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STRONG CONVERGENCE THEOREMS OF THREE-STEP ITERATIONS FOR MULTI-VALUED MAPPINGS IN BANACH SPACES

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Abstract

In this paper, two classes of three-step iteration schemes for multi-valued mappings in a uniformly convex Banach space are presented. Moreover, their strong convergence are proved.

1 Introduction

Iterative methods for approximating fixed points of multi-valued mappings in Banach spaces have been studied by some authors [2, 3, 6, 7]. In 2009, Shahzad and Zegeye [4] improved the results of [3, 6]. They introduced the following iteration scheme.

Let D be a nonempty convex subset of a Banach space X, $T : D \to CB(D)$ a quasi-nonexpansive multi-valued mapping and $\alpha_n, \beta_n \in [0, 1]$. The sequence of Ishikawa iterates is defined by $x_1 \in D$,

$$\begin{aligned} y_n &= \beta_n z_n + (1-\beta_n)x_n, \quad n \geq 1, \\ x_{n+1} &= \alpha_n z'_n + (1-\alpha_n)x_n, \quad n \geq 1, \end{aligned} \tag{1}$$

where $z_n \in T(x_n)$ and $z'_n \in T(y_n)$.

Next, they introduced the following iteration scheme.

Let D be a nonempty convex subset of a Banach space X, $T : D \to P(D)$ a multi-valued mapping and $P_T(x) = \{y \in T(x); \|x - y\| = d(x, T(x))\}$ and $\alpha_n, \beta_n \in [0, 1]$. The sequence of Ishikawa iterates is defined by $x_1 \in D$,

$$\begin{aligned} y_n &= \beta_n z_n + (1-\beta_n)x_n, \quad n \geq 1, \\ x_{n+1} &= \alpha_n z'_n + (1-\alpha_n)x_n, \quad n \geq 1, \end{aligned} \tag{2}$$

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where $z_n \in P_T(x_n)$ and $z'_n \in P_T(y_n)$.
They proved strong convergence theorems for these iteration schemes in uniformly convex Banach spaces. Inspired and motivated by the above facts, we introduce the following iteration scheme.
Let D be a nonempty convex subset of a Banach space X and $T : D \to CB(D)$ a multi-valued mapping. For a given $x_1 \in D$, compute the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ by the iteration scheme

$$\begin{aligned} z_n &= a_n w_n^{(1)} + (1 - a_n)x_n, & n \geq 1, \\ y_n &= b_n w_n^{(2)} + c_n w_n^{(3)} + (1 - b_n - c_n)x_n, & n \geq 1, \\ x_{n+1} &= \alpha_n w_n^{(4)} + \beta_n w_n^{(5)} + (1 - \alpha_n - \beta_n)x_n, & n \geq 1, \end{aligned} \tag{3}$$

where $w_n^{(1)}, w_n^{(3)} \in T(x_n), w_n^{(2)}, w_n^{(5)} \in T(z_n), w_n^{(4)} \in T(y_n)$ and $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}$ and $\{\beta_n\}$ are appropriate sequences in [0,1].
If $a_n = c_n = \beta_n = 0$, then (3) reduces to (1). Next, we introduce the following iteration scheme.
Let D be a nonempty convex subset of a Banach space X and $T : D \to P(D)$ a multi-valued mapping. For a given $x_1 \in D$, compute the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ by the iteration scheme

$$\begin{aligned} z_n &= a_n w_n^{(1)} + (1 - a_n)x_n, & n \geq 1, \\ y_n &= b_n w_n^{(2)} + c_n w_n^{(3)} + (1 - b_n - c_n)x_n, & n \geq 1, \\ x_{n+1} &= \alpha_n w_n^{(4)} + \beta_n w_n^{(5)} + (1 - \alpha_n - \beta_n)x_n, & n \geq 1, \end{aligned} \tag{4}$$

where $w_n^{(1)}, w_n^{(3)} \in P_T(x_n), w_n^{(2)}, w_n^{(5)} \in P_T(z_n), w_n^{(4)} \in P_T(y_n)$ and $\{a_n\}, \{b_n\}, \{c_n\}$, $\{\alpha_n\}$ and $\{\beta_n\}$ are appropriate sequences in [0,1].
If $a_n = c_n = \beta_n = 0$, then (4) reduces to (2). In this paper, we prove strong convergence theorems for these new iteration schemes.
Let D be a nonempty subset of a Banach space X. A single-valued mapping $T : D \to D$ is called nonexpansive if $\|T(x) - T(y)\| \leq \|x - y\|$ for $x, y \in D$. Let $CB(D)$ denotes the family of nonempty, closed and bounded subsets of D. The set D is called proximinal if for each $x \in X$, there exists an element $y \in D$ such that $\|x - y\| = d(x, D)$, let P(D) denotes nonempty, proximinal and bounded subsets of D. The Hausdorff metric on $CB(D)$ is defined by

$$H(A_1, A_2) = \max\left\{\sup_{x \in A_1} d(x, A_2), \sup_{y \in A_2} d(y, A_1)\right\},$$

