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THE ASSOCIATED SCHUR COMPLEMENTS OF $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$

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Abstract

Let $S_1 = A - BD^{\dagger}C$ and $S_2 = D - CA^{\dagger}B$ be the associated Schur complements of $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$. In this paper, we derive necessary and sufficient conditions for $S_1 = 0$ imply $S_2 = 0$ by using generalized inverses of matrices and singular value decompositions.

1 Introduction

Let $A \in \mathbb{C}_r^{m\times n}, B \in \mathbb{C}_s^{m\times q}, C \in \mathbb{C}_t^{p\times n}$ and $D \in \mathbb{C}_k^{p\times q}$. Consider the matrix

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{1}$$

and let

$$S_1 = A - BD^{\dagger}C \quad \text{and} \quad S_2 = D - CA^{\dagger}B \tag{2}$$

be the associated Schur complements.

In [2], the following problem was given:

Problem 1: Let M be the matrix in (1) with Schur complements (2). When does $S_1 = 0$ imply $S_2 = 0$?

In this paper, we derive necessary and sufficient conditions for $S_1 = 0$ imply $S_2 = 0$ by using generalized inverses of matrices and singular value decomposition.

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Lemma 1. *Let* $\begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}^{\dagger} = 0$ *with appropriate sizes, then:*

$$X_{12} = 0 \quad \text{and} \quad X_{22} = 0.$$

Proof. Substituting $\begin{bmatrix} \widehat{X}_{11} & \widehat{X}_{12} \\ \widehat{X}_{21} & \widehat{X}_{22} \end{bmatrix} := \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}^{\dagger}$ into $\begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}^{\dagger} = 0$ leads to the equivalences $\widehat{X}_{21} = 0$ and $\widehat{X}_{22} = 0$.

Because $\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} = \begin{bmatrix} \widehat{X}_{11} & \widehat{X}_{12} \\ \widehat{X}_{21} & \widehat{X}_{22} \end{bmatrix}^{\dagger} = \begin{bmatrix} \widehat{X}_{11} & \widehat{X}_{12} \\ 0 & 0 \end{bmatrix}^{\dagger}$, we drive

$$\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} = \begin{bmatrix} X_{11} & 0 \\ X_{21} & 0 \end{bmatrix},$$

i.e., $X_{12} = 0$ and $X_{22} = 0$. □

Lemma 2. *Let A, B, C, D, M, S_1 and S_2 be the same as in Problem 1. Suppose $S_1 = 0$ and $S_2 = 0$. Then*

$$\left(BD^{\dagger}C\right)^{\dagger} = C^{\dagger}DB^{\dagger}. \tag{3}$$

Proof. Let the singular value decomposition (SVD)[3] of A, B, C and D be

$$\begin{gathered}
U_A^* A V_A^* = \begin{array}{c} \begin{bmatrix} \Sigma_A & 0 \\ 0 & 0 \end{bmatrix} \begin{array}{c} r \\ m-r \end{array} \\ \begin{array}{cc} r & n-r \end{array} \end{array}, \quad U_B^* B V_B^* = \begin{array}{c} \begin{bmatrix} \Sigma_B & 0 \\ 0 & 0 \end{bmatrix} \begin{array}{c} s \\ m-s \end{array} \\ \begin{array}{cc} s & q-s \end{array} \end{array}, \\
U_C^* C V_C^* = \begin{array}{c} \begin{bmatrix} \Sigma_C & 0 \\ 0 & 0 \end{bmatrix} \begin{array}{c} t \\ p-t \end{array} \\ \begin{array}{cc} t & n-t \end{array} \end{array}, \quad U_D^* D V_D^* = \begin{array}{c} \begin{bmatrix} \Sigma_D & 0 \\ 0 & 0 \end{bmatrix} \begin{array}{c} k \\ p-k \end{array} \\ \begin{array}{cc} k & q-k \end{array} \end{array},
\end{gathered} \tag{4}$$

in which $\Sigma_A, \Sigma_B, \Sigma_C$ and Σ_D are positive diagonal matrices, and U_A, U_B, U_C, U_D, V_A, V_B, V_C and V_D are unitary matrices with appropriate sizes.

