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ON SOME WEIGHTED INTEGRAL INEQUALITIES FOR CONVEX FUNCTIONS RELATED TO FEJÉR'S RESULT

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Abstract

In this paper, we introduce some functionals associated with weighted integral means for convex functions. Some new Fejér-type inequalities are obtained as well.

1 Introduction

Throughout this paper, let $f : [a, b] \to \mathbb{R}$ be convex, $g : [a, b] \to [0, \infty)$ be integrable and symmetric to $\frac{a+b}{2}$. We define the following mappings on $[0, 1]$ that are associated with the well known *Hermite-Hadamard inequality* [1]

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}, \tag{1.1}$$

namely

$$G(t) = \frac{1}{2}\left[f\left(ta + (1-t)\frac{a+b}{2}\right) + f\left(tb + (1-t)\frac{a+b}{2}\right)\right];$$

$$Q(t) = \frac{1}{2}\left[f(ta + (1-t)b) + f(tb + (1-t)a)\right];$$

$$H(t) = \frac{1}{b-a}\int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right)dx;$$

$$H_g(t) = \int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right)g(x)\,dx;$$

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$$I(t)=\int_a^b \frac{1}{2}\left[f\left(t\frac{x+a}{2}+(1-t)\frac{a+b}{2}\right)\right.$$
$$\left.+f\left(t\frac{x+b}{2}+(1-t)\frac{a+b}{2}\right)\right]g(x)\,dx;$$

$$P(t)=\frac{1}{2(b-a)}\int_a^b\left[f\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)x\right)\right.$$
$$\left.+f\left(\left(\frac{1+t}{2}\right)b+\left(\frac{1-t}{2}\right)x\right)\right]dx;$$

$$P_g(t)=\int_a^b\frac{1}{2}\left[f\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)x\right)g\left(\frac{x+a}{2}\right)\right.$$
$$\left.+f\left(\left(\frac{1+t}{2}\right)b+\left(\frac{1-t}{2}\right)x\right)g\left(\frac{x+b}{2}\right)\right]dx;$$

$$N(t)=\int_a^b\frac{1}{2}\left[f\left(ta+(1-t)\frac{x+a}{2}\right)+f\left(tb+(1-t)\frac{x+b}{2}\right)\right]g(x)\,dx;$$

$$L(t)=\frac{1}{2(b-a)}\int_a^b\left[f(ta+(1-t)x)+f(tb+(1-t)x)\right]dx;$$

$$L_g(t)=\frac{1}{2}\int_a^b\left[f(ta+(1-t)x)+f(tb+(1-t)x)\right]g(x)\,dx$$

and

$$S_g(t)=\frac{1}{4}\int_a^b\left[f\left(ta+(1-t)\frac{x+a}{2}\right)+f\left(ta+(1-t)\frac{x+b}{2}\right)\right.$$
$$\left.+f\left(tb+(1-t)\frac{x+a}{2}\right)+f\left(tb+(1-t)\frac{x+b}{2}\right)\right]g(x)\,dx.$$

Remark 1. *We note that* $H=H_g=I, P=P_g=N$ *and* $L=L_g=S_g$ *on* $[0,1]$ *as* $g(x)=\frac{1}{b-a}\,(x\in[a,b])$.

For some results which generalize, improve, and extend the famous Hermite-Hadamard integral inequality, see [2] – [19].

In [8], Fejér established the following weighted generalization of the Hermite-Hadamard inequality (1.1) :

Theorem A. *Let* f,g *be defined as above. Then*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx\le\int_a^b f(x)\,g(x)\,dx\le\frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx. \qquad (1.2)$$

In [11], Tseng et al. established the following Fejér-type inequalities.

Theorem B. *Let f, g be defined as above. Then we have*

$$\begin{aligned} f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx &\le \frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}\int_a^b g(x)\,dx \\ &\le \int_a^b \frac{1}{2}\left[f\left(\frac{x+a}{2}\right)+f\left(\frac{x+b}{2}\right)\right]g(x)\,dx \\ &\le \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]\int_a^b g(x)\,dx \\ &\le \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx. \end{aligned} \tag{1.3}$$

In [2], Dragomir established the following Hermite-Hadamard-type inequality which refines the first inequality of (1.1).

Theorem C. *Let f, H be defined as above. Then H is convex, increasing on $[0,1]$, and for all $t \in [0,1]$, we have*

$$f\left(\frac{a+b}{2}\right) = H(0) \le H(t) \le H(1) = \frac{1}{b-a}\int_a^b f(x)\,dx. \tag{1.4}$$

In [15], Yang and Hong obtained the following Hermite-Hadamard-type inequality which is a refinement of the second inequality in (1.1).

Theorem D. *Let f, P be defined as above. Then P is convex, increasing on $[0,1]$, and for all $t \in [0,1]$, we have*

$$\frac{1}{b-a}\int_a^b f(x)\,dx = P(0) \le P(t) \le P(1) = \frac{f(a)+f(b)}{2}. \tag{1.5}$$

Yang and Tseng [16] and Tseng et al. [11] established the following Fejér-type inequalities which are weighted generalizations of Theorems C – D.

Theorem E ([16])**.** *Let f, g, H_g, P_g be defined as above. Then H_g, P_g are convex, increasing on $[0,1]$, and for all $t \in [0,1]$, we have*

$$\begin{aligned} f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx &= H_g(0) \le H_g(t) \le H_g(1) \\ &= \int_a^b f(x)\,g(x)\,dx \\ &= P_g(0) \le P_g(t) \le P_g(1) \\ &= \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx. \end{aligned} \tag{1.6}$$

Theorem F ([11])**.** *Let f, g, I, N be defined as above. Then I, N are convex, increasing on $[0,1]$, and for all $t \in [0,1]$, we have*

$$\begin{aligned} f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx &= I(0) \le I(t) \le I(1) \\ &= \int_a^b \frac{1}{2}\left[f\left(\frac{x+a}{2}\right)+f\left(\frac{x+b}{2}\right)\right]g(x)\,dx \\ &= N(0) \le N(t) \le N(1) \\ &= \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx. \end{aligned} \tag{1.7}$$

In [7], Dragomir et al. established the following Hermite-Hadamard-type inequality.