for $A_1, A_2 \in CB(D)$, where $d(x, A_1) = \inf\{\|x - y\|; y \in A_1\}$.
The multi-valued mapping $T : D \to CB(D)$ is called nonexpansive if

$$H(T(x), T(y)) \leq \|x - y\| \quad \forall x, y \in D.$$

An element $p \in D$ is called a fixed point of $T : D \to D$(respectively, $T : D \to CB(D)$) if $p = T(p)$ (respectively, $p \in T(p)$). The set of fixed points of T is represented by $F(T)$. The multi-valued mapping $T : D \to CB(D)$ is called quasi-nonexpansive [5] if $F(T) \neq \emptyset$ and $H(T(x), T(p)) \leq \|x - p\|$ for all $x \in D$ and all $p \in F(T)$. It is clear that every nonexpansive multi-valued map T with $F(T) \neq \emptyset$ is quasi-nonexpansive. But there exist quasi-nonexpansive mappings that are not nonexpansive.

Example 1. Let $D = [0, \infty)$ with the usual metric and $T : D \to CB(D)$ be defined by

$$T(x) = \begin{cases} \{0\}, & x \leq 1; \\ [x - \frac{3}{4}, x - \frac{1}{3}], & x > 1. \end{cases}$$

Then, clearly $F(T) = \{0\}$ and $H(T(x), T(0)) \leq |x - 0|$ for all x, hence T is quasi-nonexpansive. However, if $x = 2, y = 1$ we get that $H(T(x), T(y)) > |x - y| = 1$, hence T is not nonexpansive.

The mapping $T : D \to CB(D)$ with $F = F(T) \neq \emptyset$ satisfies condition (I) if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for $r \in (0, \infty)$ such that

$$d(x, T(x)) \geq f(d(x, F)) \quad \text{for all} \quad x \in D.$$

2 Main results

We use the following lemma to prove our main results.

Lemma 1. *[1] Let X be a uniformly convex Banach space and $B_r = \{x \in X : \|x\| \leq r\}, r > 0$. Then there exists a continuous, strictly increasing and convex function $g : [0, \infty) \to [0, \infty), g(0) = 0$ such that*

$$\|\alpha x + \beta y + \gamma z\|^2 \leq \alpha \|x\|^2 + \beta \|y\|^2 + \gamma \|z\|^2 - \alpha\beta g(\|x - y\|),$$

for all $x, y, z \in B_r$ and all $\alpha, \beta, \gamma \in [0, 1]$ with $\alpha + \beta + \gamma = 1$.

Lemma 2. *Let X be a uniformly convex Banach space and D a nonempty, closed and convex subset of X. Suppose $T : D \to CB(D)$ is a nonexpansive multi-valued mapping with $F = F(T) \neq \emptyset$ and $T(q) = \{q\}$ for each $q \in F$. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0, 1]$, such that $b_n + c_n$ and $\alpha_n + \beta_n$ are in $[0, 1]$ for all $n \geq 1$. For a given $x_1 \in D$, let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be the sequences defined as in (3), then $\lim_{n\to\infty} \|x_n - q\|$ exists for each $q \in F$.*

Proof. Let $q \in F$. We have

$$\begin{aligned} \|z_n - q\| &= \|a_n(w_n^{(1)} - q) + (1 - a_n)(x_n - q)\| \\ &\leq a_n\|w_n^{(1)} - q\| + (1 - a_n)\|x_n - q\| \\ &= a_n d(w_n^{(1)}, T(q)) + (1 - a_n)\|x_n - q\| \\ &\leq a_n H(T(x_n), T(q)) + (1 - a_n)\|x_n - q\| \\ &\leq a_n\|x_n - q\| + (1 - a_n)\|x_n - q\| \\ &= \|x_n - q\|, \end{aligned}$$

and

$$\begin{aligned}
\|y_n - q\| &= \|b_n(w_n^{(2)} - q) + c_n(w_n^{(3)} - q) + (1 - b_n - c_n)(x_n - q)\| \\
&\leq b_n\|w_n^{(2)} - q\| + c_n\|w_n^{(3)} - q\| + (1 - b_n - c_n)\|x_n - q\| \\
&= b_n d(w_n^{(2)}, T(q)) + c_n d(w_n^{(3)}, T(q)) + (1 - b_n - c_n)\|x_n - q\| \\
&\leq b_n H(T(z_n), T(q)) + c_n H(T(x_n), T(q)) + (1 - b_n - c_n)\|x_n - q\| \\
&\leq b_n\|z_n - q\| + c_n\|x_n - q\| + (1 - b_n - c_n)\|x_n - q\| \\
&\leq b_n\|x_n - q\| + c_n\|x_n - q\| + (1 - b_n - c_n)\|x_n - q\| \\
&= \|x_n - q\|.
\end{aligned}$$