Substituting (4) into $S_1 = 0$ leads to

$$U_B^* U_A \begin{bmatrix} \Sigma_A & 0 \\ 0 & 0 \end{bmatrix} V_A V_C^* = \begin{bmatrix} \Sigma_B & 0 \\ 0 & 0 \end{bmatrix} V_B V_D^* \begin{bmatrix} \Sigma_D^{-1} & 0 \\ 0 & 0 \end{bmatrix} U_D^* U_C \begin{bmatrix} \Sigma_C & 0 \\ 0 & 0 \end{bmatrix},$$

$$U_B^* U_A \begin{bmatrix} \Sigma_A & 0 \\ 0 & 0 \end{bmatrix} V_A V_C^* = \begin{array}{c} \begin{bmatrix} \widetilde{A} & 0 \\ 0 & 0 \end{bmatrix} \begin{array}{c} s \\ m-s \end{array} \\ \begin{array}{cc} t & n-t \end{array} \end{array}, \tag{5}$$

and

$$\left(U_B^* U_A \begin{bmatrix} \Sigma_A & 0 \\ 0 & 0 \end{bmatrix} V_A V_C^*\right)^{\dagger} = V_C V_A^* \begin{bmatrix} \Sigma_A^{-1} & 0 \\ 0 & 0 \end{bmatrix} U_A^* U_B = \begin{bmatrix} \left(\widetilde{A}\right)^{-1} & 0 \\ 0 & 0 \end{bmatrix}. \tag{6}$$

Substituting (4) into $S_2 = 0$ leads to

$$U_C^* U_D \begin{bmatrix} \Sigma_D & 0 \\ 0 & 0 \end{bmatrix} V_D V_B^* = \begin{bmatrix} \Sigma_C & 0 \\ 0 & 0 \end{bmatrix} V_C V_A^* \begin{bmatrix} \Sigma_A^{-1} & 0 \\ 0 & 0 \end{bmatrix} U_A^* U_B \begin{bmatrix} \Sigma_B & 0 \\ 0 & 0 \end{bmatrix}. \quad (7)$$

Then from (4)-(7),

$$\begin{aligned}
& C^\dagger D B^\dagger \\
&= V_C^* \begin{bmatrix} \Sigma_C^{-1} & 0 \\ 0 & 0 \end{bmatrix} U_C^* U_D \begin{bmatrix} \Sigma_D & 0 \\ 0 & 0 \end{bmatrix} V_D V_B^* \begin{bmatrix} \Sigma_B^{-1} & 0 \\ 0 & 0 \end{bmatrix} U_B^* \\
&= V_C^* \begin{bmatrix} \Sigma_C^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Sigma_C & 0 \\ 0 & 0 \end{bmatrix} V_C V_A^* \begin{bmatrix} \Sigma_A^{-1} & 0 \\ 0 & 0 \end{bmatrix} U_A^* U_B \begin{bmatrix} \Sigma_B & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Sigma_B^{-1} & 0 \\ 0 & 0 \end{bmatrix} U_B^* \\
&= V_C^* \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} V_C V_A^* \begin{bmatrix} \Sigma_A^{-1} & 0 \\ 0 & 0 \end{bmatrix} U_A^* U_B \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} U_B^* \\
&= V_C^* \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \widetilde{A} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} U_B^* \\
&= V_C^* V_C V_A^* \begin{bmatrix} \Sigma_A^{-1} & 0 \\ 0 & 0 \end{bmatrix} U_A^* U_B U_B^* \\
&= A^\dagger. \qquad (8)
\end{aligned}$$

Applying (8) and $S_1 = A - C^\dagger D B^\dagger = 0$ gives (3). □

Lemma 3. *[6] Let $A \in \mathbb{C}^{m\times n}, B \in \mathbb{C}^{n\times p}$ and $C \in \mathbb{C}^{p\times q}$ be given, and let $M = ABC$ if $R(B) \subseteq R(A)$ and $R(B^*) \subseteq R(C)$, then*

$$r\left(M^\dagger - C^\dagger B^\dagger A^\dagger\right) = r \begin{bmatrix} B \\ BCC^* \end{bmatrix} + r \begin{bmatrix} B & A^*AB \end{bmatrix} - 2r(B).$$