Theorem G. *Let f, H, G, L be defined as above. Then G is convex, increasing on $[0,1]$, L is convex on $[0,1]$, and for all $t \in [0,1]$, we have*

$$H(t) \le G(t) \le L(t) \le \frac{1-t}{b-a}\int_a^b f(x)\,dx + t\cdot\frac{f(a)+f(b)}{2} \le \frac{f(a)+f(b)}{2}. \tag{1.8}$$

In [12] – [13], Tseng et al. obtained the following theorems related to Fejér's result which in their turn are weighted generalizations of the inequality (1.8).

Theorem H ([12])**.** *Let f, g, G, H_g, L_g be defined as above. Then L_g is convex, increasing on $[0,1]$, and for all $t \in [0,1]$, we have*

$$\begin{aligned} H_g(t) &\le G(t)\int_a^b g(x)\,dx \\ &\le L_g(t) \\ &\le (1-t)\int_a^b f(x)g(x)\,dx + t\cdot\frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx \\ &\le \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx. \end{aligned} \tag{1.9}$$

Theorem I ([13])**.** *Let f, g, G, I, S_g be defined as above. Then S_g is convex, increasing on $[0,1]$, and for all $t \in [0,1]$, we have*

$$\begin{aligned} I(t) &\le G(t)\int_a^b g(x)\,dx \le S_g(t) \\ &\le (1-t)\int_a^b \frac{1}{2}\left[f\left(\frac{x+a}{2}\right)+f\left(\frac{x+b}{2}\right)\right]g(x)\,dx \\ &\qquad + t\cdot\frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx \\ &\le \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx. \end{aligned} \tag{1.10}$$

In this paper, we provide some new Fejér-type inequalities related to the mappings $G, Q, H_g, P_g, I, N, L_g, S_g$ defined above. They generalize known results obtained in relation with the Hermite-Hadamard inequality and therefore are useful in obtaining various results for means when the convex function and the weight take particular forms.

2 Main Results

The following lemmae are needed in the proofs of our main results:

Lemma 2 (see [9])**.** *Let f be defined as above and let $a \le A \le C \le D \le B \le b$ with $A + B = C + D$. Then*

$$f(C) + f(D) \le f(A) + f(B).$$

The assumptions in Lemma 2 can be weakened as in the following lemma:

Lemma 3. *Let f be defined as above and let $a \le A \le C \le B \le b$ and $a \le A \le D \le B \le b$ with $A + B = C + D$. Then*

$$f(C) + f(D) \le f(A) + f(B).$$

Lemma 4 (see [14])**.** *Let f, G, Q be defined as above. Then Q is symmetric about $\frac{1}{2}$, Q is decreasing on $\left[0, \frac{1}{2}\right]$ and increasing on $\left[\frac{1}{2}, 1\right]$,*

$$G(2t) \le Q(t) \qquad \left(t \in \left[0, \frac{1}{4}\right]\right),$$

$$G(2t) \ge Q(t) \qquad \left(t \in \left[\frac{1}{4}, \frac{1}{2}\right]\right),$$

$$G(2(1-t)) \ge Q(t) \qquad \left(t \in \left[\frac{1}{2}, \frac{3}{4}\right]\right)$$

and

$$G(2(1-t)) \le Q(t) \qquad \left(t \in \left[\frac{3}{4}, 1\right]\right).$$

Now, we are ready to state and prove our results.

Theorem 5. *Let $f, g, G, H_g, P_g, L_g, S_g$ be defined as above. Then:*

1. *The inequality*

$$\begin{aligned} \int_a^b f(x) g(x) \, dx &\le 2 \left[\int_a^{\frac{3a+b}{4}} f(x) g(2x - a) \, dx + \int_{\frac{a+3b}{4}}^b f(x) g(2x - b) \, dx \right] \qquad (2.1) \\ &\le \int_0^1 P_g(t) \, dt \\ &\le \frac{1}{2} \left[\int_a^b f(x) g(x) \, dx + \frac{f(a) + f(b)}{2} \int_a^b g(x) \, dx \right] \end{aligned}$$

holds.

2. *The inequalities*

$$\begin{aligned} L_g(t) &\leq P_g(t) \\ &\leq (1-t)\int_a^b f(x)\,g(x)\,dx + t\cdot\frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx \\ &\leq \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx \end{aligned} \tag{2.2}$$

and

$$0 \leq N(t) - G(t)\int_a^b g(x)\,dx \leq \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx - N(t) \tag{2.3}$$

hold for all $t \in [0,1]$.

3. *If* f *is differentiable on* $[a,b]$, *then we have the inequalities*

$$\begin{aligned} 0 &\leq t\left[\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right]\cdot \inf_{x\in[a,b]} g(x) \\ &\leq P_g(t) - \int_a^b f(x)\,g(x)\,dx; \end{aligned} \tag{2.4}$$

$$\begin{aligned} 0 &\leq P_g(t) - f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx \\ &\leq \frac{(f'(b)-f'(a))(b-a)}{4}\int_a^b g(x)\,dx; \end{aligned} \tag{2.5}$$

$$0 \leq L_g(t) - H_g(t) \leq \frac{(f'(b)-f'(a))(b-a)}{4}\int_a^b g(x)\,dx; \tag{2.6}$$

$$0 \leq P_g(t) - L_g(t) \leq \frac{(f'(b)-f'(a))(b-a)}{4}\int_a^b g(x)\,dx; \tag{2.7}$$

$$0 \leq P_g(t) - H_g(t) \leq \frac{(f'(b)-f'(a))(b-a)}{4}\int_a^b g(x)\,dx; \tag{2.8}$$

$$0 \leq N(t) - I(t) \leq \frac{(f'(b)-f'(a))(b-a)}{4}\int_a^b g(x)\,dx \tag{2.9}$$

and

$$0 \leq S_g(t) - I(t) \leq \frac{(f'(b)-f'(a))(b-a)}{4}\int_a^b g(x)\,dx \tag{2.10}$$

for all $t \in [0,1]$.

