is called a statistical cluster point of a sequence (x_n) of real numbers provided that for every $\varepsilon > 0$

$$\delta(\{n \in \mathbb{N} : |x_n - \gamma| < \varepsilon\}) \neq 0.$$

The set of all statistical cluster points of a sequence (x_n) is denoted by Γ_x [13].

Theorem 1 ([13])**.** *If (x_n) is a sequence of real numbers which has a bounded nonthin subsequence, then (x_n) has a statistical cluster point.*

Theorem 2 ([23])**.** *Let (x_n), (y_n) and (z_n) be sequences of real numbers. If $x_n \leq y_n \leq z_n$ for all $n \in K \subset \mathbb{N}$ such that $\delta(K) = 1$ and $st-\lim x_n = st-\lim z_n = l$, then we have $st-\lim y_n = l$.*

Corollary 1. *Let $K \subseteq \mathbb{N}$ and $\delta(K) \neq 0$. If $a \leq x_n \leq b$ for each $n \in K$ and $\delta(\{n \in K : |x_n - l| < \varepsilon\}) \neq 0$, then we have $a \leq l \leq b$; where $a, b \in \mathbb{R}$ and $a \leq b$.*

Theorem 3 ([23]). *Let $x_n \leq y_n$ for all $n \in K \subset \mathbb{N}$, with $\delta(K) = 1$. If $st - \lim x_n$ and $st - \lim y_n$ exist, then $st - \lim x_n \leq st - \lim y_n$.*

Corollary 2. *Let $K \subseteq \mathbb{N}$ and $\delta(K) \neq 0$. If $x_n \leq y_n$ for each $n \in K$, $\delta(\{n \in K : |x_n - l| < \varepsilon\}) \neq 0$ and $\delta(\{n \in K : |y_n - l'| < \varepsilon\}) \neq 0$ then $l \leq l'$.*

Now, we briefly recall some of the basic notions in the theory of fuzzy numbers and refer readers to [4, 8, 18] for more details.

A fuzzy number is a function X from $\mathbb{R}$ to $[0,1]$, satisfying: X is normal, fuzzy convex, upper semi-continuous and the closure of $\{x \in \mathbb{R} : X(x) > 0\}$, denoted by X^0, is compact. These properties imply that for each $\alpha \in (0,1]$, the $\alpha-$level set $X^\alpha := \{x \in \mathbb{R} : X(x) \geq \alpha\} = \left[\underline{X}^\alpha, \overline{X}^\alpha\right]$ is a non-empty compact convex subset of $\mathbb{R}$. We denote the set of all fuzzy numbers by $L(\mathbb{R})$.

Theorem 4 ([14]). *Let $X \in L(\mathbb{R})$. Then the following are satisfied:*
1. *$\underline{X}^\alpha$ is a bounded left continuous nondecreasing function on $(0,1]$;*
2. *$\overline{X}^\alpha$ is a bounded left continuous nonincreasing function on $(0,1]$;*
3. *$\underline{X}^\alpha$ and $\overline{X}^\alpha$ are right continuous at $\alpha = 0$;*
4. *$\underline{X}^1 \leq \overline{X}^1$.*
Conversely, if the pair of functions $f(\alpha)$ and $g(\alpha)$ satisfy the conditions (1)-(4), then there exists a unique $X \in L(\mathbb{R})$ such that $X^\alpha = [f(\alpha), g(\alpha)]$ for each $\alpha \in [0,1]$.

With respect to order-theoretical aspect, there are several possibilities to define an order relation on the set of fuzzy numbers. As it is customary, we will basically follow the order relation on $L(\mathbb{R})$ defined by

$$X \leq Y \text{ iff } \underline{X}^\alpha \leq \underline{Y}^\alpha \text{ and } \overline{X}^\alpha \leq \overline{Y}^\alpha \text{ for any } \alpha \in [0,1].$$

The fuzzy numbers X and Y are said to be incomparable if neither $X \leq Y$ nor $Y \leq X$ holds. In this case, we use the notation $X \nsim Y$. We say that $X < Y$, if $X \leq Y$ and there exists an $\alpha_0 \in [0,1]$ such that "$\underline{X}^{\alpha_0} < \underline{Y}^{\alpha_0}$ or $\overline{X}^{\alpha_0} < \overline{Y}^{\alpha_0}$."