Therefore

$$\begin{aligned}
\|x_{n+1} - q\| &= \|\alpha_n(w_n^{(4)} - q) + \beta_n(w_n^{(5)} - q) + (1 - \alpha_n - \beta_n)(x_n - q)\| \\
&\leq \alpha_n\|w_n^{(4)} - q\| + \beta_n\|w_n^{(5)} - q\| + (1 - \alpha_n - \beta_n)\|x_n - q\| \\
&= \alpha_n d(w_n^{(4)}, T(q)) + \beta_n d(w_n^{(5)}, T(q)) + (1 - \alpha_n - \beta_n)\|x_n - q\| \\
&\leq \alpha_n H(T(y_n), T(q)) + \beta_n H(T(z_n), T(q)) \\
&\quad + (1 - \alpha_n - \beta_n)\|x_n - q\| \\
&\leq \alpha_n\|y_n - q\| + \beta_n\|z_n - q\| + (1 - \alpha_n - \beta_n)\|x_n - q\| \\
&\leq \alpha_n\|x_n - q\| + \beta_n\|x_n - q\| + (1 - \alpha_n - \beta_n)\|x_n - q\| \\
&= \|x_n - q\|.
\end{aligned} \tag{5}$$

Hence $\{\|x_n - q\|\}$ is a nonincreasing sequence, so $\lim_{n\to\infty} \|x_n - q\|$ exists for any $q \in F(T)$. Also $\{x_n\}$ is bounded. □

Theorem 1. *Let X be a uniformly convex Banach space and D a nonempty, closed and convex subset of X. Suppose $T : D \to CB(D)$ is a nonexpansive multi-valued mapping with $F = F(T) \neq \emptyset$ and $T(q) = \{q\}$ for each $q \in F$. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0, 1]$, such that $b_n + c_n$ and $\alpha_n + \beta_n$ are in $[0, 1]$ for all $n \geq 1$ and $\lim_{n\to\infty} c_n = 0$. For a given $x_1 \in D$, let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be the sequences defined as in (3). Suppose T satisfies condition (I). If $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty}(b_n + c_n) < 1$ and $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$, then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. Let $q \in F(T)$. Then, as in the proof of Lemma 2, $\{x_n\}, \{y_n\}$ and $\{z_n\}$ are bounded. Therefore, there exists $R > 0$ such that $x_n - q, y_n - q, z_n - q \in B_R(0)$ for all $n \geq 1$. Applying Lemma 1, there is a continuous, strictly increasing and convex function $g : [0, \infty) \to [0, \infty), g(0) = 0$, such that

$$\begin{aligned}
\|y_n - q\|^2 &= \|b_n(w_n^{(2)} - q) + c_n(w_n^{(3)} - q) + (1 - b_n - c_n)(x_n - q)\|^2 \\
&\leq b_n\|w_n^{(2)} - q\|^2 + c_n\|w_n^{(3)} - q\|^2 + (1 - b_n - c_n)\|x_n - q\|^2 \\
&\quad -b_n(1 - b_n - c_n)g(\|w_n^{(2)} - x_n\|) \\
&= b_n(d(w_n^{(2)}, T(q)))^2 + c_n(d(w_n^{(3)}, T(q)))^2 + (1 - b_n - c_n)\|x_n - q\|^2 \\
&\quad -b_n(1 - b_n - c_n)g(\|w_n^{(2)} - x_n\|) \\
&\leq b_n(H(T(z_n), T(q)))^2 + c_n(H(T(x_n), T(q)))^2 \\
&\quad +(1 - b_n - c_n)\|x_n - q\|^2 - b_n(1 - b_n - c_n)g(\|w_n^{(2)} - x_n\|) \\
&\leq b_n\|z_n - q\|^2 + c_n\|x_n - q\|^2 + (1 - b_n - c_n)\|x_n - q\|^2 \\
&\quad -b_n(1 - b_n - c_n)g(\|w_n^{(2)} - x_n\|).
\end{aligned} \tag{6}$$

