

## GENERALIZED COUNTABLE ITERATED FUNCTION SYSTEMS

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### Abstract

One of the most common and most general way to generate fractals is by using iterated function systems which consists of a finite or infinitely many maps. Generalized countable iterated function systems (GCIFS) are a generalization of countable iterated function systems by considering contractions from  $X \times X$  into  $X$  instead of contractions on the metric space  $X$  to itself, where  $(X, d)$  is a compact metric space. If all contractions of a GCIFS are Lipschitz with respect to a parameter and the supremum of the Lipschitz constants is finite, then the associated attractor depends continuously on the respective parameter.

## 1 Introduction

In the famous paper [2], J.E. Hutchinson proves that, given a set of contractions  $(\omega_n)_{n=1}^N$  in a complete metric space  $X$ , there exists a unique nonempty compact set  $A \subset X$ , named the *attractor* of IFS. This attractor is, generally, a fractal set. These ideas has been extended to infinitely many contractions, a such generalization can be found, for example, in [3] and, for Countable Iterated Function Systems (CIFS) on a compact metric space, in [6]. There is a current effort to extend the classical Hutchinson's framework to more general spaces and infinite iterated function systems or, more generally, to multifunction systems. In [7] it is shown that any compact subset of a metric space can be obtained as attractor of a CIFS. M. Barnsley (in [1]) and others show that, if the contractions of an IFS depend continuously on a parameter, then the corresponding attractor also depends continuously of the respective parameter with respect to the Hausdorff metric. The result has been extended to the countable case (see [8]).

We start with a short description of a Hausdorff metric on a metric space and on a product of two metric spaces and we prove some of its properties which will

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be used in the sequel. Next, the notions of iterated function system (IFS) and countable iterated function system on a complete and, respectively compact metric space  $(X, d)$  together with some its properties are presented.

In [4], A. Mihail introduces the Recurrent Iterated Functions Systems (RIFS) which is a finite family of contractions  $\omega_n : X \times X \rightarrow X$ ,  $n = 1, \dots, N$ , where  $(X, d)$  is a complete metric space and he proves some of its properties. That construction is extended in [5] by A. Mihail and R. Miculescu to a finite family of contractive mappings from  $X^m$  ( $m \in \mathbb{N}$ ) into  $X$ , the space  $(X, d)$  being compact.

The main results of that paper are given in section 3 when it is introduced the Generalized Countable Iterated Functions Systems (GCIFS) of order two. A GCIFS consists of a sequence of contractions  $\omega_n : X \times X \rightarrow X$ ,  $n = 1, 2, \dots$ , where  $(X, d)$  is a compact metric space. Notice that the treatment of GCIFS of any order  $m \in \mathbb{N}$ ,  $m \geq 3$ , (when the considered contractions are defined on  $X^m$  space) can be made in an analogous way as in the case when  $m = 2$ .

It is described some ways to characterize the attractor of a GCIFS as a limiting process and by means of the fixed points of a proper family of contractions. If the contractions which compose the GCIFS obey some continuity conditions with respect to a parameter, then the corresponding attractor depends continuously with respect to that parameter.

Some ways to write the attractor of a GCIFS as a limit of a sequence of sets are presented. They can be very beneficial in certain cases to use the computer to approximate the attractor. Finally, some examples in the compact subspace  $X$  of  $\mathbb{R}$  and, respectively  $\mathbb{R}^2$ , is given.

## 2 Preliminary Facts

In this section we give some well known aspects on Fractal Theory used in the sequel (more complete and rigorous treatments may be found in [2], [1], [6], [8]).

Let us consider a function  $f : X \rightarrow Y$ , where  $(X, d)$ ,  $(Y, \delta)$  are two metric spaces, and we define

$$\text{Lip}(f) := \sup_{\substack{x, y \in X \\ x \neq y}} \frac{\delta(f(x), f(y))}{d(x, y)} \in \overline{\mathbb{R}}_+.$$

$f$  is said to be a Lipschitz function if  $\text{Lip}(f) < \infty$  and a contraction if  $\text{Lip}(f) < 1$ . If  $f$  is contraction, then any  $r \in (0, \text{Lip}(f))$  is called contraction ratio.

