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A POSTERIORI BOUNDS OF APPROXIMATE SOLUTION TO VARIATIONAL AND QUASI-VARIATIONAL INEQUALITIES

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Abstract

In this paper we present some bounds of an approximate solution to variational and quasi-variational inequalities. The measures of errors can be used for construction of iterative and continuous procedures for solving variational (quasi-variational) inequalities and formulation of corresponding stopping rules. We will also present some methods based on linearization for solving quasi-variational inequalities.

We are concerned with variational inequality: find $x_* \in C$ such that

$$\langle F(x_*), y - x_* \rangle \geq 0, \ \forall y \in C, \tag{1}$$

where C is a closed convex set in Hilbert space H and $F : H \mapsto H$ is an operator of H. In case $F(x) = c \in R^n$ and $C = \{x \in R^n : Ax \leq b, Bx = d\}$ variational inequality (1) is a linear programming problem. It is well known that if $F(x) = f'(x)$ is potential operator, then (1) can be understood as a necessary condition of optimality in the problem of minimization of the function f on C. As a consequence this fact, many methods for solving optimization problems can be adapted for solving variational inequalities.

We will also consider so called quasi-variational inequality, when $C : H \mapsto 2^H$ is a set-valued mapping with nonempty closed and convex values. In this case the problem can be formulated as it follows:

Find $x_* \in C(x_*)$ such that

$$\langle F(x_*), y - x_* \rangle \geq 0, \ \forall y \in C(x_*), \tag{2}$$

Let us note that the approximation theory for quasi varational inequalities requires a variational inequality and a fixed point problem should be solved simultaneously. Consequently, many techniques for variational inequalities are not convenient for quasi-variational inequalities [5], [2].

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There are many results that can be considered as the bounds how close is an arbitrary point to the the set of solution of (1). For example Pang in [6], proposed three so called projection measures of closeness of an arbitrary vector to the unique solution of (1) when C is polyhedron, Jianghua and Xiaoguo in [4] proposed some bounds measuring the distance between any point and the solution set for cocoercive variational inequalities. Some modifications of the results from [6] were presented in [3]. In this paper we will present the measures of closeness that include both the measures from [6] and [3].

1 Gap Functions and Projection Measures

Note that almost all questions and results that are presented in this paper can be formulated in terms of gap functions.

A function $r : C \mapsto R$ *is a gap function for (1) if*
(i) $r(x) \geq 0,$ *for all* $x \in C$;
(ii) $(r(x) = 0$ *if and only if* x *is a solution of (1)*

Using gap function r, inequality (1) can be formulated as an optimization problem

$$\text{minimize } r(x) \text{ on } C.$$

The first example of gap function

$$r(x) = \sup\{\langle F(x), x - y\rangle : y \in C\}$$

was proposed by Auslender [1] and it has been extensively studied in various context (continuity, differentiability, convexity ...). Regularized gap functions for (1) of the type

$$r(x) = \sup\left\{\langle F(x), x - y\rangle - \frac{1}{2}\langle y - x, G(y - x)\rangle : y \in C\right\},$$

where $G : H \mapsto H$ is a positive symmetric linear operator were proposed by Fukushima (see for example [2], [7]). In [5] regularized gap functions were used for construction of methods for solving variational and quasi-variational inequalities.

Construction of a gap functions of a projection type is based on well known fact that x_* is a solution of (1) if and only if

$$x_* = \Pi_C(x_* - \beta F(x_*)),\ \beta > 0,$$

where $\Pi_C : H \mapsto H$ is the operator of projection on the set C. As a consequence of this fact we have that z ia a solution of (1) if and only if

$$r(z) := z - \Pi_C(z - \beta F(z)) = 0. \tag{3}$$

Hence, $\|r(z)\|$ can be used as a measure of closeness of z to the set C_* of solutions of (1). But, very simple examples show it is possible that $\|r(z)\|$ is small while at the same time the distance $d(z, C_*)$ can be very large. So, $r(z)$ can be used as a residual measure only for some classes of the variational inequalities (1).

Let us start with one theorem related to one projection measure based on the methods and estimates from [5].