A sequence $\{X_n\}$ of fuzzy numbers is said to be statistically bounded from above if there exists a fuzzy number μ such that $\delta(\{n \in \mathbb{N} : X_n > \mu\} \cup \{n \in \mathbb{N} : X_n \nsim \mu\}) = 0$. Statistical boundedness from below can be defined similarly. If a sequence $\{X_n\}$ is both statistically bounded from above and below, then it is called statistically bounded. It is easy to see that if a sequence is bounded, it is also statistically bounded.

Let $\{X_n\}$ be a sequence of fuzzy numbers. Then $\{X_n\}$ is said to be levelwise convergent to X_0, written as $l - \lim X_n = X_0$, provided that

$$\lim_{n\to\infty} \underline{X_n}^\alpha = \underline{X_0}^\alpha \qquad \text{and} \qquad \lim_{n\to\infty} \overline{X_n}^\alpha = \overline{X_0}^\alpha$$

for every $\alpha \in [0,1]$ (see [9, 16, 24]).

$\{X_n\}$ is said to be levelwise statistically convergent to X_0, written as $st_l - \lim_n X_n = X_0$, if for every $\varepsilon > 0$ and $\alpha \in [0,1]$, the set

$$\left\{n \in \mathbb{N} : \max\left\{\left|\underline{X}_n^\alpha - \underline{X}_0^\alpha\right|, \left|\overline{X}_n^\alpha - \overline{X}_0^\alpha\right|\right\} \geq \varepsilon\right\} \tag{1}$$

has natural density zero.

The above condition (1) is equivalent to saying that

$$st-\lim_n \underline{X}_n^\alpha = \underline{X}_0^\alpha \qquad \text{and} \qquad st-\lim_n \overline{X}_n^\alpha = \overline{X}_0^\alpha$$

for each $\alpha \in [0,1]$.

Since the natural density of a finite set is zero, levelwise convergence implies levelwise statistical convergence. However, the converse does not hold in general (see [6]).

$\{X_n\}$ is said to be a levelwise statistically Cauchy sequence if for every $\varepsilon > 0$ and $\alpha \in [0,1]$ there exists a number $N = N(\varepsilon, \alpha) \in \mathbb{N}$ such that the set

$$\left\{ n \in \mathbb{N} : \max \left\{ |\underline{X}_n^\alpha - \underline{X}_N^\alpha|, \left|\overline{X}_n^\alpha - \overline{X}_N^\alpha\right| \right\} \geq \varepsilon \right\}$$

has natural density zero (see [6]).

A fuzzy number μ is called a levelwise statistical cluster point of $\{X_n\}$ provided that

$$\delta(\left\{ n \in \mathbb{N} : \max \left\{ \left|\underline{X}_n^\alpha - \underline{\mu}_0^\alpha\right|, \left|\overline{X}_n^\alpha - \overline{\mu}_0^\alpha\right| \right\} < \varepsilon \right\}) \neq 0 \tag{2}$$

for every $\varepsilon > 0$ and $\alpha \in [0,1]$. Condition (2) means that

$$\bar{\delta}(\left\{ n \in \mathbb{N} : \max \left\{ \left|\underline{X}_n^\alpha - \underline{\mu}_0^\alpha\right|, \left|\overline{X}_n^\alpha - \overline{\mu}_0^\alpha\right| \right\} < \varepsilon \right\}) > 0 \quad \text{(see [7]).}$$

We denote the set of all levelwise statistical cluster points of $\{X_n\}$ by Γ_X^l.

3 A Characterization Theorem

The levelwise statistical limit of a sequence $\{X_n\}$ of fuzzy numbers may not exist even though $st-\lim \underline{X}_n^\alpha$ and $st-\lim \overline{X}_n^\alpha$ exist for each $\alpha \in [0,1]$. In other words, for a sequence $\{X_n\}$ of fuzzy numbers, the pair of functions $st-\lim \underline{X}_n^\alpha$ and $st-\lim \overline{X}_n^\alpha$ may not determine a fuzzy number as can be seen by the following example.