### 2.1 Hausdorff metric

Let  $(X, d)$  be a metric space and  $\mathcal{K}(X)$  be the class of all compact non-empty subsets of  $X$ . The function  $h : \mathcal{K}(X) \times \mathcal{K}(X) \rightarrow \mathbb{R}_+$ ,  $h(A, B) = \max \{D(A, B), D(B, A)\}$ , where  $D(A, B) = \sup_{x \in A} (\inf_{y \in B} d(x, y))$ , for all  $A, B \in \mathcal{K}(X)$ , is a metric, namely the

*Hausdorff metric*. If  $(X, d)$  is complete, then  $\mathcal{K}(X)$  is a complete metric space with respect to this metric  $h$ . Also,  $(\mathcal{K}(X), h)$  is a compact metric space provided that  $(X, d)$  is compact.

Some simple standard facts in the space  $(\mathcal{K}(X), h)$ , which will be used in the sequel, are described in the following two lemmas:

LEMMA 2.1. *If  $(E_i)_{i \in \mathfrak{S}}, (F_i)_{i \in \mathfrak{S}}$  are two sequences of sets in  $\mathcal{K}(X)$ , then*

$$h\left(\overline{\bigcup_{i \in \mathfrak{S}} E_i}, \overline{\bigcup_{i \in \mathfrak{S}} F_i}\right) = h\left(\bigcup_{i \in \mathfrak{S}} E_i, \bigcup_{i \in \mathfrak{S}} F_i\right) \leq \sup_{i \in \mathfrak{S}} h(E_i, F_i).$$

LEMMA 2.2. [6, Th.1.1] *Let  $(A_n)_n$  be a sequence of nonempty compact subsets of  $X$ .*

(a) *If  $A_n \subset A_{n+1}$ , for all  $n \geq 1$ , and the set  $A := \bigcup_{n \geq 1} A_n$  is relatively compact, then*

$$\overline{A} = \overline{\bigcup_{n \geq 1} A_n} = \lim_n A_n,$$

*the limiting process being taken with respect to the Hausdorff metric and the bar means the closure;*

(b) *If  $A_{n+1} \subset A_n$ , for any  $n \geq 1$ , then  $\lim_n A_n = \bigcap_{n \geq 1} A_n$ .*

We now consider another metric space  $(Y, \delta)$  and we use the same notation  $h$  for the Hausdorff metric on  $\mathcal{K}(X)$  and  $\mathcal{K}(Y)$  and, analogously, for the function  $D$ . We equip the space  $X \times Y$  with the "max" metric  $d_2$ , namely

$$d_2((x_1, y_1), (x_2, y_2)) := \max\{d(x_1, x_2), \delta(y_1, y_2)\}.$$

It is known that  $d_2$  is a metric and the space  $(X \times Y, d_2)$  is complete, respectively compact, whenever  $(X, d)$  and  $(Y, \delta)$  are complete, respectively compact.

Let  $h_2$  be the Hausdorff metric on  $\mathcal{K}(X) \times \mathcal{K}(Y)$  induced by  $d_2$ . We denote  $D_2$  the corresponding set function from the definition of Hausdorff metric,

$$D_2((B_1, C_1), (B_2, C_2)) := \sup_{\substack{x_1 \in B_1 \\ y_1 \in C_1}} \inf_{\substack{x_2 \in B_2 \\ y_2 \in C_2}} d_2((x_1, y_1), (x_2, y_2)).$$

LEMMA 2.3. *Under the above conditions,*

$$h_2((B_1, C_1), (B_2, C_2)) = \max\{h(B_1, B_2), h(C_1, C_2)\},$$

*for any  $B_1, B_2 \in \mathcal{K}(X)$  and any  $C_1, C_2 \in \mathcal{K}(Y)$ .*