Therefore

$$\begin{aligned}
\|x_{n+1} - q\|^2 &= \|\alpha_n(w_n^{(4)} - q) + \beta_n(w_n^{(5)} - q) + (1 - \alpha_n - \beta_n)(x_n - q)\|^2 \\
&\leq \alpha_n\|w_n^{(4)} - q\|^2 + \beta_n\|w_n^{(5)} - q\|^2 + (1 - \alpha_n - \beta_n)\|x_n - q\|^2 \\
&\quad -\alpha_n(1 - \alpha_n - \beta_n)g(\|w_n^{(4)} - x_n\|) \\
&= \alpha_n(d(w_n^{(4)}, T(q)))^2 + \beta_n(d(w_n^{(5)}, T(q)))^2 \\
&\quad +(1 - \alpha_n - \beta_n)\|x_n - q\|^2 - \alpha_n(1 - \alpha_n - \beta_n)g(\|w_n^{(4)} - x_n\|) \\
&\leq \alpha_n(H(T(y_n), T(q)))^2 + \beta_n(H(T(z_n), T(q)))^2 \\
&\quad +(1 - \alpha_n - \beta_n)\|x_n - q\|^2 - \alpha_n(1 - \alpha_n - \beta_n)g(\|w_n^{(4)} - x_n\|) \\
&\leq \alpha_n\|y_n - q\|^2 + \beta_n\|z_n - q\|^2 + (1 - \alpha_n - \beta_n)\|x_n - q\|^2 \\
&\quad -\alpha_n(1 - \alpha_n - \beta_n)g(\|w_n^{(4)} - x_n\|) \\
&\leq \alpha_n(b_n\|z_n - q\|^2 + c_n\|x_n - q\|^2 + (1 - b_n - c_n)\|x_n - q\|^2 \\
&\quad -b_n(1 - b_n - c_n)g(\|w_n^{(2)} - x_n\|)) + \beta_n\|z_n - q\|^2 \\
&\quad +(1 - \alpha_n - \beta_n)\|x_n - q\|^2 - \alpha_n(1 - \alpha_n - \beta_n)g(\|w_n^{(4)} - x_n\|) \\
&\leq \alpha_n(b_n\|x_n - q\|^2 + c_n\|x_n - q\|^2 + (1 - b_n - c_n)\|x_n - q\|^2 \\
&\quad -b_n(1 - b_n - c_n)g(\|w_n^{(2)} - x_n\|)) + \beta_n\|x_n - q\|^2 \\
&\quad +(1 - \alpha_n - \beta_n)\|x_n - q\|^2 \\
&\quad -\alpha_n(1 - \alpha_n - \beta_n)g(\|w_n^{(4)} - x_n\|) \\
&= \|x_n - q\|^2 - \alpha_n b_n(1 - b_n - c_n)g(\|w_n^{(2)} - x_n\|) \\
&\quad -\alpha_n(1 - \alpha_n - \beta_n)g(\|w_n^{(4)} - x_n\|).
\end{aligned} \tag{7}$$

Hence

$$\|x_{n+1} - q\|^2 \leq \|x_n - q\|^2 - \alpha_n b_n(1 - b_n - c_n)g(\|w_n^{(2)} - x_n\|). \tag{8}$$

Since $\liminf_{n\to\infty} \alpha_n > 0$ and $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty}(b_n + c_n) < 1$, there exist positive integers $r_1, r_2 \in (0, 1)$ and $N_1 > 0$ such that

$$0 < r_1 < b_n, 0 < r_1 < \alpha_n \quad \text{and} \quad b_n + c_n < r_2 < 1, \quad \forall n \geq N_1.$$

By (8) we obtain

$$\begin{aligned} {r_1}^2(1-r_2)\Sigma_{n=N_1}^{\infty} g(\|w_n^{(2)}-x_n\|) &< \Sigma_{n=N_1}^{\infty} \alpha_n b_n(1-b_n-c_n) g(\|w_n^{(2)}-x_n\|) \\ &\leq \Sigma_{n=N_1}^{\infty}(\|x_n-q\|^2-\|x_{n+1}-q\|^2) \\ &< \infty. \end{aligned} \tag{9}$$

Therefore $\lim_{n\to\infty} g(\|w_n^{(2)}-x_n\|)=0$ and hence

$$\lim_{n\to\infty}\|w_n^{(2)}-x_n\|=0. \tag{10}$$

In addition, by (7) we obtain

$$\begin{aligned} \|x_{n+1}-q\|^2 &\leq \|x_n-q\|^2-\alpha_n b_n(1-b_n-c_n) g(\|w_n^{(2)}-x_n\|) \\ &\quad -\alpha_n(1-\alpha_n-\beta_n) g(\|w_n^{(4)}-x_n\|) \\ &\leq \|x_n-q\|^2-\alpha_n(1-\alpha_n-\beta_n) g(\|w_n^{(4)}-x_n\|). \end{aligned} \tag{11}$$

Since $0<\liminf_{n\to\infty}\alpha_n \leq \limsup_{n\to\infty}(\alpha_n+\beta_n)<1$, there exist positive integers $r_3, r_4 \in (0,1)$ and $N_2>0$ such that

$$0<r_3<\alpha_n \quad \text{and} \quad \alpha_n+\beta_n<r_4<1, \quad \forall n \geq N_2.$$

By (11) we obtain

$$\begin{aligned} r_3(1-r_4)\Sigma_{n=N_2}^{\infty} g(\|w_n^{(4)}-x_n\|) &< \Sigma_{n=N_2}^{\infty} \alpha_n(1-\alpha_n-\beta_n) g(\|w_n^{(4)}-x_n\|) \\ &\leq \Sigma_{n=N_2}^{\infty}(\|x_n-q\|^2-\|x_{n+1}-q\|^2) \\ &< \infty. \end{aligned} \tag{12}$$