Example 1. *Define a sequence $\{X_n\}$ of fuzzy numbers as*

$$X_n(x) := \begin{cases} \left.\begin{array}{ll} 1 + \frac{x-1}{n} & , \text{ if } x \in [0,1] \\ 0 & , \text{ otherwise} \end{array}\right\} & , \text{ if } n \text{ is nonsquare} \\ \left.\begin{array}{ll} 0 & , \text{ if } x \in (-\infty, n-1) \cup (n+1, \infty) \\ x-(n-1) & , \text{ if } x \in [n-1, n] \\ -x+(n+1) & , \text{ if } x \in (n, n+1] \end{array}\right\} & , \text{ otherwise} \end{cases}.$$

It is easy to see that $st-\lim \underline{X}_n^\alpha$ and $st-\lim \overline{X}_n^\alpha$ exist for each $\alpha \in [0,1]$. However, the function $st-\lim \underline{X}_n^\alpha$ does not correspond to the left endpoint of the α-level set of the fuzzy number

$$\mu(x) := \begin{cases} 1 & , \text{ if } 0 \leq x \leq 1 \\ 0 & , \text{ otherwise} \end{cases}$$

when $\alpha = 1$, since we have $\underline{\mu}^1 = 0 \neq st - \lim \underline{X}_n^1 = 1$. Consequently, the sequence $\{X_n\}$ is not levelwise statistically convergent to the fuzzy number μ.

Now the question arises: Under what conditions do the pair of functions $st-\lim \underline{X}_n^\alpha$ and $st-\lim \overline{X}_n^\alpha$ determine a fuzzy number? We try to give an answer to this question by Theorem 5. First, we define the notions of statistical equi-left and right-continuity of a sequence of functions.

Definition 1. *Let (f_n) be a sequence of functions defined on $[a,b]$ and $\alpha_0 \in (a,b]$. Then (f_n) is said to be statistically equi-left continuous (SELC) at α_0 if for each $\varepsilon > 0$ there exists $\varepsilon' > 0$ such that*

$$\delta\left(\{n \in \mathbb{N} : |f_n(\alpha) - f_n(\alpha_0)| \geq \varepsilon\}\right) = 0$$

whenever $\alpha \in (\alpha_0 - \varepsilon', \alpha_0]$.

Statistical equi-right continuity (SERC) at $\alpha_0 \in [a,b)$ can be defined similarly.

Theorem 5 (Characterization Theorem)**.** *Let $\{X_n\} \subset L(\mathbb{R})$ and*

$$\begin{aligned}\delta\left(\{n \in \mathbb{N} : |\underline{X}_n^\alpha - f(\alpha)| \geq \varepsilon\}\right) &= 0,\\ \delta\left(\left\{n \in \mathbb{N} : \left|\overline{X}_n^\alpha - g(\alpha)\right| \geq \varepsilon\right\}\right) &= 0\end{aligned}$$

for each $\varepsilon > 0$ and $\alpha \in [0,1]$. Then the pair of functions $f(\alpha)$ and $g(\alpha)$ determine a fuzzy number iff the sequences of functions $(\underline{X}_n^\alpha)$ and $\left(\overline{X}_n^\alpha\right)$ are SELC at each $\alpha \in (0,1]$ and SERC at $\alpha = 0$.

Proof. $(\Rightarrow)$ Suppose that there exists $X_0 \in L(\mathbb{R})$ such that

$$\begin{aligned}\delta\left(\{n \in \mathbb{N} : |\underline{X}_n^\alpha - \underline{X}_0^\alpha| \geq \varepsilon\}\right) &= 0 \quad \text{and} \\ \delta\left(\left\{n \in \mathbb{N} : \left|\overline{X}_n^\alpha - \overline{X}_0^\alpha\right| \geq \varepsilon\right\}\right) &= 0\end{aligned} \tag{3}$$

hold for each $\varepsilon > 0$ and each $\alpha \in [0,1]$, where $\underline{X}_0^\alpha = f(\alpha)$ and $\overline{X}_0^\alpha = g(\alpha)$. Since $\underline{X}_0^\alpha$ is left continuous at $\alpha \in (0,1]$, we can say that for all $\varepsilon > 0$ there exists $\alpha_1 \in (0,\alpha)$ such that