*Proof.* We first prove that

$$D_2\{(B_1, C_1), (B_2, C_2)\} = \max\{D(B_1, B_2), D(C_1, C_2)\}, \quad (2.1)$$

that is

$$\begin{aligned} & \sup_{\substack{x_1 \in B_1 \\ y_1 \in C_1}} \inf_{\substack{x_2 \in B_2 \\ y_2 \in C_2}} \max\{d(x_1, x_2), \delta(y_1, y_2)\} \\ &= \max\left\{ \sup_{x_1 \in B_1} \inf_{x_2 \in B_2} d(x_1, x_2), \sup_{y_1 \in C_1} \inf_{y_2 \in C_2} \delta(y_1, y_2) \right\}. \end{aligned}$$

We suppose, by contradiction, that one has

$$D_2\{(B_1, C_1), (B_2, C_2)\} > \max\{D(B_1, B_2), D(C_1, C_2)\}.$$

There exists  $r \in \mathbb{R}$  such that

$$\begin{aligned} & \sup_{\substack{x_1 \in B_1 \\ y_1 \in C_1}} \inf_{\substack{x_2 \in B_2 \\ y_2 \in C_2}} \max\{d(x_1, x_2), \delta(y_1, y_2)\} > r, \\ r & > \sup_{x_1 \in B_1} \inf_{x_2 \in B_2} d(x_1, x_2) \text{ and } r > \sup_{y_1 \in C_1} \inf_{y_2 \in C_2} \delta(y_1, y_2). \end{aligned}$$

Thence:

$$\begin{aligned} & \exists x_1^0 \in B_1, \exists y_1^0 \in C_1 \text{ such that } \forall x_2 \in B_2, \forall y_2 \in C_2 \\ & \Rightarrow d(x_1^0, x_2) > r \text{ or } \delta(y_1^0, y_2) > r. \end{aligned} \quad (2.2)$$

At the same time,

$$\begin{aligned} & \forall x_1 \in B_1, \exists x_2 \in B_2 \text{ such that } d(x_1, x_2) < r, \text{ and} \\ & \forall y_1 \in C_1, \exists y_2 \in C_2 \text{ such that } \delta(y_1, y_2) < r. \end{aligned}$$

Next, for  $x_1^0$  and  $y_1^0$ , there are  $x_2^0 \in B_2$  and  $y_2^0 \in C_2$  with  $d(x_1^0, x_2^0) < r$  and  $\delta(y_1^0, y_2^0) < r$  contradicting (2.2). It follows that, in (2.1), one has the inequality " $\leq$ ".

By using the similar arguments as before, we deduce the other inequality.

Finally, by symmetry, we find

$$D_2\{(C_1, B_1), (C_2, B_2)\} = \max\{D(B_2, B_1), D(C_2, C_1)\}$$

and thence, with (2.1), the equality of statement comes. □

**THEOREM 2.1.** *Let  $(X, d), (Y, \delta), (Z, \rho)$  be three metric spaces and  $\omega : X \times Y \rightarrow Z$  be a function. Then*

(i) *if  $\omega$  is a Lipschitz map, one has*

$$h(\omega(B_1, C_1), \omega(B_2, C_2)) \leq \text{Lip}(\omega) h_2((B_1, C_1), (B_2, C_2));$$

(ii) *if  $\omega$  is uniform continuous, then the set function  $F_\omega : \mathcal{K}(X) \times \mathcal{K}(Y) \rightarrow \mathcal{K}(Z)$ ,  $F_\omega(B, C) := \omega(B, C)$ , is continuous (for simplicity, we use the same notation  $h$  for the Hausdorff metric on  $\mathcal{K}(X), \mathcal{K}(Y)$  and  $\mathcal{K}(Z)$ ).*

*Proof.* (i) By using Lemma 2.3, we have

$$\begin{aligned} & \sup_{\substack{x_1 \in B_1 \\ y_1 \in C_1}} \inf_{\substack{x_2 \in B_2 \\ y_2 \in C_2}} \rho(\omega(x_1, y_1), \omega(x_2, y_2)) \leq \text{Lip}(\omega) \sup_{\substack{x_1 \in B_1 \\ y_1 \in C_1}} \inf_{\substack{x_2 \in B_2 \\ y_2 \in C_2}} \max\{d(x_1, x_2), \delta(y_1, y_2)\} \\ & \leq \text{Lip}(\omega) \max\{D(B_1, B_2), D(C_1, C_2)\} \leq \text{Lip}(\omega) \max\{h(B_1, B_2), h(C_1, C_2)\} \end{aligned}$$

$$= \text{Lip}(\omega)h_2((B_1, C_1), (B_2, C_2)).$$

Therefrom, we deduce

$$D(\omega(B_1, C_1), \omega(B_2, C_2)) \leq \text{Lip}(\omega)h_2((B_1, C_1), (B_2, C_2)),$$

consequently the assertion (i) follows.