Proof. (1) By using simple integration techniques and the hypothesis of g, we have the following identities

$$\int_a^b f(x)\,g(x)\,dx = 2\int_a^{\frac{a+b}{2}}\int_0^{\frac{1}{2}} \left[f(x)+f(a+b-x)\right] g(x)\,dtdx; \tag{2.11}$$

$$\begin{aligned} &2\left[\int_a^{\frac{3a+b}{4}} f(x)\,g(2x-a)\,dx + \int_{\frac{a+3b}{4}}^b f(x)\,g(2x-b)\,dx\right] \\ &= 2\int_a^{\frac{3a+b}{4}} \left[f(x)+f(a+b-x)\right] g(2x-a)\,dx \\ &= 2\int_a^{\frac{a+b}{2}}\int_0^{\frac{1}{2}} \left[f\left(\frac{a+x}{2}\right)+f\left(\frac{a+2b-x}{2}\right)\right] g(x)\,dtdx; \end{aligned} \tag{2.12}$$

$$\begin{aligned} \int_0^1 P_g(t)\,dt &= \int_a^{\frac{a+b}{2}}\int_0^1 f(ta+(1-t)x)\,g(x)\,dtdx \\ &\quad + \int_{\frac{a+b}{2}}^b\int_0^1 f(tb+(1-t)x)\,g(x)\,dtdx \\ &= \int_a^{\frac{a+b}{2}}\int_0^1 f(ta+(1-t)x)\,g(x)\,dtdx \\ &\quad + \int_a^{\frac{a+b}{2}}\int_0^1 f(tb+(1-t)(a+b+x))\,g(x)\,dtdx \\ &= \int_a^{\frac{a+b}{2}}\int_0^{\frac{1}{2}} \left[f(tx+(1-t)a)+f(ta+(1-t)x)\right] g(x)\,dtdx \\ &\quad + \int_a^{\frac{a+b}{2}}\int_0^{\frac{1}{2}} \left[f(tb+(1-t)(a+b-x))\right. \\ &\quad \left. +f(t(a+b-x)+(1-t)b)\right] g(x)\,dtdx \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} &\frac{1}{2}\left[\int_a^b f(x)\,g(x)\,dx + \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx\right] \\ &\qquad = \int_a^{\frac{a+b}{2}}\int_0^{\frac{1}{2}} \left[f(a)+f(x)\right] g(x)\,dtdx \\ &\qquad\quad + \int_a^{\frac{a+b}{2}}\int_0^{\frac{1}{2}} \left[f(a+b-x)+f(b)\right] g(x)\,dtdx. \end{aligned} \tag{2.14}$$

By Lemma 2, the following inequalities hold for all $t \in \left[0, \frac{1}{2}\right]$ and $x \in \left[a, \frac{a+b}{2}\right]$.

$$f(x)+f(a+b-x) \le f\left(\frac{a+x}{2}\right)+f\left(\frac{a+2b-x}{2}\right) \tag{2.15}$$

holds when $A=\frac{a+x}{2}$, $C=x$, $D=a+b-x$ and $B=\frac{a+2b-x}{2}$ in Lemma 2.

$$f\left(\frac{a+x}{2}\right)\leq\frac{1}{2}\left[f\left(tx+(1-t)\,a\right)+f\left(ta+(1-t)\,x\right)\right] \tag{2.16}$$

holds when $A=tx+(1-t)\,a$, $C=D=\frac{a+x}{2}$ and $B=ta+(1-t)\,x$ in Lemma 2.

$$\begin{aligned}&f\left(\frac{a+2b-x}{2}\right)\\&\qquad\leq\frac{1}{2}\left[f\left(tb+(1-t)\,(a+b-x)\right)+f\left(t\,(a+b-x)+(1-t)\,b\right)\right]\end{aligned} \tag{2.17}$$

holds when $A=tb+(1-t)\,(a+b-x)\,,$ $C=D=\frac{a+2b-x}{2}$ and $B=t\,(a+b-x)+(1-t)\,b$ in Lemma 2.

$$\frac{1}{2}\left[f\left(tx+(1-t)\,a\right)+f\left(ta+(1-t)\,x\right)\right]\leq\frac{f\,(a)+f\,(x)}{2} \tag{2.18}$$

holds when $A=a$, $C=tx+(1-t)\,a$, $D=ta+(1-t)\,x$ and $B=x$ in Lemma 2.

$$\begin{aligned}&\frac{1}{2}\left[f\left(tb+(1-t)\,(a+b-x)\right)+f\left(t\,(a+b-x)+(1-t)\,b\right)\right]\\&\qquad\qquad\leq\frac{f\,(a+b-x)+f\,(b)}{2}\end{aligned} \tag{2.19}$$

holds as $A=a+b-x$, $C=tb+(1-t)\,(a+b-x)\,,$ $D=t\,(a+b-x)+(1-t)\,b$ and $B=b$ in Lemma 2. Multiplying the inequalities (2.15) – (2.19) by $g\,(x)$ and integrating them over t on $\left[0,\frac{1}{2}\right]$, over x on $\left[a,\frac{a+b}{2}\right]$ and using identities (2.11) – (2.14), we derive (2.1).