$$\left|\underline{X}_0^\beta - \underline{X}_0^\alpha\right| < \frac{\varepsilon}{3} \tag{4}$$

for each $\beta \in (\alpha_1, \alpha]$. Taking $0 < \widetilde{\varepsilon} < \alpha - \alpha_1$, we get $\alpha - \widetilde{\varepsilon} \in (\alpha_1, \alpha]$. By the fact (3) , we can write

$$\begin{aligned}\delta\left(\left\{n \in \mathbb{N} : |\underline{X}_n^\alpha - \underline{X}_0^\alpha| \geq \frac{\varepsilon}{3}\right\}\right) &= 0 \quad \text{and} \\ \delta\left(\left\{n \in \mathbb{N} : \left|\underline{X}_n^{\alpha-\widetilde{\varepsilon}} - \underline{X}_0^{\alpha-\widetilde{\varepsilon}}\right| \geq \frac{\varepsilon}{3}\right\}\right) &= 0.\end{aligned} \tag{5}$$

Let us denote $K := K_1^c \cap K_2^c$ where $K_1 = K_1(\alpha,\varepsilon) := \left\{n \in \mathbb{N} : |\underline{X}_n^\alpha - \underline{X}_0^\alpha| \geq \frac{\varepsilon}{3}\right\}$ and $K_2 = K_2(\alpha,\varepsilon,\widetilde{\varepsilon}) := \left\{n \in \mathbb{N} : \left|\underline{X}_n^{\alpha-\widetilde{\varepsilon}} - \underline{X}_0^{\alpha-\widetilde{\varepsilon}}\right| \geq \frac{\varepsilon}{3}\right\}$. Since $\underline{X}_n^\alpha$ and $\underline{X}_0^\alpha$ are nondecreasing, using (3) and (5) we get

$$\begin{aligned}0 &\leq \underline{X}_n^\alpha - \underline{X}_n^\beta < \frac{\varepsilon}{3} + \underline{X}_0^\alpha - \underline{X}_n^\beta \leq \frac{\varepsilon}{3} + \underline{X}_0^\alpha - \underline{X}_n^{\alpha-\widetilde{\varepsilon}} < \underline{X}_0^{\alpha-\widetilde{\varepsilon}} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} - \underline{X}_n^{\alpha-\widetilde{\varepsilon}} \\ &= \underline{X}_0^{\alpha-\widetilde{\varepsilon}} - \underline{X}_n^{\alpha-\widetilde{\varepsilon}} + \frac{2\varepsilon}{3} < \underline{X}_n^{\alpha-\widetilde{\varepsilon}} + \frac{\varepsilon}{3} - \underline{X}_n^{\alpha-\widetilde{\varepsilon}} + \frac{2\varepsilon}{3} = \varepsilon\end{aligned} \tag{6}$$

for each $\beta \in (\alpha - \widetilde{\varepsilon}, \alpha]$ and each $n \in K$. We also have $\delta(K) = 1$ since $\delta(K_1) = \delta(K_2) = 0$ implies $\delta(K_1^c) = \delta(K_2^c) = 1$. By the relation (6) we can write

$$\delta\left(\left\{n \in \mathbb{N} : \left|\underline{X}_n^\alpha - \underline{X}_n^\beta\right| < \varepsilon\right\}\right) = 1$$

implies

$$\delta\left(\left\{n \in \mathbb{N} : \left|\underline{X}_n^\alpha - \underline{X}_n^\beta\right| \geq \varepsilon\right\}\right) = 0$$

for each $\beta \in (\alpha - \widetilde{\varepsilon}, \alpha]$. Hence, the sequence $(\underline{X}_n^\alpha)$ is SELC at $\alpha \in (0, 1]$.

Similarly, it can be proved that $\left(\overline{X}_n^\alpha\right)$ is SELC at $\alpha \in (0, 1]$.