Therefore $\lim_{n\to\infty} g(\|w_n^{(4)}-x_n\|)=0$ and hence

$$\lim_{n\to\infty}\|w_n^{(4)}-x_n\|=0. \tag{13}$$

It is a simple consequence of the definition of $H(T(x_n),T(y_n))$ that for each positive integer $n \geq 1$, there exists $\hat{w}_n \in T(x_n)$ such that

$$\|\hat{w}_n-w_n^{(4)}\| \leq H(T(x_n),T(y_n))+\frac{1}{n}, \tag{14}$$

Next, consider

$$\begin{aligned} \|w_n^{(3)}-x_n\| &\leq \|w_n^{(3)}-q\|+\|x_n-q\| \\ &= d(w_n^{(3)},T(q))+\|x_n-q\| \\ &\leq H(T(x_n),T(q))+\|x_n-q\| \\ &\leq \|x_n-q\|+\|x_n-q\| \\ &= 2\|x_n-q\|. \end{aligned} \tag{15}$$

Since $\{\|x_n - q\|\}$ is bounded, we have $\{\|w_n^{(3)} - x_n\|\}$ is a bounded sequence.
By (14) we obtain

$$\begin{aligned}\|\hat{w}_n - x_n\| &\le \|\hat{w}_n - w_n^{(4)}\| + \|w_n^{(4)} - x_n\| \\ &\le H(T(x_n), T(y_n)) + \tfrac{1}{n} + \|w_n^{(4)} - x_n\| \\ &\le \|x_n - y_n\| + \tfrac{1}{n} + \|w_n^{(4)} - x_n\| \\ &\le b_n\|w_n^{(2)} - x_n\| + c_n\|w_n^{(3)} - x_n\| + \tfrac{1}{n} + \|w_n^{(4)} - x_n\|.\end{aligned}$$

As, $n \to \infty$ in the above inequality, by (10) and (13) we obtain $\lim_{n\to\infty} \|\hat{w}_n - x_n\| = 0$. Since $d(x_n, T(x_n)) \le \|\hat{w}_n - x_n\|$, it follows that $\lim_{n\to\infty} d(x_n, T(x_n)) = 0$. Since T satisfies condition (I), we have $\lim_{n\to\infty} d(x_n, F) = 0$. Thus there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\|x_{n_k} - p_k\| < \frac{1}{2^k}$ for some $\{p_k\} \subset F$ and all k.
Since

$$\|x_{n_{k+1}} - p_k\| \le \|x_{n_k} - p_k\| < \frac{1}{2^k},$$

we get

$$\begin{aligned}\|p_{k+1} - p_k\| &\le \|p_{k+1} - x_{n_{k+1}}\| + \|x_{n_{k+1}} - p_k\| \\ &< \tfrac{1}{2^{k+1}} + \tfrac{1}{2^k} \\ &< \tfrac{1}{2^{k-1}}.\end{aligned}$$

Therefore $\{p_k\}$ is a Cauchy sequence in D and thus converges to $q \in D$.
Since

$$\begin{aligned}d(p_k, T(q)) &\le H(T(p_k), T(q)) \\ &\le \|p_k - q\|,\end{aligned}$$

and $p_k \to q$ as $k \to \infty$, we have $d(q, T(q)) = 0$ and thus $q \in F$. Since $\{x_{n_k}\}$ converges strongly to q and $\lim_{n\to\infty} \|x_n - q\|$ exists, we have $\{x_n\}$ converges strongly to q. □

Theorem 2. *Let X be a uniformly convex Banach space and D a nonempty, closed and convex subset of X. Suppose $T : D \to P(D)$ is a multi-valued mapping with $F = F(T) \ne \emptyset$ such that P_T is nonexpansive. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0, 1]$, such that $b_n + c_n$ and $\alpha_n + \beta_n$ are in $[0, 1]$ for all $n \ge 1$ and $\lim_{n\to\infty} c_n = 0$. For a given $x_1 \in D$, let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be the sequences defined as in (4). Suppose T satisfies condition (I). If $0 < \liminf_{n\to\infty} b_n \le \limsup_{n\to\infty}(b_n + c_n) < 1$ and $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$, then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. Let $q \in F$. Then $P_T(q) = \{q\}$, We have

$$\begin{aligned}\|z_n - q\| &= \|a_n(w_n^{(1)} - q) + (1 - a_n)(x_n - q)\| \\ &\le a_n\|w_n^{(1)} - q\| + (1 - a_n)\|x_n - q\| \\ &= a_n d(w_n^{(1)}, P_T(q)) + (1 - a_n)\|x_n - q\| \\ &\le a_n H(P_T(x_n), P_T(q)) + (1 - a_n)\|x_n - q\| \\ &\le a_n\|x_n - q\| + (1 - a_n)\|x_n - q\| \\ &= \|x_n - q\|,\end{aligned}$$

and

$$\begin{aligned}\|y_n - q\| &= \|b_n(w_n^{(2)} - q) + c_n(w_n^{(3)} - q) + (1 - b_n - c_n)(x_n - q)\| \\ &\leq b_n\|w_n^{(2)} - q\| + c_n\|w_n^{(3)} - q\| + (1 - b_n - c_n)\|x_n - q\| \\ &= b_n d(w_n^{(2)}, P_T(q)) + c_n d(w_n^{(3)}, P_T(q)) + (1 - b_n - c_n)\|x_n - q\| \\ &\leq b_n H(P_T(z_n), P_T(q)) + c_n H(P_T(x_n), P_T(q)) \\ &\quad +(1 - b_n - c_n)\|x_n - q\| \\ &\leq b_n\|z_n - q\| + c_n\|x_n - q\| + (1 - b_n - c_n)\|x_n - q\| \\ &\leq b_n\|x_n - q\| + c_n\|x_n - q\| + (1 - b_n - c_n)\|x_n - q\| \\ &= \|x_n - q\|.\end{aligned}$$