(2) Using substitution rules for integration and the hypothesis of g, we have the following identities

$$\begin{aligned}P_g\,(t)&=\int_a^{\frac{a+b}{2}}f\left(ta+(1-t)\,x\right)g\,(x)\,dx\\&\qquad+\int_{\frac{a+b}{2}}^{b}f\left(tb+(1-t)\,x\right)g\,(x)\,dx\\&=\int_a^{\frac{a+b}{2}}\left[f\left(ta+(1-t)\,x\right)\right.\\&\qquad\left.+\,f\left(tb+(1-t)\,(a+b-x)\right)\right]g\,(x)\,dx\end{aligned} \tag{2.20}$$

and

$$
\begin{aligned}
L_g(t) = \frac{1}{2}\Bigg[\int_a^{\frac{a+b}{2}} & f(ta+(1-t)x)\,g(x)\,dx \\
&+ \int_{\frac{a+b}{2}}^{b} f(tb+(1-t)x)\,g(x)\,dx\Bigg] \\
&+ \frac{1}{2}\Bigg[\int_{\frac{a+b}{2}}^{b} f(ta+(1-t)x)\,g(x)\,dx \\
&+ \int_a^{\frac{a+b}{2}} f(tb+(1-t)x)\,g(x)\,dx\Bigg] \\
= \frac{1}{2}P_g(t) + \frac{1}{2}\int_a^{\frac{a+b}{2}} & [f(ta+(1-t)(a+b-x)) \\
&+ f(tb+(1-t)x)]\,g(x)\,dx
\end{aligned} \tag{2.21}
$$

for all $t \in [0,1]$.

If we choose $A = ta+(1-t)x$, $C = ta+(1-t)(a+b-x)$, $D = tb+(1-t)x$ and $B = tb+(1-t)(a+b-x)$ in Lemma 3, then the inequality

$$
\begin{aligned}
& f(ta+(1-t)(a+b-x)) + f(tb+(1-t)x) \\
& \qquad \le f(ta+(1-t)x) + f(tb+(1-t)(a+b-x))
\end{aligned} \tag{2.22}
$$

holds for all $t \in [0,1]$ and $x \in \left[a, \frac{a+b}{2}\right]$. Multiplying the inequality (2.22) by $g(x)$, integrating both sides over x on $\left[a, \frac{a+b}{2}\right]$ and using identities (2.20) – (2.21), we derive the first inequality of (2.2). The second and third inequalities of (2.2) can be obtained by the convexity of f and (1.2). This proves (2.2).

Again, using substitution rules for integration and the hypothesis of g, we have the following identity

$$
\begin{aligned}
N(t) = \int_a^b \frac{1}{2}\Bigg[& f\left(ta+(1-t)\frac{x+a}{2}\right) \\
&+ f\left(tb+(1-t)\frac{a+2b-x}{2}\right)\Bigg] g(x)\,dx
\end{aligned}
$$

$$
\begin{aligned}
&= \int_a^{\frac{a+b}{2}} [f(ta+(1-t)x) \\
&\qquad + f(tb+(1-t)(a+b-x))]\, g(2x-a)\, dx \qquad (2.23) \\
&= \int_a^{\frac{3a+b}{4}} [f(ta+(1-t)x) \\
&\qquad + f\left(ta+(1-t)\left(\frac{3a+b}{2}-x\right)\right) \\
&\qquad + f\left(tb+(1-t)\left(\frac{b-a}{2}+x\right)\right) \\
&\qquad + f(tb+(1-t)(a+b-x))]\, g(2x-a)\, dx \qquad (2.24)
\end{aligned}
$$

for all $t \in [0,1]$. By Lemma 2, the following inequalities hold for all $t \in [0,1]$ and $x \in \left[a, \frac{3a+b}{4}\right]$.

$$
\begin{aligned}
&f(ta+(1-t)x) + f\left(ta+(1-t)\left(\frac{3a+b}{2}-x\right)\right) \\
&\qquad \le f(a) + f\left(ta+(1-t)\frac{a+b}{2}\right) \qquad (2.25)
\end{aligned}
$$

holds when $A = a$, $C = ta+(1-t)x$, $D = ta+(1-t)\left(\frac{3a+b}{2}-x\right)$ and $B = ta+(1-t)\frac{a+b}{2}$ in Lemma 2.

$$
\begin{aligned}
&f\left(tb+(1-t)\left(\frac{b-a}{2}+x\right)\right) + f(tb+(1-t)(a+b-x)) \\
&\qquad \le f\left(tb+(1-t)\frac{a+b}{2}\right) + f(b). \qquad (2.26)
\end{aligned}
$$

holds when $A = tb+(1-t)\frac{a+b}{2}$, $C = tb+(1-t)\left(\frac{b-a}{2}+x\right)$, $D = tb+(1-t)(a+b-x)$ and $B = b$ in Lemma 2. Multiplying the inequalities (2.25) – (2.26) by $g(2x-a)$ and integrating them over x on $\left[a, \frac{3a+b}{4}\right]$ and using (2.24), we have

$$
N(t) \le \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + G(t)\right]\int_a^b g(x)\, dx \qquad (2.27)
$$

for all $t \in [0,1]$. Using (2.27), we derive the second inequality of (2.3).

Again, using Lemma 2, we have

$$
\begin{aligned}
&f\left(ta+(1-t)\frac{a+b}{2}\right) + f\left(tb+(1-t)\frac{a+b}{2}\right) \\
&\qquad \le f(ta+(1-t)x) + f(tb+(1-t)(a+b-x)) \qquad (2.28)
\end{aligned}
$$

for all $t \in [0,1]$ and $x \in \left[a, \frac{a+b}{2}\right]$. Multiplying the inequality (2.28) by $g(2x-a)$, integrating both sides over x on $\left[a, \frac{a+b}{2}\right]$ and using (2.23), we derive the first inequality of (2.3).

This proves (2.3).

(3) Integrating by parts, we have

$$\frac{1}{b-a}\int_a^{\frac{a+b}{2}}\left[(a-x)\,f'(x)+(x-a)\,f'(a+b-x)\right]dx$$
$$=\frac{1}{b-a}\int_a^b f(x)\,dx-f\left(\frac{a+b}{2}\right). \quad (2.29)$$

Using substitution rules for integration, we have the following identity

$$\frac{1}{b-a}\int_a^b f(x)\,dx=\frac{1}{b-a}\int_a^{\frac{a+b}{2}}\left[f(x)+f(a+b-x)\right]dx. \quad (2.30)$$

Now, using the convexity of f and $g(x)\geq 0$ on $[a,b]$, the inequality

$$\begin{aligned}&\left[f(ta+(1-t)\,x)-f(x)\right]g(x)\\&\qquad+\left[f(tb+(1-t)(a+b-x))-f(a+b-x)\right]g(x)\\&\geq t(a-x)\,f'(x)\,g(x)+t(x-a)\,f'(a+b-x)\,g(x)\\&=t(x-a)\left[f'(a+b-x)-f'(x)\right]g(x)\\&\geq t(x-a)\left[f'(a+b-x)-f'(x)\right]\inf_{x\in[a,b]}g(x)\end{aligned}$$

holds for all $t\in[0,1]$ and $x\in\left[a,\frac{a+b}{2}\right]$. Integrating the above inequality over x on $\left[a,\frac{a+b}{2}\right]$, dividing both sides by $(b-a)$ and using (1.1), (2.20), (2.29) and (2.30), we derive (2.4).