Now we show that $(\underline{X}_n^\alpha)$ is SERC at 0. Since $\underline{X}_0^\alpha$ is right continuous at $\alpha = 0$, we can say that for all $\varepsilon > 0$ there exists $\widetilde{\varepsilon_1} > 0$ such that

$$\left|\underline{X}_0^\beta - \underline{X}_0^0\right| < \frac{\varepsilon}{3}$$

for each $\beta \in [0, \widetilde{\varepsilon_1})$. Since $\underline{X}_0^\alpha$ is nondecreasing, we have

$$0 \leq \underline{X}_0^\beta - \underline{X}_0^0 < \frac{\varepsilon}{3}\ . \tag{7}$$

Let us denote $\widetilde{\varepsilon} := \frac{\widetilde{\varepsilon_1}}{2}$. Using (3) , we get

$$\begin{aligned}
\delta\left(\left\{n \in \mathbb{N} : \left|\underline{X}_n^0 - \underline{X}_0^0\right| \geq \frac{\varepsilon}{3}\right\}\right) &= 0 \quad \text{and} \\
\delta\left(\left\{n \in \mathbb{N} : \left|\underline{X}_n^{\widetilde{\varepsilon}} - \underline{X}_0^{\widetilde{\varepsilon}}\right| \geq \frac{\varepsilon}{3}\right\}\right) &= 0.
\end{aligned} \tag{8}$$

Now denote $K' := K_3^c \cap K_4^c$ where $K_3 = K_3(\varepsilon) := \left\{n \in \mathbb{N} : \left|\underline{X}_n^0 - \underline{X}_0^0\right| \geq \frac{\varepsilon}{3}\right\}$ and $K_4 = K_4(\varepsilon) := \left\{n \in \mathbb{N} : \left|\underline{X}_n^{\widetilde{\varepsilon}} - \underline{X}_0^{\widetilde{\varepsilon}}\right| \geq \frac{\varepsilon}{3}\right\}$. Since $\underline{X}_n^\alpha$ and $\underline{X}_0^\alpha$ are nondecreasing, using (7) and (8) we get

$$\begin{aligned}
0 &\leq \underline{X}_n^\beta - \underline{X}_n^0 \leq \underline{X}_n^{\widetilde{\varepsilon}} - \underline{X}_n^0 < \underline{X}_n^{\widetilde{\varepsilon}} + \frac{\varepsilon}{3} - \underline{X}_0^0 < \frac{\varepsilon}{3} + \underline{X}_0^{\widetilde{\varepsilon}} + \frac{\varepsilon}{3} - \underline{X}_0^0 \\
&= \underline{X}_0^{\widetilde{\varepsilon}} - \underline{X}_0^0 + \frac{2\varepsilon}{3} < \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} = \varepsilon
\end{aligned} \tag{9}$$

for each $\beta \in [0, \widetilde{\varepsilon})$ and each $n \in K'$. We also have $\delta(K') = 1$ since $\delta(K_3) = \delta(K_4) = 0$ implies $\delta(K_3^c) = \delta(K_4^c) = 1$. By the relation (6) we can write

$$\delta\left(\left\{n \in \mathbb{N} : \left|\underline{X}_n^\beta - \underline{X}_n^0\right| < \varepsilon\right\}\right) = 1$$

which implies that

$$\delta\left(\left\{n \in \mathbb{N} : \left|\underline{X}_n^\beta - \underline{X}_n^0\right| \geq \varepsilon\right\}\right) = 0$$

for each $\beta \in [0, \widetilde{\varepsilon})$. Hence, the sequence $(\underline{X}_n^\alpha)$ is SERC at $\alpha = 0$.

Similarly, it can be proved that $\left(\overline{X}_n^\alpha\right)$ is SERC at $\alpha = 0$.

($\Leftarrow$) In this part of the proof we show that the conditions of Theorem 4 are satisfied for the pair of functions $f(\alpha)$ and $g(\alpha)$.