In particular,

$$(ABC)^\dagger = C^\dagger B^\dagger A^\dagger$$

if and only if

$$R(A^*AB) \subseteq R(B) \quad \text{and} \quad R\left((BCC^*)^*\right) \subseteq R(B^*).$$

Lemma 4. *Let B, C and D be the same as in Problem 1. Then the equalities $\left(I - CC^\dagger\right) D = 0$, $D\left(I - B^\dagger B\right) = 0$ and (3) holds if and only if*

$$r \begin{bmatrix} B \\ D \end{bmatrix} = r(B), r \begin{bmatrix} C & D \end{bmatrix} = r(C) \quad \text{and} \quad r \begin{bmatrix} D \\ DB^*B \end{bmatrix} + r \begin{bmatrix} D & CC^*D \end{bmatrix} = 2r(D) \quad (9)$$

Proof. It is well know that the equalities $\left(I - CC^\dagger\right) D = 0$ and $D\left(I - B^\dagger B\right) = 0$ holds if and only if

$$R(D^*) \subseteq R(B^*) \quad \text{and} \quad R(D) \subseteq R(C) \quad (10)$$

if and only if

$$r\begin{bmatrix} B \\ D \end{bmatrix} = r(B) \quad \text{and} \quad r\begin{bmatrix} C & D \end{bmatrix} = r(C), \tag{11}$$

and the equalities $R(D^*) \subseteq R\left(B^\dagger\right)$ and $R(D) \subseteq R\left(\left(C^\dagger\right)^*\right)$ holds if and only if

$$R(D^*) \subseteq R(B^*) \quad \text{and} \quad R(D) \subseteq R(C) \tag{12}$$

and

$$\begin{aligned} 0 &= r\begin{bmatrix} D^\dagger \\ D^\dagger CC^* \end{bmatrix} + r\begin{bmatrix} D^\dagger & B^*BD^\dagger \end{bmatrix} - 2r\left(D^\dagger\right) \\ &= r\begin{bmatrix} D^* \\ D^*CC^* \end{bmatrix} + r\begin{bmatrix} D^* & B^*BD^* \end{bmatrix} - 2r(D) \\ &= r\begin{bmatrix} D \\ DB^*B \end{bmatrix} + r\begin{bmatrix} D & CC^*D \end{bmatrix} - 2r(D). \end{aligned} \tag{13}$$

Applying Lemma 3 and (10-13) to (9) gives $\left(I - CC^\dagger\right)D = 0$, $D\left(I - B^\dagger B\right) = 0$ and (3) holds if and only if
$r\begin{bmatrix} B \\ D \end{bmatrix} = r(B), r\begin{bmatrix} C & D \end{bmatrix} = r(C)$ and $r\begin{bmatrix} D^\dagger \\ D^\dagger CC^* \end{bmatrix} + r\begin{bmatrix} D^\dagger & B^*BD^\dagger \end{bmatrix} = 2r\left(D^\dagger\right)$
holds if and only if
$r\begin{bmatrix} B \\ D \end{bmatrix} = r(B), r\begin{bmatrix} C & D \end{bmatrix} = r(C)$ and $r\begin{bmatrix} D \\ DB^*B \end{bmatrix} + r\begin{bmatrix} D & CC^*D \end{bmatrix} = 2r(D)$.

2 Main Result

Theorem 1. *Let A, B, C, D, M, S_1 and S_2 be the same as in Problem 1. Suppose $S_1 = 0$. Then the following conditions are equivalent:*

1) $S_2 = 0$;
2) $\left(I - CC^\dagger\right)D = 0, D\left(I - B^\dagger B\right) = 0$ *and* $\left(BD^\dagger C\right)^\dagger = C^\dagger DB^\dagger$;
3) $r\begin{bmatrix} B \\ D \end{bmatrix} = r(B)$, $r\begin{bmatrix} C & D \end{bmatrix} = r(C)$ *and* $r\begin{bmatrix} D \\ DB^*B \end{bmatrix} + r\begin{bmatrix} D & CC^*D \end{bmatrix} = 2r(D)$.