On the other hand, we have

$$\begin{aligned}\frac{f(a)-f\left(\frac{a+b}{2}\right)}{2}\int_a^b g(x)\,dx&\leq\frac{1}{2}\left(a-\frac{a+b}{2}\right)f'(a)\int_a^b g(x)\,dx\\&=\frac{a-b}{4}f'(a)\int_a^b g(x)\,dx\end{aligned}$$

and

$$\begin{aligned}\frac{f(b)-f\left(\frac{a+b}{2}\right)}{2}\int_a^b g(x)\,dx&\leq\frac{1}{2}\left(b-\frac{a+b}{2}\right)f'(b)\int_a^b g(x)\,dx\\&=\frac{b-a}{4}f'(b)\int_a^b g(x)\,dx\end{aligned}$$

and taking their sum we obtain:

$$\left[\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right]\int_a^b g(x)\,dx$$
$$\leq\frac{(f'(b)-f'(a))(b-a)}{4}\int_a^b g(x)\,dx. \quad (2.31)$$

Finally, $(2.5)-(2.10)$ follow from (1.6), (1.7), (1.9), (1.10), (2.2) and (2.31).

This completes the proof. ∎

Let $g(x) = \frac{1}{b-a}$ $(x \in [a,b])$. Then the following Hermite-Hadamard-type inequalities, which are also given in [14], are natural consequences of Theorem 5.

Corollary 6. *Let f, G, H, L, P be defined as above. Then:*

1. *The inequality*

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)\,dx &\le \frac{2}{b-a}\int_{\left[a,\frac{3a+b}{4}\right]\cup\left[\frac{a+3b}{4},b\right]} f(x)\,dx \\ &\le \int_0^1 P(t)\,dt \\ &\le \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f(x)\,dx + \frac{f(a)+f(b)}{2}\right]\end{aligned}$$

holds.

2. *The inequalities*

$$L(t) \le P(t) \le \frac{1-t}{b-a}\int_a^b f(x)\,dx + t\cdot\frac{f(a)+f(b)}{2} \le \frac{f(a)+f(b)}{2}$$

and

$$0 \le P(t) - G(t) \le \frac{f(a)+f(b)}{2} - P(t)$$

hold for all $t \in [0,1]$.

3. *If f is differentiable on $[a,b]$, then we have the inequalities*

$$\begin{aligned}0 &\le t\left[\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right] \\ &\le P(t) - \frac{1}{b-a}\int_a^b f(x)\,dx;\end{aligned}$$

$$0 \le P(t) - f\left(\frac{a+b}{2}\right) \le \frac{(f'(b)-f'(a))(b-a)}{4};$$

$$0 \le L(t) - H(t) \le \frac{(f'(b)-f'(a))(b-a)}{4};$$

$$0 \le P(t) - L(t) \le \frac{(f'(b)-f'(a))(b-a)}{4}$$

and

$$0 \le P(t) - H(t) \le \frac{(f'(b)-f'(a))(b-a)}{4}$$

for all $t \in [0,1]$.

Remark 7. *In Theorem 5, the inequality* (2.1) *gives a new refinement of the Fejér inequality* (1.2) *.*

Remark 8. *In Theorem 5, the inequality* (2.2) *refines the Fejér-type inequality* (1.9) *.*

In the next theorem, we point out some inequalities for the functions G, Q, H_g, P_g, S_g considered above:

Theorem 9. *Let $f, g, G, Q, H_g, P_g, S_g$ be defined as above. Then:*

1. *The inequalities*

$$\begin{aligned} H_g(t) &\leq Q(t)\int_a^b g(x)\,dx \\ &\leq \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx \qquad \left(t\in\left[0,\frac{1}{3}\right]\right) \end{aligned} \tag{2.32}$$

and

$$\begin{aligned} f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx &\leq Q(t)\int_a^b g(x)\,dx \\ &\leq P_g(t) \qquad \left(t\in\left[\frac{1}{3},1\right]\right) \end{aligned} \tag{2.33}$$

hold for all $t\in[0,1]$.

2. *The inequality*

$$\begin{aligned} 0 &\leq S_g(t) - G(t)\int_a^b g(x)\,dx \\ &\leq \frac{1}{2}\left[\frac{f(a)+f(b)}{2}+Q(t)\right]\int_a^b g(x)\,dx - S_g(t) \end{aligned} \tag{2.34}$$

holds for all $t\in[0,1]$.

Proof. (1) We discuss the following two cases.
Case 1. $t\in\left[0,\frac{1}{3}\right]$.

Using substitution rules for integration and the hypothesis of g, we have the following identity

$$\begin{aligned} H(t) = \int_a^{\frac{a+b}{2}} \Bigg[f\left(tx+(1-t)\frac{a+b}{2}\right) \\ + f\left(t(a+b-x)+(1-t)\frac{a+b}{2}\right)\Bigg] g(x)\,dx. \end{aligned} \tag{2.35}$$

If we choose $A=(1-t)\,a+tb$, $C=tx+(1-t)\,\frac{a+b}{2}$, $D=t\,(a+b-x)+(1-t)\,\frac{a+b}{2}$ and $B=ta+(1-t)\,b$ in Lemma 2, then the inequality

$$f\left(tx+(1-t)\,\frac{a+b}{2}\right)+f\left(t\,(a+b-x)+(1-t)\,\frac{a+b}{2}\right) \leq f\left((1-t)\,a+tb\right)+f\left(ta+(1-t)\,b\right) \quad (2.36)$$

holds for all $t\in\left[0,\frac{1}{3}\right]$ and $x\in\left[a,\frac{a+b}{2}\right]$. Multiplying the inequality (2.36) by $g\,(x)$, integrating both sides over x on $\left[a,\frac{a+b}{2}\right]$ and using identity (2.35), we derive the first inequality of (2.32). From Lemma 4, we have

$$\sup_{t\in\left[0,\frac{1}{3}\right]} Q\,(t)=\frac{f\,(a)+f\,(b)}{2}.$$

Then the second inequality of (2.32) can be obtained. This proves (2.32).