Theorem 1. *Suppose that $C \subseteq H$ is a closed and convex set and the operator $F : H \mapsto H$ satisfies the conditions*

$$\langle F(x) - F(y), x - y \rangle \geq \alpha \|x - y\|^2, \ (\alpha > 0) \tag{4}$$

$$\|F(x) - F(y)\| \leq L\|x - y\|, \forall x, y \in R^n, (L > 0) \tag{5}$$

Then

$$\|z - x_*\| \leq k_1 \|r(z)\|, \ \ \|r(z)\| \leq \alpha\beta k_1 \|z - x_*\|, \tag{6}$$

where x_* is a unique solution of (1) and

$$k_1 = k(\alpha, L, \beta) = \frac{L\beta + 1 + \sqrt{(L\beta + 1)^2 - 4\alpha\beta}}{2\alpha\beta.} \tag{7}$$

Proof. By definition of $r(z)$,

$$\langle \beta F(z) - r(z), y - (z - r(z)) \rangle \geq 0, \ \forall y \in C,$$

from where, for $y = x_* \in C$, we obtain

$$\langle \beta F(z) - r(z), x_* - (z - r(z)) \rangle \geq 0.$$

Therefore, taking into account condition (4), we get

$$\langle -r(z), x_* - (z - r(z)) \rangle \geq \beta \langle F(z), -r(z) \rangle + \beta \langle F(z), z - x_* \rangle =$$

$$\beta \langle F(z), -r(z) \rangle + \beta \langle F(z) - F(x_*), z - x_* \rangle + \beta \langle F(x_*), z - x_* \rangle \geq$$

$$\alpha\beta \|z - x_*\|^2 + \beta \langle F(z) - F(x_*), -r(z) \rangle + \beta \langle F(x_*), z - r(z) - x_* \rangle \geq$$

$$\alpha\beta \|z - x_*\|^2 + \beta \langle F(z) - F(x_*), -r(z) \rangle.$$

Hence,

$$\langle -r(z), x_* - z \rangle - \|r(z)\|^2 \geq \alpha\beta \|z - x_*\|^2 - \beta L \|z - x_*\| \|r(z)\|,$$

and

$$\alpha\beta \|z - x_*\|^2 + \|r(z)\|^2 \leq (L\beta + 1) \|r(z)\| \|z - x_*\|,$$

from where (6) follows. □

Remark 1. Note that $(L\beta+1)^2 \geq 4\alpha\beta$ is an immediate consequence of $L \geq \alpha$.

Remark 2. From (6) it follows the estimate $\|z - x_*\| \leq \frac{L\beta+1}{\alpha\beta}\|r(z\|$; for $\beta = 1$, we obtain the estimate from ([6]).

Remark 3. In [9], [10] and [4] authors consider problem of the bounds measuring the distance between any point and the set of solution C_* of so-called cocoercive variational inequality (1). Let us mention that the map $F : H$
mapstoH is said to be cocoercive if there exists $\gamma > 0$ such that

$$\langle f(x) - f(y), x - y\rangle \geq \gamma\|f(x) - f(y)\|^2, \ \forall x, y \in H.$$

The cocoercivity plays an important role in the convergence analysis of algorithms. Note that any cocoercive map with modulus γ is monotone (but not strongly monotone) and Lipschitz continuous with Lipschitz constant $L = \frac{1}{\gamma}$. For solving such variational inequalities one can apply Tikhonov regularization method, replacing the map $F(x)$ with $F(x) + \varepsilon x$, $x \in H$, $\varepsilon > 0$, which is strongly monotone with modulus ε. If x_ε is a unique point such that

$$\langle F(x_\varepsilon) + \varepsilon x_\varepsilon, y - x_\varepsilon\rangle \geq 0, \ \forall y \in C,$$

and

$$r_\varepsilon(z) = z - \Pi_C(z - \beta(F(z) + \varepsilon z)),$$

then,

$$\|z - x_\varepsilon\| \leq k_2\|r_\varepsilon(z)\|,$$

where $k_2 = k(\varepsilon, \gamma^{-1} + \varepsilon, \beta)$. In [9] and [4] is proved that, in case of $H = \mathbb{R}^n$, if the set of solutions of (1) is nonempty and bounded, then for sufficiently small ε, the following estimate holds

$$dist(x_\varepsilon, C_*) \leq \beta$$

Consequently, in this case, as a measure of the closeness of any point z to the set C_* one can use the estimate

$$dist(z, C_*) \leq k_2\|r_\varepsilon(z)\| + \beta.$$

2 Linearization and Projection Measures

Application of the projection measures is possible only if it is not difficult to realize a projection onto C. If the set C is given by nonlinear convex differentiable constraints $g_i : H \mapsto R$,