Proof. Suppose $S_2 = 0$, i.e., $D = CA^\dagger B$.

Multiply $D = CA^\dagger B$ from the left-hand (right-hand) side in $CC^\dagger$ $\left(B^\dagger B\right)$, then

$$\begin{aligned} CC^\dagger DB^\dagger B &= CC^\dagger CA^\dagger BB^\dagger B \\ &= CA^\dagger B \\ &= D. \end{aligned} \tag{14}$$

Then

$$D\left(I-B^{\dagger}B\right)=0, \tag{15}$$

and

$$\left(I-CC^{\dagger}\right)D=0. \tag{16}$$

Applying (15), (16) and Lemma 2 gives 2).

Conversely, denote $V_B V_D^{*}\begin{bmatrix}\Sigma_D^{-1} & 0\\ 0 & 0\end{bmatrix}U_D^{*}U_C := \begin{array}{cc} \begin{bmatrix}\widetilde{D} & \widetilde{D}_{12}\\ \widetilde{D}_{21} & \widetilde{D}_{22}\end{bmatrix} & \begin{matrix} t \\ p-t \end{matrix} \\ \begin{matrix} s & q-s \end{matrix} & \end{array}$, then

$$\begin{aligned} V_B V_D^{*}\begin{bmatrix}\Sigma_D & 0\\ 0 & 0\end{bmatrix}U_D^{*}U_C &= \left(V_B V_D^{*}\begin{bmatrix}\Sigma_D^{-1} & 0\\ 0 & 0\end{bmatrix}U_D^{*}U_C\right)^{\dagger *}\\ &= \left(\begin{bmatrix}\widetilde{D} & \widetilde{D}_{12}\\ \widetilde{D}_{21} & \widetilde{D}_{22}\end{bmatrix}^{\dagger}\right)^{*}. \end{aligned} \tag{17}$$

Applying (11) and (12) gives the equivalences

$$\left(U_D\begin{bmatrix}\Sigma_D & 0\\ 0 & 0\end{bmatrix}V_D V_B^{*}\begin{bmatrix}0 & 0\\ 0 & I_B\end{bmatrix}V_B\right)^{*} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} = \left(I-B^{\dagger}B\right)D^{*},$$

and

$$\begin{aligned} \begin{bmatrix}0 & 0\\ 0 & I_B\end{bmatrix}V_B V_D^{*}\begin{bmatrix}\Sigma_D & 0\\ 0 & 0\end{bmatrix}U_D^{*}U_C &= \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\\ &= \begin{bmatrix}0 & 0\\ 0 & I_B\end{bmatrix}\left(\begin{bmatrix}\widetilde{D} & \widetilde{D}_{12}\\ \widetilde{D}_{21} & \widetilde{D}_{22}\end{bmatrix}^{\dagger}\right)^{*}. \end{aligned} \tag{18}$$

Substituting Lemma 1 into (18) leads to the equivalences

$$\widetilde{D}_{21}=0, \widetilde{D}_{22}=0.$$

Similarly, we can show

$$\widetilde{D}_{12}=0.$$

Then:

$$V_B V_D^{*}\begin{bmatrix}\Sigma_D^{-1} & 0\\ 0 & 0\end{bmatrix}U_D^{*}U_C = \begin{bmatrix}\widetilde{D} & \widetilde{D}_{12}\\ \widetilde{D}_{21} & \widetilde{D}_{22}\end{bmatrix} = \begin{bmatrix}\widetilde{D} & 0\\ 0 & 0\end{bmatrix}. \tag{19}$$

Substituting (5) and (19) into (7) gives

$$\begin{bmatrix}\widetilde{A} & 0\\ 0 & 0\end{bmatrix} = \begin{bmatrix}\Sigma_B\widetilde{D}\Sigma_C & 0\\ 0 & 0\end{bmatrix}. \tag{20}$$