Case 2. $t\in\left[\frac{1}{3},1\right]$.

If we choose $A=ta+(1-t)\,x$, $C=ta+(1-t)\,b$, $D=(1-t)\,a+tb$ and $B=tb+(1-t)\,(a+b-x)$ in Lemma 3, then the inequality

$$f\left(ta+(1-t)\,b\right)+f\left(tb+(1-t)\,a\right) \leq f\left(ta+(1-t)\,x\right)+f\left(tb+(1-t)\,(a+b-x)\right) \quad (2.37)$$

holds for all $t\in\left[\frac{1}{3},1\right]$ and $x\in\left[a,\frac{a+b}{2}\right]$. Multiplying the inequality (2.37) by $g\,(x)$, integrating both sides over x on $\left[a,\frac{a+b}{2}\right]$ and using identity (2.20), we obtain the second inequality of (2.33). From Lemma 4, we have

$$\inf_{t\in\left[\frac{1}{3},1\right]} Q\,(t)=f\left(\frac{a+b}{2}\right).$$

Then the first inequality of (2.33) can be obtained. This proves (2.33).

(2) Using substitution rules for integration and the hypothesis of g, we have the

following identity

$$
\begin{aligned}
2S_g(t) &= \int_a^{\frac{a+b}{2}} \left[f(ta+(1-t)x)+f(tb+(1-t)x)\right] g(2x-a)\,dx \qquad (2.38)\\
&\quad + \int_{\frac{a+b}{2}}^{b} \left[f(ta+(1-t)x)+f(tb+(1-t)x)\right] g(2x-b)\,dx \\
&= \int_a^{\frac{a+b}{2}} \left[f(ta+(1-t)x)+f(tb+(1-t)x)\right. \\
&\quad \left. + f(ta+(1-t)(a+b-x)) + f(tb+(1-t)(a+b-x))\right] \\
&\quad \times g(2x-a)\,dx \\
&= \int_a^{\frac{3a+b}{4}} \left[f(ta+(1-t)x) + f\left(ta+(1-t)\left(\frac{3a+b}{2}-x\right)\right)\right. \\
&\quad + f\left(ta+(1-t)\left(\frac{b-a}{2}+x\right)\right) + f(ta+(1-t)(a+b-x)) \\
&\quad + f(tb+(1-t)x) + f\left(tb+(1-t)\left(\frac{3a+b}{2}-x\right)\right) \\
&\quad \left. + f\left(tb+(1-t)\left(\frac{b-a}{2}+x\right)\right) + f(tb+(1-t)(a+b-x))\right] \\
&\quad \times g(2x-a)\,dx
\end{aligned}
$$

for all $t \in [0,1]$.

By Lemma 2, the following inequalities hold for all $t \in [0,1]$ and $x \in \left[a, \frac{3a+b}{4}\right]$.

$$
\begin{aligned}
& f(ta+(1-t)x) + f\left(ta+(1-t)\left(\frac{3a+b}{2}-x\right)\right) \\
& \qquad \le f(a) + f\left(ta+(1-t)\frac{a+b}{2}\right) \qquad (2.39)
\end{aligned}
$$

holds when $A = a$, $C = ta+(1-t)x$, $D = ta+(1-t)\left(\frac{3a+b}{2}-x\right)$ and $B = ta+(1-t)\frac{a+b}{2}$ in Lemma 2.

$$
\begin{aligned}
& f\left(ta+(1-t)\left(\frac{b-a}{2}+x\right)\right) + f(ta+(1-t)(a+b-x)) \\
& \qquad \le f\left(ta+(1-t)\frac{a+b}{2}\right) + f(ta+(1-t)b) \qquad (2.40)
\end{aligned}
$$

holds when $A = ta+(1-t)\frac{a+b}{2}$, $C = ta+(1-t)\left(\frac{b-a}{2}+x\right)$, $D = ta+(1-t)(a+b-x)$

and $B = ta + (1-t)\,b$ in Lemma 2.

$$f\left(tb+(1-t)\,x\right)+f\left(tb+(1-t)\left(\frac{3a+b}{2}-x\right)\right) \leq f\left(tb+(1-t)\,a\right)+f\left(tb+(1-t)\,\frac{a+b}{2}\right) \quad (2.41)$$

holds when $A = tb+(1-t)\,a$, $C = tb+(1-t)\,x$, $D = tb+(1-t)\left(\frac{3a+b}{2}-x\right)$ and $B = tb+(1-t)\,\frac{a+b}{2}$ in Lemma 2.

$$f\left(tb+(1-t)\left(\frac{b-a}{2}+x\right)\right)+f\left(tb+(1-t)\,(a+b-x)\right) \leq f\left(tb+(1-t)\,\frac{a+b}{2}\right)+f\,(b) \quad (2.42)$$

holds when $A = tb+(1-t)\,\frac{a+b}{2}$, $C = tb+(1-t)\left(\frac{b-a}{2}+x\right)$, $D = tb+(1-t)\,(a+b-x)$ and $B = b$ in Lemma 2. Multiplying the inequalities (2.39) – (2.42) by $g\,(2x-a)$, integrating them over x on $\left[a, \frac{3a+b}{4}\right]$ and using identity (2.38), we have

$$2S_g\,(t) \leq G\,(t)\int_a^b g\,(x)\,dx + \frac{1}{2}\left[\frac{f\,(a)+f\,(b)}{2}+Q\,(t)\right]\int_a^b g\,(x)\,dx \quad (2.43)$$

for all $t \in [0,1]$. Using (1.10) and (2.43), we derive (2.34). This completes the proof. ∎

Let $g\,(x) = \frac{1}{b-a}$ $(x \in [a,b])$. Then the following Hermite-Hadamard-type inequalities, which are given in [14], are natural consequences of Theorem 9.