$$C = \{x \in C_0 : g(x) \leq 0\}, \ g = (g_1, g_2, \dots, g_m) : H \mapsto R^m, \tag{8}$$

where $C_0 \subseteq H$ is a closed and convex set of simple structure (for example C_0 is a ball or polyhedron in R^n), then one can replace the projection onto C with projection onto its linear approximations

$$C(z) = \{y \in C_0 : g(z) + \langle g'(z), y - z\rangle \leq 0\}$$

Then a posteriori estimate of the arbitrary vector z to set C_* of the the solutions of (1) may be calculated using a residual [3]

$$r_{L,G}(z) = z - \Pi_{C(z)}(z - G^{-1}F(z)),$$

where $G : H \mapsto H$ is a symmetric linear operator satisfying to

$$m\|\xi\|^2 \leq \langle G\xi, \xi\rangle \leq M\|ksi\|^2,\ 0 \leq m < M, \tag{9}$$

and $\Pi_{L,G}$ is a projection onto $C(z)$ in the norm $\|x\|_G := \sqrt{\langle Gx, x\rangle}$. Let us note that in case of $C = C_0$, this problem was considered in Theorem 1.

Idea of linearization is widely used for construction of the numerical algorithm for solving equations, variational inequalities, problems of optimization ([2], [3], [8]).

In the course of the proof of the next theorem we will also prove that z is a solution of (1) if and only if $r_{L,G}(z) = 0$.

Theorem 2. *Suppose:*
$C_0 \subseteq H$ is a closed and convex set;
$F : H \mapsto H$ is strongly monotone Lipschitz continuous operator:

$$\langle F(x) - F(y), x - y\rangle \geq \alpha\|x - y\|^2,\ ,(\alpha > 0)$$

$$\|F(x) - F(y)\| \leq L\|x - y\|,\ (L > 0);$$

$g_i : H \mapsto R$, $i = 1, \dots, m$ are convex and differentiable function such that

$$\|g_i'(x) - g_i'(y)\| \leq L_i\|x - y\|,\ \forall x, y \in H;$$

there is $\overline{x} \in C_0$, such that $g_i(\overline{x}) < 0, i = 1, 2, \dots, m$ (Slater condition);
C is given by (8).
Then

$$\|z - x_*\|^2 \leq k_L\|r_{L,G}(z)\|^2, \tag{10}$$

where x_* is a unique solution of (1),

$$k_L = \left(\frac{L+\alpha}{4} - m + \frac{M\varepsilon}{2} + \sum_{i=1}^{m} \lambda_i^* L_i\right) : \left(\frac{L\alpha}{L+\alpha} - \frac{M}{2\varepsilon}\right)$$

and $\lambda^* = (\lambda_1*, \dots, \lambda_m^*)$ are Lagrange multipliers for the point x_*.

Proof. Since x_* is a (unique) solution of (1) and $z - r_{L,G}(z) \in C_0$, (see [8], Lemma 5.5., p. 117) there exist $\lambda^* = (\lambda_1, \lambda_2^*, \dots, \lambda_m^*)$, $\lambda^* \neq 0$, such that

$$\left\langle F(x_*) + \sum_{i=1}^{m} \lambda_i^* g_i'(x_*), z - r_{L,G}(z) - x_*\right\rangle \geq 0, \tag{11}$$

$$\lambda_i^* g_i(x_*) = 0,\ \lambda_i^* \geq 0,\ g_i(x_*) \leq 0,\ i = 1, 2, \dots, m. \tag{12}$$

Further, from the condition of convexity, it follows that $g_i(z)+\langle g_i'(z), x-z\rangle \leq g_i(x)$, for all $x \in C_0$. Therefore, $C \subseteq C(z)$ and Slater condition is also fulfilled for the set $C(z)$. Applying Kuhn-Tucker theorem to the problem of projection of the point $z - G^{-1}F(z)$ onto $C(z)$, we obtain that there is $\xi^* = (\xi_1^*, \xi_2^*, \ldots, \xi_m^*) \in R^m$, $\xi^* \neq 0$, such that

$$\left\langle -Gr_{L,G}(z) + F(z) + \sum_{i=1}^{m} \xi_i^* g_i'(z), y - (z - r_{L,G}(z)) \right\rangle \geq 0 \text{ for all } y \in C_0. \quad (13)$$