(i) Since the sequences $(\underline{X}_n^\alpha)$ and $\left(\overline{X}_n^\alpha\right)$ are SELC at each $\alpha \in (0,1]$, then for any $\varepsilon > 0$ there exists $\widetilde{\varepsilon} > 0$ such that

$$\begin{aligned} \delta\left(\left\{n \in \mathbb{N} : \left|\underline{X}_n^\alpha - \underline{X}_n^\beta\right| \geq \varepsilon\right\}\right) &= 0 \quad \text{and} \\ \delta\left(\left\{n \in \mathbb{N} : \left|\overline{X}_n^\alpha - \overline{X}_n^\beta\right| \geq \varepsilon\right\}\right) &= 0 \end{aligned}$$

hold for each $\beta \in (\alpha - \widetilde{\varepsilon}, \alpha]$. Denote $M := K_5^c \cap K_6^c$ where $K_5 = K_5(\alpha,\beta,\varepsilon) := \left\{n \in \mathbb{N} : \left|\underline{X}_n^\alpha - \underline{X}_n^\beta\right| \geq \varepsilon\right\}$ and $K_6 = K_6(\alpha,\beta,\varepsilon) := \left\{n \in \mathbb{N} : \left|\overline{X}_n^\alpha - \overline{X}_n^\beta\right| \geq \varepsilon\right\}$. Since $\underline{X}_n^\alpha$ is nondecreasing and $\overline{X}_n^\alpha$ is nonincreasing with respect to α, we have

$$\begin{aligned} 0 &\leq \underline{X}_n^\alpha - \underline{X}_n^\beta < \varepsilon \quad \text{and} \\ 0 &\leq \overline{X}_n^\beta - \overline{X}_n^\alpha < \varepsilon \end{aligned} \tag{10}$$

for each $\beta \in (\alpha - \widetilde{\varepsilon}, \alpha]$ and each $n \in M$. Hence we get $\underset{n\in M,\ n\to\infty}{st-\lim} \underline{X}_n^\alpha = f(\alpha)$, $\underset{n\in M,\ n\to\infty}{st-\lim} \underline{X}_n^\beta = f(\beta)$, $\underset{n\in M,\ n\to\infty}{st-\lim} \overline{X}_n^\alpha = g(\alpha)$ and $\underset{n\in M,\ n\to\infty}{st-\lim} \overline{X}_n^\beta = g(\beta)$ for each $\alpha, \beta \in [0,1]$, since $\delta(M) = 1$. By Theorem 2 and the linearity of the statistical limit operator, we get

$$\begin{aligned} 0 &\leq f(\alpha) - f(\beta) < \varepsilon, \\ 0 &\leq g(\beta) - g(\alpha) < \varepsilon \end{aligned}$$

for each $\beta \in (\alpha - \widetilde{\varepsilon}, \alpha]$. Thus the functions $f(\alpha)$ and $g(\alpha)$ are left continuous at each $\alpha \in (0,1]$.

(ii) Since the sequences $(\underline{X}_n^\alpha)$ and $\left(\overline{X}_n^\alpha\right)$ are SERC at $\alpha = 0$, then for any $\varepsilon > 0$ there exists $\widetilde{\varepsilon} > 0$ such that

$$\begin{aligned} \delta\left(\left\{n \in \mathbb{N} : \left|\underline{X}_n^0 - \underline{X}_n^\beta\right| \geq \varepsilon\right\}\right) &= 0 \quad \text{and} \\ \delta\left(\left\{n \in \mathbb{N} : \left|\overline{X}_n^0 - \overline{X}_n^\beta\right| \geq \varepsilon\right\}\right) &= 0 \end{aligned}$$

hold for each $\beta \in [0, \widetilde{\varepsilon})$. Let us write $M' := K_7^c \cap K_8^c$ where $K_7 = K_7(\beta,\varepsilon) := \left\{n \in \mathbb{N} : \left|\underline{X}_n^0 - \underline{X}_n^\beta\right| \geq \varepsilon\right\}$ and $K_8 = K_8(\beta,\varepsilon) := \left\{n \in \mathbb{N} : \left|\overline{X}_n^0 - \overline{X}_n^\beta\right| \geq \varepsilon\right\}$. Since $\underline{X}_n^\alpha$ is nondecreasing and $\overline{X}_n^\alpha$ is nonincreasing with respect to α, we get

$$\begin{aligned} 0 &\leq \underline{X}_n^\beta - \underline{X}_n^0 < \varepsilon, \\ 0 &\leq \overline{X}_n^0 - \overline{X}_n^\beta < \varepsilon \end{aligned}$$

for each $\beta \in [0, \widetilde{\varepsilon})$, and each $n \in M'$. By Theorem 2, we have

$$\begin{aligned} 0 &\leq f(\beta) - f(0) < \varepsilon, \\ 0 &\leq g(0) - g(\beta) < \varepsilon \end{aligned}$$

for each $\beta \in [0, \widehat{\varepsilon})$ since $\delta(M') = 1$. Hence the functions $f(\alpha)$ and $g(\alpha)$ are right continuous at $\alpha = 0$.