(ii) We us consider the sequence of sets  $(B_n, C_n)_n$  with  $B_n \in \mathcal{K}(X)$ ,  $C_n \in \mathcal{K}(Y)$  converging to  $(B, C) \in \mathcal{K}(X) \times \mathcal{K}(Y)$  with respect to the Hausdorff metric  $h_2$ . Then  $h(B_n, B) \rightarrow 0$  and  $h(C_n, C) \rightarrow 0$ . We suppose by reductio ad absurdum that  $(\omega(B_n, C_n))_n$  do not converging to  $\omega(B, C)$ . Then there exists  $\varepsilon_0 > 0$  such that

$$\forall n \in \mathbb{N}, \exists k_n \geq n \text{ such that } h(\omega(B_{k_n}, C_{k_n}), \omega(B, C)) \geq \varepsilon_0.$$

That is, for each  $n = 1, 2, \dots$ , one has

$$\sup_{\substack{x_1 \in B_{k_n} \\ y_1 \in C_{k_n}}} \inf_{\substack{x_2 \in B \\ y_2 \in C}} \rho(\omega(x_1, y_1), \omega(x_2, y_2)) \geq \varepsilon_0$$

or

$$\sup_{\substack{x_2 \in B \\ y_2 \in C}} \inf_{\substack{x_1 \in B_{k_n} \\ y_1 \in C_{k_n}}} \rho(\omega(x_1, y_1), \omega(x_2, y_2)) \geq \varepsilon_0.$$

Case I: By considering, eventually, a subsequence, we can suppose that, for any  $n \geq 1$ ,

$$\sup_{\substack{x_1 \in B_n \\ y_1 \in C_n}} \inf_{\substack{x_2 \in B \\ y_2 \in C}} \rho(\omega(x_1, y_1), \omega(x_2, y_2)) \geq \varepsilon_0.$$

So, for each  $n \geq 1$ , one can find  $(x_n, y_n) \in (B_n, C_n)$  such that, for any  $(x', y') \in (B, C)$ , we have

$$\rho(\omega(x_n, y_n), \omega(x', y')) \geq \varepsilon_0. \quad (2.3)$$

Now, let be  $\varepsilon > 0$ ,  $\varepsilon < \varepsilon_0$ . By the uniform continuity of  $\omega$ , there is  $\eta > 0$  so that

$$\begin{aligned} \forall (x, y), (x', y') \in X \times Y \text{ with } \max\{d(x, x'), \delta(y, y')\} < \eta \\ \Rightarrow \rho(\omega(x, y), \omega(x', y')) < \varepsilon. \end{aligned}$$

Next, by hypothesis, we have  $B_n \rightarrow B$  and  $C_n \rightarrow C$ . Thence, there is  $n_\eta \geq 1$  so that  $h(B_n, B) < \eta$  and  $h(C_n, C) < \eta$  for all  $n \geq n_\eta$ . It follows that

$$\sup_{x \in B_n} \left( \inf_{x' \in B} d(x, x') \right) < \eta$$

and as well as  $\sup_{y \in C_n} \left( \inf_{y' \in C} \delta(y, y') \right) < \eta$ , for any  $n \geq n_\eta$ . Therefrom, for any  $x \in B_n$  and any  $y \in C_n$ , there exist  $x' \in B$  and  $y' \in C$  with  $d(x, x') < \eta$  and more  $\delta(y, y') < \eta$ .

In particular,  $d(x_n, x') < \eta$ ,  $\delta(y_n, y') < \eta$  and hence

$$\rho(\omega(x_n, y_n), \omega(x'_n, y'_n)) < \varepsilon < \varepsilon_0,$$

contradicting (2.3).