Corollary 10. *Let f, G, H, L, P be defined as above. Then:*

1. *The inequalities*

$$H\,(t) \leq Q\,(t) \leq \frac{f\,(a)+f\,(b)}{2} \qquad \left(t \in \left[0, \frac{1}{3}\right]\right)$$

and

$$f\left(\frac{a+b}{2}\right) \leq Q\,(t) \leq P\,(t) \qquad \left(t \in \left[\frac{1}{3}, 1\right]\right)$$

hold for all $t \in [0,1]$.

2. *The inequality*

$$0 \leq L\,(t) - G\,(t) \leq \frac{1}{2}\left[\frac{f\,(a)+f\,(b)}{2}+Q\,(t)\right] - L\,(t)$$

holds for all $t \in [0,1]$.

The following Fejér-type inequalities are natural consequences of Theorems A – B, E – I, 5, 9 and Lemma 4 and we shall omit their proofs.

Theorem 11. *Let $f, g, G, H_g, P_g, I, L_g, S_g$ be defined as above.*

$$\begin{aligned} f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx &\le H_g(t) \le G(t)\int_a^b g(x)\,dx \le S_g(t) \\ &\le (1-t)\int_a^b \frac{1}{2}\left[f\left(\frac{x+a}{2}\right)+f\left(\frac{x+b}{2}\right)\right]g(x)\,dx \\ &\quad + t\cdot\frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx \\ &\le \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx \end{aligned}$$

and

$$\begin{aligned} f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx &\le I(t) \le G(t)\int_a^b g(x)\,dx \\ &\le L_g(t) \le P_g(t) \\ &\le (1-t)\int_a^b f(x)\,g(x)\,dx + t\cdot\frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx \\ &\le \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx. \end{aligned}$$

Theorem 12. *Let f, g, G, Q, H_g, I be defined as above. Then, for all $t \in \left[0, \frac{1}{4}\right]$, we have*

$$\begin{aligned} f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx &\le H_g(t) \le H_g(2t) \le G(2t)\int_a^b g(x)\,dx \\ &\le Q(t)\int_a^b g(x)\,dx \le \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx \end{aligned}$$

and

$$\begin{aligned} f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx &\le I(t) \le I(2t) \le G(2t)\int_a^b g(x)\,dx \\ &\le Q(t)\int_a^b g(x)\,dx \le \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx. \end{aligned}$$

Theorem 13. *Let $f, g, G, Q, H_g, P_g, L_g, S_g$ be defined as above. Then, for all $t \in$*

$\left[\frac{1}{4}, \frac{1}{3}\right]$, *we have*

$$\begin{aligned} f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx &\le H_g(t) \le Q(t)\int_a^b g(x)\,dx \le G(2t)\int_a^b g(x)\,dx \\ &\le L_g(2t) \le P_g(2t) \\ &\le (1-2t)\int_a^b f(x)\,g(x)\,dx + 2t\cdot\frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx \\ &\le \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx \end{aligned}$$

and

$$\begin{aligned} f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx &\le H_g(t) \le Q(t)\int_a^b g(x)\,dx \\ &\le G(2t)\int_a^b g(x)\,dx \le S_g(2t) \\ &\le (1-2t)\int_a^b \frac{1}{2}\left[f\left(\frac{x+a}{2}\right)+f\left(\frac{x+b}{2}\right)\right]g(x)\,dx \\ &\qquad + 2t\cdot\frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx \\ &\le \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx. \end{aligned}$$

Theorem 14. *Let $f, g, G, Q, P_g, L_g, S_g$ be defined as above. Then, for all $t \in \left[\frac{1}{3}, \frac{1}{2}\right]$, we have*

$$\begin{aligned} f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,d &\le Q(t)\int_a^b g(x)\,dx \\ &\le G(2t)\int_a^b g(x)\,dx \le L_g(2t) \le P_g(2t) \\ &\le (1-2t)\int_a^b f(x)\,g(x)\,dx + 2t\cdot\frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx \\ &\le \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx; \end{aligned}$$

$$
\begin{aligned}
f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,d &\le Q(t)\int_a^b g(x)\,dx \\
&\le G(2t)\int_a^b g(x)\,dx \le S_g(2t) \\
&\le (1-2t)\int_a^b \frac{1}{2}\left[f\left(\frac{x+a}{2}\right)+f\left(\frac{x+b}{2}\right)\right]g(x)\,dx \\
&\qquad + 2t\cdot\frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx \\
&\le \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx
\end{aligned}
$$

and

$$
\begin{aligned}
f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx &\le Q(t)\int_a^b g(x)\,dx \le P_g(t) \le P_g(2t) \\
&\le (1-2t)\int_a^b f(x)\,g(x)\,dx + 2t\cdot\frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx \\
&\le \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx.
\end{aligned}
$$

Theorem 15. *Let $f, g, G, Q, P_g, L_g, S_g$ be defined as above. Then, for all $t \in \left[\frac{1}{2}, \frac{2}{3}\right]$, we have*

$$
\begin{aligned}
f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx &\le Q(t)\int_a^b g(x)\,dx \le G(2(1-t))\int_a^b g(x)\,dx \\
&\le L_g(2(1-t)) \le P_g(2(1-t)) \\
&\le (2t-1)\int_a^b f(x)\,g(x)\,dx + 2(1-t)\cdot\frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx \\
&\le \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx
\end{aligned}
$$

and

$$
\begin{aligned}
f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx &\le Q(t)\int_a^b g(x)\,dx \\
&\le G(2(1-t))\int_a^b g(x)\,dx \le S_g(2(1-t)) \\
&\le (2t-1)\int_a^b \frac{1}{2}\left[f\left(\frac{x+a}{2}\right)+f\left(\frac{x+b}{2}\right)\right]g(x)\,dx \\
&\qquad + 2(1-t)\cdot\frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx \\
&\le \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx.
\end{aligned}
$$