$$\xi_i^*(g_i(z) + \langle g_i'(z), -r_{L,G}(z)\rangle) = 0,\ i = 1, 2, \ldots, m. \quad (14)$$

$$\xi^* \geq 0, g_i(z) + \langle g_i'(z), -r_{L,G}(z) \leq 0, \quad (15)$$

Putting $y = x_*$ in (13), and adding (11) and (13) we obtain

$$\langle Gr_{L,G}(z), z - r_{L,G}(z) - x_*\rangle + \langle F(z) - F(x_*), x_* - z + r_{L,G}(z)\rangle +$$

$$\left\langle \sum_{i=1}^{m} \lambda_i^* g_i'(x_*), z - r_{L,G}(z) - x_* \right\rangle + \left\langle \sum_{i=1}^{m} \xi_i^* g_i'(z), x_* - z + r_{L,G}(z) \right\rangle \geq 0. \quad (16)$$

Now, we are going to estimate all terms in (16). First, combining (14) and (15), having in mind the convexity of the functions g_i, we have

$$\left\langle \sum_{i=1}^{m} \xi_i^* g_i'(z), x_* - z + r_{L,G}(z) \right\rangle =$$

$$\left\langle \sum_{i=1}^{m} \xi_i^* g_i'(z), x_* - z \right\rangle + \sum_{i=1}^{m} \xi_i^* g_i(z) \leq \sum_{i=1}^{m} \xi_i^* g_i(x_*) \leq 0 \quad (17)$$

In the similar, combining (11) and the condition of convexity (see [8], Lemma 2.3.1, p. 93), we obtain

$$\left\langle \sum_{i=1}^{m} \lambda_i^* g_i'(x_*), z - r_{L,G}(z) - x_* \right\rangle =$$

$$\left\langle \sum_{i=1}^{m} \lambda_i^* (g_i(x_*) + g_i'(x_*), z - r_{L,G}(z) - x_* \right\rangle \leq \frac{1}{2} \sum_{i=1}^{m} \lambda_i^* L_i \|r_{L,G}(z)\|^2. \quad (18)$$

The second term in (16) can be estimated by

$$\langle Gr_{L,G}(z), z - r_{L,G}(z) - x_*\rangle = \langle Gr_{L,G}(z), -r_{L,G}\rangle + \langle Gr_{L,G}(z), z - x_*\rangle \leq$$

$$-m\|r_{L,G}(z)\|^2 + \frac{M\varepsilon\|r_{L,G}(z)\|^2}{2} + \frac{M\|z - x_*\|^2}{2\varepsilon}, \quad (19)$$

where $\varepsilon > 0$.

Finally, from the conditions (4) and (5), we obtain the following inequality (see [8], p. 181, proof of Theorem 3.4.5.)

$$\langle F(z) - F(x_*), x_* - z - r_{L,G}(z)\rangle \leq \frac{L+\alpha}{4}\|r_{L,G}(z)\|^2 - \frac{L\alpha}{L+\alpha}\|z - x_*\|^2. \quad (20)$$

Now, combining (17) - (20), we get

$$\left(\frac{L\alpha}{L+\alpha}-\frac{M}{2\varepsilon}\right)\|z-x_*\|^2 \leq \left(\frac{L+\alpha}{4}-m+\frac{M\varepsilon}{2}+\sum_{i=1}^{m}\lambda_i^* L_i\right)\|r_{L,G}(z)\|^2. \quad (21)$$

The previous estimates are valid if the parameters are chosen such that

$$\frac{L\alpha}{L+\alpha}-\frac{M}{2\varepsilon}>0, \text{ and } \frac{L+\alpha}{4}-m+\frac{M}{2\varepsilon}+\sum_{i=1}^{m}\lambda_i^* L_i>0.$$

In this case, the statement of Theorem is a consequence of (21). □

Remark 4. Let us observe that an applications of this estimate requires the knowledge of Lagrange multipliers. Sometimes, they have physical or geometrical meaning, and this fact can be useful to get the estimates of their values.

Remark 5. For $C = C_0$, we have new estimates for the distance between any point and the set C_* of the solutions of (1).