Therefore

$$\begin{aligned}\|x_{n+1} - q\| &= \|\alpha_n(w_n^{(4)} - q) + \beta_n(w_n^{(5)} - q) + (1 - \alpha_n - \beta_n)(x_n - q)\| \\ &\leq \alpha_n\|w_n^{(4)} - q\| + \beta_n\|w_n^{(5)} - q\| + (1 - \alpha_n - \beta_n)\|x_n - q\| \\ &= \alpha_n d(w_n^{(4)}, P_T(q)) + \beta_n d(w_n^{(5)}, P_T(q)) + (1 - \alpha_n - \beta_n)\|x_n - q\| \\ &\leq \alpha_n H(P_T(y_n), P_T(q)) + \beta_n H(P_T(z_n), P_T(q)) \\ &\quad +(1 - \alpha_n - \beta_n)\|x_n - q\| \\ &\leq \alpha_n\|y_n - q\| + \beta_n\|z_n - q\| + (1 - \alpha_n - \beta_n)\|x_n - q\| \\ &\leq \alpha_n\|x_n - q\| + \beta_n\|x_n - q\| + (1 - \alpha_n - \beta_n)\|x_n - q\| \\ &= \|x_n - q\|.\end{aligned} \tag{16}$$

Hence $\{\|x_n - q\|\}$ is a nonincreasing sequence. Therefore, $\lim_{n\to\infty} \|x_n - q\|$ exists for each $q \in F(T)$. So $\{x_n\}$ is bounded. Also $\{y_n\}$ and $\{z_n\}$ are bounded. Therefore, there exists $R > 0$ such that $x_n - q, y_n - q, z_n - q \in B_R(0)$ for all $n \geq 1$. By Lemma 1, there is a continuous, strictly increasing and convex function $g : [0, \infty) \to [0, \infty), g(0) = 0$, such that

$$\begin{aligned}\|y_n - q\|^2 &= \|b_n(w_n^{(2)} - q) + c_n(w_n^{(3)} - q) + (1 - b_n - c_n)(x_n - q)\|^2 \\ &\leq b_n\|w_n^{(2)} - q\|^2 + c_n\|w_n^{(3)} - q\|^2 + (1 - b_n - c_n)\|x_n - q\|^2 \\ &\quad -b_n(1 - b_n - c_n)g(\|w_n^{(2)} - x_n\|) \\ &= b_n(d(w_n^{(2)}, P_T(q)))^2 + c_n(d(w_n^{(3)}, P_T(q)))^2 + (1 - b_n - c_n)\|x_n - q\|^2 \\ &\quad -b_n(1 - b_n - c_n)g(\|w_n^{(2)} - x_n\|) \\ &\leq b_n(H(P_T(z_n), P_T(q)))^2 + c_n(H(P_T(x_n), P_T(q)))^2 \\ &\quad +(1 - b_n - c_n)\|x_n - q\|^2 \\ &\quad -b_n(1 - b_n - c_n)g(\|w_n^{(2)} - x_n\|) \\ &\leq b_n\|z_n - q\|^2 + c_n\|x_n - q\|^2 + (1 - b_n - c_n)\|x_n - q\|^2 \\ &\quad -b_n(1 - b_n - c_n)g(\|w_n^{(2)} - x_n\|).\end{aligned} \tag{17}$$