Theorem 16. *Let $f, g, G, Q, H_g, P_g, L_g, S_g$ be defined as above. Then, for all $t \in \left[\frac{2}{3}, \frac{3}{4}\right]$, we have*

$$\begin{aligned}
f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx &\leq Q(t)\int_a^b g(x)\,dx \\
&\leq G(2(1-t))\int_a^b g(x)\,dx \\
&\leq G(t)\int_a^b g(x)\,dx \leq L_g(t) \leq P_g(t) \\
&\leq (1-t)\int_a^b f(x)\,g(x)\,dx + t\cdot\frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx \\
&\leq \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx
\end{aligned}$$

and

$$\begin{aligned}
f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx &\leq Q(t)\int_a^b g(x)\,dx \leq G(2(1-t))\int_a^b g(x)\,dx \\
&\leq G(t)\int_a^b g(x)\,dx \leq S_g(t)
\end{aligned}$$

$$\leq (1-t)\int_a^b \frac{1}{2}\left[f\left(\frac{x+a}{2}\right)+f\left(\frac{x+b}{2}\right)\right]g(x)\,dx$$
$$+t\cdot\frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx$$
$$\leq \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx.$$

Theorem 17. *Let $f, g, G, Q, H_g, P_g, I, S_g$ be defined as above. Then, for all $t \in \left[\frac{3}{4}, 1\right]$, we have*

$$\begin{aligned}
f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx &\leq H_g(2(1-t)) \leq G(2(1-t))\int_a^b g(x)\,dx \\
&\leq Q(t)\int_a^b g(x)\,dx \leq P_g(t) \\
&\leq \frac{1-t}{b-a}\int_a^b f(x)\,g(x)\,dx + t\cdot\frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx \\
&\leq \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx
\end{aligned}$$

and

$$\begin{aligned}
f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx &\leq I(2(1-t)) \leq G(2(1-t))\int_a^b g(x)\,dx \\
&\leq Q(t)\int_a^b g(x)\,dx \leq P_g(t) \\
&\leq \frac{1-t}{b-a}\int_a^b f(x)\,g(x)\,dx + t\cdot\frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx \\
&\leq \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx.
\end{aligned}$$

Let $g(x) = \frac{1}{b-a}\,(x \in [a,b])$. Then the following Hermite-Hadamard-type inequalities are natural consequences of Theorems 11 – 17, which are given in [14].

Corollary 18. *Let f, Q, G, H, P, L be defined as above. Then we have:*

1. *For all $t \in \left[0, \frac{1}{4}\right]$ one has the inequality*

$$f\left(\frac{a+b}{2}\right) \leq H(t) \leq H(2t) \leq G(2t) \leq Q(t) \leq \frac{f(a)+f(b)}{2}.$$

2. For all $t \in \left[\frac{1}{4}, \frac{1}{3}\right]$ *one has the inequality*

$$
\begin{aligned}
f\left(\frac{a+b}{2}\right) &\leq H(t) \leq Q(t) \leq G(2t) \leq L(2t) \leq P(2t) \\
&\leq \frac{1-2t}{b-a} \int_a^b f(x)\,dx + 2t \cdot \frac{f(a)+f(b)}{2} \\
&\leq \frac{f(a)+f(b)}{2}.
\end{aligned}
$$

3. For all $t \in \left[\frac{1}{3}, \frac{1}{2}\right]$ *one has the inequalities*

$$
\begin{aligned}
f\left(\frac{a+b}{2}\right) &\leq Q(t) \leq G(2t) \leq L(2t) \leq P(2t) \\
&\leq \frac{1-2t}{b-a} \int_a^b f(x)\,dx + 2t \cdot \frac{f(a)+f(b)}{2} \\
&\leq \frac{f(a)+f(b)}{2}
\end{aligned}
$$

and

$$
\begin{aligned}
f\left(\frac{a+b}{2}\right) &\leq Q(t) \leq P(t) \leq P(2t) \\
&\leq \frac{1-2t}{b-a} \int_a^b f(x)\,dx + 2t \cdot \frac{f(a)+f(b)}{2} \\
&\leq \frac{f(a)+f(b)}{2}.
\end{aligned}
$$

4. For all $t \in \left[\frac{1}{2}, \frac{2}{3}\right]$ *one has the inequality*

$$
\begin{aligned}
f\left(\frac{a+b}{2}\right) &\leq Q(t) \leq G(2(1-t)) \leq L(2(1-t)) \leq P(2(1-t)) \\
&\leq \frac{2t-1}{b-a} \int_a^b f(x)\,dx + 2(1-t) \cdot \frac{f(a)+f(b)}{2} \\
&\leq \frac{f(a)+f(b)}{2}.
\end{aligned}
$$

5. For all $t \in \left[\frac{2}{3}, \frac{3}{4}\right]$ *one has the inequality*

$$
\begin{aligned}
f\left(\frac{a+b}{2}\right) &\leq Q(t) \leq G(2(1-t)) \leq G(t) \leq L(t) \leq P(t) \\
&\leq \frac{1-t}{b-a} \int_a^b f(x)\,dx + t \cdot \frac{f(a)+f(b)}{2} \leq \frac{f(a)+f(b)}{2}.
\end{aligned}
$$

6. For all $t \in \left[\frac{3}{4}, 1\right]$ *one has the inequality*

$$f\left(\frac{a+b}{2}\right) \leq H\left(2\left(1-t\right)\right) \leq G\left(2\left(1-t\right)\right) \leq Q\left(t\right) \leq P\left(t\right)$$
$$\leq \frac{1-t}{b-a}\int_a^b f\left(x\right)dx + t\cdot\frac{f\left(a\right)+f\left(b\right)}{2} \leq \frac{f\left(a\right)+f\left(b\right)}{2}.$$

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