(iii) Note that the function $\underline{X}_n^\alpha$ is nondecreasing for α, that is, we have $\underline{X}_n^{\alpha_1} \le \underline{X}_n^{\alpha_2}$ whenever $\alpha_1 \le \alpha_2$. By Theorem 3, we get $f(\alpha_1) \le f(\alpha_2)$. Hence, the function $f(\alpha)$ is nondecreasing. Similarly it can be proved that the function $g(\alpha)$ is nonincreasing.

(iv) Since X_n is a fuzzy number, we have

$$\underline{X}_n^1 \le \overline{X}_n^1$$

for each $n \in \mathbb{N}$. By Theorem 3, we get

$$f(1) \le g(1).$$

Consequently, the pair of functions $f(\alpha)$ and $g(\alpha)$ determine a fuzzy number by Theorem 4. □

4 Applications of Characterization Theorem

In this section, we apply certain conditions similar to those of Theorem 5 to two results which are valid for sequences of real numbers but not for sequences of fuzzy numbers. Hence we obtain modifications of these results which are also valid for sequences of fuzzy numbers as can be seen by Theorems 6 and 7. These conditions can be applied to some of the other results which are valid for sequences of real numbers but are non-valid for sequences of fuzzy numbers.

The sequence $\{X_n\}$ considered in Example 1 is statistically bounded but we have $\Gamma_X^l = \varnothing$. Using the same conditions and techniques similar to those of Theorem 5 we can make the set Γ_X^l non-empty for a sequence $\{X_n\}$ of fuzzy numbers.

Theorem 6. *Let $\{X_n\}$ be a statistically bounded sequence of fuzzy numbers and M be a subset of $\mathbb{N}$ such that $\delta(M) \neq 0$. Then we have $\Gamma_X^l \neq \varnothing$, if the following conditions are satisfied:*

(i) For each $\alpha_0 \in (0,1]$ and each $\varepsilon > 0$ there exists $\varepsilon' > 0$ such that

$$|\underline{X}_n^\alpha - \underline{X}_n^{\alpha_0}| < \varepsilon \qquad \text{and} \qquad \left|\overline{X}_n^\alpha - \overline{X}_n^{\alpha_0}\right| < \varepsilon$$

whenever $\alpha \in (\alpha_0 - \varepsilon', \alpha_0]$ and $n \in M$;

(ii) There exists $\varepsilon'' > 0$ such that

$$\left|\underline{X}_n^\alpha - \underline{X}_n^0\right| < \varepsilon \qquad \text{and} \qquad \left|\overline{X}_n^\alpha - \overline{X}_n^0\right| < \varepsilon$$

whenever $\alpha \in [0, \varepsilon'')$ and $n \in M$.

Proof. Since $\{X_n\}$ is statistically bounded there exists a real number A such that

$$\delta\left(\left\{n \in \mathbb{N} : |\underline{X}_n^\alpha| < A \text{ and } \left|\overline{X}_n^\alpha\right| < A\right\}\right) = 1 \tag{11}$$