Therefore

$$\begin{aligned}
\|x_{n+1}-q\|^2 &= \|\alpha_n(w_n^{(4)}-q)+\beta_n(w_n^{(5)}-q)+(1-\alpha_n-\beta_n)(x_n-q)\|^2 \\
&\le \alpha_n\|w_n^{(4)}-q\|^2+\beta_n\|w_n^{(5)}-q\|^2+(1-\alpha_n-\beta_n)\|x_n-q\|^2 \\
&\quad -\alpha_n(1-\alpha_n-\beta_n)g(\|w_n^{(4)}-x_n\|) \\
&= \alpha_n(d(w_n^{(4)},P_T(q)))^2+\beta_n(d(w_n^{(5)},P_T(q)))^2 \\
&\quad +(1-\alpha_n-\beta_n)\|x_n-q\|^2 \\
&\quad -\alpha_n(1-\alpha_n-\beta_n)g(\|w_n^{(4)}-x_n\|) \\
&\le \alpha_n(H(P_T(y_n),P_T(q)))^2+\beta_n(H(P_T(z_n),P_T(q)))^2 \\
&\quad +(1-\alpha_n-\beta_n)\|x_n-q\|^2-\alpha_n(1-\alpha_n-\beta_n)g(\|w_n^{(4)}-x_n\|) \\
&\le \alpha_n\|y_n-q\|^2+\beta_n\|z_n-q\|^2+(1-\alpha_n-\beta_n)\|x_n-q\|^2 \\
&\quad -\alpha_n(1-\alpha_n-\beta_n)g(\|w_n^{(4)}-x_n\|) \\
&\le \alpha_n(b_n\|z_n-q\|^2+c_n\|x_n-q\|^2+(1-b_n-c_n)\|x_n-q\|^2 \\
&\quad -b_n(1-b_n-c_n)g(\|w_n^{(2)}-x_n\|))+\beta_n\|z_n-q\|^2 \\
&\quad +(1-\alpha_n-\beta_n)\|x_n-q\|^2-\alpha_n(1-\alpha_n-\beta_n)g(\|w_n^{(4)}-x_n\|) \\
&\le \alpha_n(b_n\|x_n-q\|^2+c_n\|x_n-q\|^2+(1-b_n-c_n)\|x_n-q\|^2 \\
&\quad -b_n(1-b_n-c_n)g(\|w_n^{(2)}-x_n\|))+\beta_n\|x_n-q\|^2 \\
&\quad +(1-\alpha_n-\beta_n)\|x_n-q\|^2-\alpha_n(1-\alpha_n-\beta_n)g(\|w_n^{(4)}-x_n\|) \\
&= \|x_n-q\|^2-\alpha_nb_n(1-b_n-c_n)g(\|w_n^{(2)}-x_n\|) \\
&\quad -\alpha_n(1-\alpha_n-\beta_n)g(\|w_n^{(4)}-x_n\|).
\end{aligned} \tag{18}$$

Hence

$$\begin{aligned}
\|x_{n+1}-q\|^2 &\le \|x_n-q\|^2-\alpha_nb_n(1-b_n-c_n)g(\|w_n^{(2)}-x_n\|) \\
&\quad -\alpha_n(1-\alpha_n-\beta_n)g(\|w_n^{(4)}-x_n\|) \\
&\le \|x_n-q\|^2-\alpha_nb_n(1-b_n-c_n)g(\|w_n^{(2)}-x_n\|),
\end{aligned} \tag{19}$$

Since $\liminf_{n\to\infty}\alpha_n>0$ and $0<\liminf_{n\to\infty}b_n\le\limsup_{n\to\infty}(b_n+c_n)<1$, there exist positive integers $l_1,l_2\in(0,1)$ and $N_1>0$ such that

$$0<l_1<b_n, 0<l_1<\alpha_n \quad\text{and}\quad b_n+c_n<l_2<1, \quad \forall n\ge N_1.$$

By (19) we obtain

$$\begin{aligned}
{l_1}^2(1-l_2)\Sigma_{n=N_1}^{\infty}g(\|w_n^{(2)}-x_n\|) &< \Sigma_{n=N_1}^{\infty}\alpha_nb_n(1-b_n-c_n)g(\|w_n^{(2)}-x_n\|) \\
&\le \Sigma_{n=N_1}^{\infty}(\|x_n-q\|^2-\|x_{n+1}-q\|^2) \\
&< \infty.
\end{aligned} \tag{20}$$

Therefore $\lim_{n\to\infty}g(\|w_n^{(2)}-x_n\|)=0$ and hence

$$\lim_{n\to\infty}\|w_n^{(2)}-x_n\|=0. \tag{21}$$

In addition, by (18) we obtain

$$\begin{aligned}\|x_{n+1}-q\|^2 &\le \|x_n-q\|^2-\alpha_n b_n(1-b_n-c_n)g(\|w_n^{(2)}-x_n\|)\\ &\quad -\alpha_n(1-\alpha_n-\beta_n)g(\|w_n^{(4)}-x_n\|)\\ &\le \|x_n-q\|^2-\alpha_n(1-\alpha_n-\beta_n)g(\|w_n^{(4)}-x_n\|).\end{aligned}\tag{22}$$

If $0<\liminf_{n\to\infty}\alpha_n\le\limsup_{n\to\infty}(\alpha_n+\beta_n)<1$, there exist positive integers $l_3,l_4\in(0,1)$ and $N_2>0$ such that

$$0<l_3<\alpha_n \quad\text{and}\quad \alpha_n+\beta_n<l_4<1,\quad \forall n\ge N_2.$$

By (22) we obtain

$$\begin{aligned}l_3(1-l_4)\Sigma_{n=N_2}^{\infty}g(\|w_n^{(4)}-x_n\|) &< \Sigma_{n=N_2}^{\infty}\alpha_n(1-\alpha_n-\beta_n)g(\|w_n^{(4)}-x_n\|)\\ &\le \Sigma_{n=N_2}^{\infty}(\|x_n-q\|^2-\|x_{n+1}-q\|^2)\\ &<\infty.\end{aligned}\tag{23}$$