3 Projection Measures for Quasi-variational Inequalites

The theorem about existence of solutions show a notable difference between variational and quasi-variational inequalities. For example, if F is strongly monotone on closed and convex set, then variational inequality (1) has a unique solution. On the other hand, in our knowledge the following statement is the best result related to the existence of solutions of quasi-variational inequaliites (2) (see [5]):

If the map F is strongly monotone and Lipschitz continuous with constants $\alpha > 0$, $L \geq 0$, and $C : R^n \mapsto R^n$ is a set-valued mapping with nonempty closed and convex values, such that

$$\|\Pi_{C(x)}(z)-\Pi_{C(y)}(z)\| \leq \gamma\|x-y\|,\ \gamma<\frac{\alpha}{L},\ \forall x,y \text{ and } z, \quad (22)$$

then quasi-variational (2) has a unique solution.

Assuming that the conditions of existence are fulfilled, we will derive the estimates of the closeness of any point to the solution of (2).

Theorem 3. *If the conditions (4), (5) and (22) are satisfied, then*

(i)

$$\|z-x_*\| \leq k_Q\|r_Q(z)\|^2, \quad (23)$$

where

$$r_Q(z)=z-\Pi_{C(z)}(z-\beta F(z)),\ \beta>0,$$

and

$$k_Q=\left(1-\beta\gamma L-\sqrt{1-2\beta\gamma+L^2\beta^2}\right)^{-1}$$

(ii)

$$\|z-x_*\| \leq k_T\frac{\alpha}{\alpha-\gamma L}\|r_{T(z)}(z)\|, \quad (24)$$

where $T(z) \in C(z)$,

$$\langle F(T(z)), y - T(z)\rangle \geq 0 \ \forall y \in C(z),$$

$$r_{T(z)}(z) = z - \Pi_{C(z)}(z - \beta F(z)),$$

and k_T *is the constant* k_1 *from Theorem 1 for the variational inequality (1) on the set* $C(z)$.

Proof. (i) We suppose that the parameter $\beta > 0$ is sufficiently small, such that a constant k_Q is positive.

Let us start with notice that x_* is a solution of $(refQVI)$ if and only if $x_* = r_Q(x_*)$. Then, using (22) and (5), we have

$$\begin{gathered}\|z - x_*\| = \|r(z) + \Pi_{C(z)}(z - \beta F(z)) - \Pi_{C(x_*)}(x_* - \beta F(x_*))\| \leq \\ \|r(z)\| + \|\Pi_{C(z)}(z - \beta F(z)) - \Pi_{C(x_*)}(z - \beta F(z))\| + \\ \|\Pi_{C(x_*)}(z - \beta F(z)) - \Pi_{C(x_*)}(x_* - \beta F(x_*))\| \leq \\ \|r(z)\| + \beta\gamma L\|z - x_*\| + \|\Pi_{C(x_*)}(z - \beta F(z)) - \Pi_{C(x_*)}(x_* - \beta F(x_*))\|. \qquad (25)\end{gathered}$$

Since F is strongly monotone and Lipschitz continuous, we have

$$\begin{gathered}\|\Pi_{C(x_*)}(z - \beta F(z)) - \Pi_{C(x_*)}(x_* - \beta F(x_*))\|^2 \leq \\ \|(z - x_*) - \beta(F(z)) - F(x_*))\|^2 = \\ \|z - x_*\|^2 - 2\beta\langle F(z)) - F(x_*), z - x_*\rangle + \beta^2\|F(z)) - F(x_*)\|^2 \leq \\ (1 - 2\beta L + \beta^2 L^2)\|z - x_*\|^2.\end{gathered}$$

Now, from (25), we obtain (23).

(ii) In [5] has been proved that the operator $T : H \mapsto H$ defined by conditions

$$\langle F(T(z)), y - T(z)\rangle \geq 0, \ \forall y \in C(z)$$

is a contraction with modulus of contraction $\gamma\frac{L}{\alpha} < 1$. Furthermore, x_* is a solution of (2) if and only if $x_* = T(x_*)$. Consequently,

$$\|z - x_*\| = \|z - T(x_*)\| \leq \|z - T(z)\| + \|T(z) - T(x_*)\| \leq \|z - T(z)\| + \frac{\gamma L}{\alpha}\|z - x_*\|.$$

Finally, since $T(z)$ is a unique solution of (1) on the set $C(z)$, we have

$$\|z - x_*\| \leq \left(1 - \frac{\gamma L}{\alpha}\right)\|z - T(z)\| \leq k_T\|r_{T(z)}\|,$$

from where estimate (24) follows. □

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