for each $\alpha \in [0,1]$. Denote $M_1 := \left\{n \in M : |\underline{X}_n^\alpha| < A \text{ and } \left|\overline{X}_n^\alpha\right| < A\right\}$. Using (11) and the fact that $\delta(M) \neq 0$ we get $\delta(M_1) \neq 0$. Thus there exist bounded non-thin subsequences of $(\underline{X}_n^\alpha)$ and $\left(\overline{X}_n^\alpha\right)$. By Theorem 1, the sequences $(\underline{X}_n^\alpha)$ and $\left(\overline{X}_n^\alpha\right)$ have statistical cluster points $f(\alpha)$ and $g(\alpha)$ for each $\alpha \in [0,1]$, respectively. In the proof of Theorem 5, replacing the sets K_5, K_6, K_7 and K_8 by the sets $\left\{n \in M_1 : \left|\underline{X}_n^\alpha - \underline{X}_n^\beta\right| \geq \varepsilon\right\}$, $\left\{n \in M_1 : \left|\overline{X}_n^\alpha - \overline{X}_n^\beta\right| \geq \varepsilon\right\}$, $\left\{n \in M_1 : \left|\underline{X}_n^0 - \underline{X}_n^\beta\right| \geq \varepsilon\right\}$ and $\left\{n \in M_1 : \left|\overline{X}_n^0 - \overline{X}_n^\beta\right| \geq \varepsilon\right\}$, respectively; and the role of Theorems 2 and 3 by Corollaries 1 and 2 respectively, it can be proved that the pair of functions $f(\alpha)$ and $g(\alpha)$ determine a fuzzy number. Note that for the same $n\,(\in M_1)$, the subsequences $(\underline{X}_n^\alpha)$ and $\left(\overline{X}_n^\alpha\right)$ have statistical cluster points and are equi-left-continuous at each $\alpha \in (0,1]$ and equi-right-continuous at $\alpha = 0$. Let us denote $\underline{X}_0^\alpha := f(\alpha)$ and $\overline{X}_0^\alpha := g(\alpha)$ where $X_0 \in L(\mathbb{R})$.

Now we show that $X_0 \in \Gamma_X^l$. Since we have

$$\begin{aligned} \delta\left(\{n \in M_1 : |\underline{X}_n^\alpha - \underline{X}_0^\alpha| < \varepsilon\}\right) &\neq 0 \quad \text{and} \\ \delta\left(\left\{n \in M_1 : \left|\overline{X}_n^\alpha - \overline{X}_0^\alpha\right| < \varepsilon\right\}\right) &\neq 0 \end{aligned}$$

for each $\alpha \in [0,1]$, we then get

$$\delta\left(\left\{n \in \mathbb{N} : \max\left\{\left|\overline{X}_n^\alpha - \overline{X}_0^\alpha\right|, |\underline{X}_n^\alpha - \underline{X}_0^\alpha|\right\} < \varepsilon\right\}\right) \neq 0.$$

Hence by definition of Γ_X^l we have $X_0 \in \Gamma_X^l$. □

As a direct consequence of this theorem we state the following result.

Corollary 3. *Under the hypothesis of Theorem 6, the sequence $\{X_n\}$ has a levelwise statistically convergent subsequence, i.e. levelwise statistical analogue of the Bolzano Weierstrass Theorem is valid.*

We can also apply Theorem 5 in order to get the levelwise statistical Cauchy criteria for sequences of fuzzy numbers. It can be easily seen that the sequence $\{X_n\}$ given in Example 1 is a levelwise statistically Cauchy sequence but it does not converge levelwise statistically to any fuzzy number. If a levelwise statistically Cauchy sequence satisfies the condition of Theorem 5, then it converges levelwise statistically. We state this result as follows.

Theorem 7. *Let $\{X_n\}$ be a sequence of fuzzy numbers. If the sequences of functions $(\underline{X}_n^\alpha)$ and $\left(\overline{X}_n^\alpha\right)$ are SELC at each $\alpha \in (0,1]$ and SERC at $\alpha = 0$, and the sequence $\{X_n\}$ is a levelwise statistically Cauchy sequence then the sequence $\{X_n\}$ converges levelwise statistically.*

Proof. Since $\{X_n\}$ is a levelwise statistically Cauchy sequence, the sequences $(\underline{X}_n^\alpha)$ and $\left(\overline{X}_n^\alpha\right)$ are statistically Cauchy sequences of real numbers for each $\alpha \in [0,1]$.

Since a statistically Cauchy sequence of real numbers is statistically convergent, we say that $st-\lim \underline{X}_n^\alpha$ and $st-\lim \overline{X}_n^\alpha$ exist for each $\alpha \in [0,1]$. By Theorem 5, there exists a fuzzy number X_0 such that $st-\lim \underline{X}_n^\alpha = \underline{X}_0^\alpha$ and $st-\lim \overline{X}_n^\alpha = \overline{X}_0^\alpha$ for each $\alpha \in [0,1]$. Therefore the sequence $\{X_n\}$ is levelwise statistically convergent to the fuzzy number X_0. $\square$

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