Therefore $\lim_{n\to\infty}g(\|w_n^{(4)}-x_n\|)=0$ and hence

$$\lim_{n\to\infty}\|w_n^{(4)}-x_n\|=0.\tag{24}$$

In addition

$$\begin{aligned}\|w_n^{(3)}-x_n\| &\le \|w_n^{(3)}-q\|+\|x_n-q\|\\ &= d(w_n^{(3)},P_T(q))+\|x_n-q\|\\ &\le H(P_T(x_n),P_T(q))+\|x_n-q\|\\ &\le \|x_n-q\|+\|x_n-q\|\\ &= 2\|x_n-q\|.\end{aligned}\tag{25}$$

Since $\{\|x_n-q\|\}$ is bounded, we have $\{\|w_n^{(3)}-x_n\|\}$ is a bounded sequence.
It is a simple consequence of the definition of $H(P_T(x_n),P_T(y_n))$ that for each positive integer $n\ge 1$, there exists $\hat{w}_n\in P_T(x_n)$ such that

$$\|\hat{w}_n-w_n^{(4)}\|\le H(P_T(x_n),P_T(y_n))+\frac{1}{n}.$$

Next, consider

$$\begin{aligned}\|\hat{w}_n-x_n\| &\le \|\hat{w}_n-w_n^{(4)}\|+\|w_n^{(4)}-x_n\|\\ &\le H(P_T(x_n),P_T(y_n))+\tfrac{1}{n}+\|w_n^{(4)}-x_n\|\\ &\le \|x_n-y_n\|+\tfrac{1}{n}+\|w_n^{(4)}-x_n\|\\ &\le b_n\|w_n^{(2)}-x_n\|+c_n\|w_n^{(3)}-x_n\|+\tfrac{1}{n}+\|w_n^{(4)}-x_n\|.\end{aligned}$$

As, $n\to\infty$ in the above inequality, by (21) and (24) we obtain $\lim_{n\to\infty}\|\hat{w}_n-x_n\|=0$. Since $\hat{w}_n\in P_T(x_n)$, it follows that $\lim_{n\to\infty}d(x_n,T(x_n))=0$. Since T satisfies condition (I), we have $\lim_{n\to\infty}d(x_n,F)=0$. Thus, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\|x_{n_k}-p_k\|<\frac{1}{2^k}$ for some $\{p_k\}\subset F$ and all k. Since

$$\|x_{n_{k+1}}-p_k\|\le\|x_{n_k}-p_k\|<\frac{1}{2^k},$$

we get

$$\begin{aligned}\|p_{k+1}-p_k\| &\leq \|p_{k+1}-x_{n_{k+1}}\|+\|x_{n_{k+1}}-p_k\| \\ &< \tfrac{1}{2^{k+1}}+\tfrac{1}{2^k} \\ &< \tfrac{1}{2^{k-1}}.\end{aligned}$$

Therefore $\{p_k\}$ is a Cauchy sequence in D and thus converges to $q \in D$.
Since

$$\begin{aligned}d(p_k, T(q)) &\leq H(P_T(p_k), P_T(q)) \\ &\leq \|p_k-q\|,\end{aligned}$$

and $p_k \to q$ as $k \to \infty$, we have $d(q, T(q)) = 0$, and thus $q \in F$. Since $\{x_{n_k}\}$ converges strongly to q and $\lim_{n\to\infty} \|x_n - q\|$ exists, we have $\{x_n\}$ converges strongly to q. □

References

[1] Y.J. Cho, H. Zhou, G. Gou, *Weak and strong convergence theorems for three-step iterations with errors for asymptotically nonexpansive mappings*, Comput. Math. Appl. 47 (2004), 707-717.

[2] J.S. Jung, *Strong convergence theorems for multivalued nonexpansive nonself-mappings in Banach spaces*, Nonlinear Anal. 66 (2007), 2345-2354.

[3] B. Panyanak, *Mann and Ishikawa iterative processes for multivalued mappings in Banach spaces*, Comput. Math. Appl. 54 (2007), 872-877.

[4] N. Shahzad, H. Zegeye,*On Mann and Ishikawa iteration schemes for multi-valued maps in Banach spaces*, Nonlinear Anal. 71 (2009), 838-844.

[5] C. Shiau, K.K. Tan, C.S. Wong, *Quasi-nonexpansive multi-valued maps and selections*, Fund. Math. 87 (1975), 109-119.

[6] Y. Song, H. Wang, *Erratum to "Mann and Ishikawa iterative processes for multivalued mappings in Banach spaces" [Comput. Math. Appl. 54 (2007), 872-877]*, Comput. Math. Appl. 55 (2008), 2999-3002.

[7] Y. Song, H. Wang, *Convergence of iterative algorithms for multivalued mappings in Banach spaces*, Nonlinear Anal. 70 (2009), 1547-1556.

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