Substituting (4) into $\left(BD^{\dagger}C\right)^{\dagger}$ and $C^{\dagger}DB^{\dagger}$ leads to

$$\begin{aligned}\left(BD^{\dagger}C\right)^{\dagger} &= \left(U_B \begin{bmatrix} \Sigma_B & 0 \\ 0 & 0 \end{bmatrix} V_B V_D^* \begin{bmatrix} \Sigma_D^{-1} & 0 \\ 0 & 0 \end{bmatrix} U_D^* U_C \begin{bmatrix} \Sigma_C & 0 \\ 0 & 0 \end{bmatrix} V_C\right)^{\dagger} \\ &= V_C^* \left(\begin{bmatrix} \Sigma_B & 0 \\ 0 & 0 \end{bmatrix} V_B V_D^* \begin{bmatrix} \Sigma_D^{-1} & 0 \\ 0 & 0 \end{bmatrix} U_D^* U_C \begin{bmatrix} \Sigma_C & 0 \\ 0 & 0 \end{bmatrix}\right)^{\dagger} U_B^* \end{aligned} \tag{21}$$

and

$$C^{\dagger}DB^{\dagger} = V_C^* \begin{bmatrix} \Sigma_C^{-1} & 0 \\ 0 & 0 \end{bmatrix} U_C^* U_D \begin{bmatrix} \Sigma_D & 0 \\ 0 & 0 \end{bmatrix} V_D V_B^* \begin{bmatrix} \Sigma_B^{-1} & 0 \\ 0 & 0 \end{bmatrix} U_B^*. \tag{22}$$

Applying (21), (22) and $\left(BD^{\dagger}C\right)^{\dagger} - C^{\dagger}DB^{\dagger} = 0$ gives

$$\begin{aligned} &\left(\begin{bmatrix} \Sigma_B & 0 \\ 0 & 0 \end{bmatrix} V_B V_D^* \begin{bmatrix} \Sigma_D^{-1} & 0 \\ 0 & 0 \end{bmatrix} U_D^* U_C \begin{bmatrix} \Sigma_C & 0 \\ 0 & 0 \end{bmatrix}\right)^{\dagger} - \begin{bmatrix} \Sigma_C^{-1} & 0 \\ 0 & 0 \end{bmatrix} U_C^* U_D \\ &\qquad \times \begin{bmatrix} \Sigma_D & 0 \\ 0 & 0 \end{bmatrix} V_D V_B^* \begin{bmatrix} \Sigma_B^{-1} & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \left(\Sigma_B \widetilde{D} \Sigma_C\right)^{\dagger} & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} \Sigma_C^{-1} \widetilde{D}^{\dagger} \Sigma_B^{-1} & 0 \\ 0 & 0 \end{bmatrix} \\ &= 0 \end{aligned}$$

i.e.,

$$\Sigma_C^{-1} \widetilde{D}^{\dagger} \Sigma_B^{-1} = \left(\Sigma_C \widetilde{D} \Sigma_B\right)^{\dagger}. \tag{23}$$

Substituting (4), (19) and (20) into $S_2 = D - CA^{\dagger}B$ leads to

$$\begin{aligned} &U_C^* S_2 V_B^* \\ &= U_C^* U_D \begin{bmatrix} \Sigma_D & 0 \\ 0 & 0 \end{bmatrix} V_D V_B^* - \begin{bmatrix} \Sigma_C & 0 \\ 0 & 0 \end{bmatrix} V_C V_A^* \begin{bmatrix} \Sigma_A^{-1} & 0 \\ 0 & 0 \end{bmatrix} U_A^* U_B \begin{bmatrix} \Sigma_B & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \widetilde{D}^{\dagger} & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} \Sigma_C & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \widetilde{A}^{\dagger} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Sigma_B & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \widetilde{D}^{\dagger} - \Sigma_C \widetilde{A}^{\dagger} \Sigma_B & 0 \\ 0 & 0 \end{bmatrix} \end{aligned} \tag{24}$$

Substituting (20) and (23) into (24) gives

$$\widetilde{D}^{\dagger} - \Sigma_C \widetilde{A}^{\dagger} \Sigma_B = \widetilde{D}^{\dagger} - \Sigma_C \left(\Sigma_B \widetilde{D} \Sigma_C\right)^{\dagger} \Sigma_B = 0$$

and

$$U_C^* S_2 V_B^* = 0$$

i.e., $S_2 = 0$.

The equivalence of 2) and 3) is based on Lemma 4. $\square$

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