Case **II**: We proceed in a similar way as in the preceding case denoting, for simplicity,  $(B_n, C_n)_n$  instead of  $(B_{k_n}, C_{k_n})_n$ .

Suppose that  $\sup_{\substack{x_2 \in B \\ y_2 \in C}} \inf_{\substack{x_1 \in B_n \\ y_1 \in C_n}} \rho(\omega(x_1, y_1), \omega(x_2, y_2)) \geq \varepsilon_0$ , for all  $n \geq 1$ . Then, for every  $n = 1, 2, \dots$ , there is  $(x_0, y_0) \in (B, C)$  such that, for any  $(x', y') \in (B_n, C_n)$ , one has

$$\rho(\omega(x_0, y_0), \omega(x', y')) \geq \varepsilon_0. \quad (2.4)$$

At the same time, for an arbitrary  $\varepsilon > 0$ ,  $\varepsilon < \varepsilon_0$ , there exists  $\eta > 0$  such that, for all  $(x, y), (x', y') \in X \times Y$  with  $\max\{d(x, x'), \delta(y, y')\} < \eta$ , we have

$$\rho(\omega(x, y), \omega(x', y')) < \varepsilon. \quad (2.5)$$

Next,

$$B_n \rightarrow B, C_n \rightarrow C \Rightarrow \exists n_\eta \in \mathbb{N} \text{ such that}$$

$$\sup_{\substack{x \in B \\ y \in C}} \inf_{\substack{x' \in B_n \\ y' \in C_n}} \rho(\omega(x, y), \omega(x', y')) < \eta, \forall n \geq n_\eta,$$

namely, for any  $(x, y) \in B \times C$ , there is  $(x', y') \in B_n \times C_n$  so that  $d(x, x') < \eta$  and  $\delta(y, y') < \eta$ . In particular, taking  $x = x_0$  and  $y = y_0$ , we have, in view of (2.5),  $\delta(\omega(x'_0), \omega(x)) < \varepsilon$  contradicting the relation (2.4).

Consequently,  $F_\omega$  is continuous in the arbitrary point  $(B, C)$ , so it is continuous.

The proof is complete.  $\square$

As a consequence of Lemma 2.1 and the above theorem, we have obviously:

**COROLLARY 2.1.** *We consider a sequence of Lipschitz functions  $\omega_n : X \times Y \rightarrow Z$ , the metric space  $(Z, \rho)$  being compact. We define a set function  $\mathcal{S} : \mathcal{K}(X) \times \mathcal{K}(Y) \rightarrow \mathcal{K}(Z)$  by*

$$\mathcal{S}(B, C) := \overline{\bigcup_{n \geq 1} \omega_n(B, C)}. \quad (2.6)$$

*Then  $\text{Lip}(\mathcal{S}) \leq \sup_n \text{Lip}(\omega_n)$ . In particular, if  $\sup_n \text{Lip}(\omega_n) < \infty$ , then  $\mathcal{S}$  is a Lipschitz function.*

From Theorem 2.1 and Lemma 2.1 it follows easily:

**REMARK 2.1.** *If we have a finite set of uniform continuous functions  $(\omega_n)_{n=1}^N$ , then the set function  $\mathcal{S}_N : \mathcal{K}(X) \times \mathcal{K}(Y) \rightarrow \mathcal{K}(Z)$ ,  $\mathcal{S}_N(B, C) = \bigcup_{n=1}^N \omega_n(B, C)$ , is continuous.*

## 2.2 Iterated Function Systems, Countable Iterated Function Systems

Let us consider a complete metric space  $(X, d)$ . A finite set of contractions  $\omega_n : X \rightarrow X$ ,  $n = 1, 2, \dots, N$ , is called iterated function system, shortly IFS. Then the set function  $\mathcal{S}_N : \mathcal{K}(X) \rightarrow \mathcal{K}(X)$ ,  $\mathcal{S}_N(B) := \bigcup_{n=1}^N \omega_n(B)$ , is a contraction in the space  $(\mathcal{K}(X), h)$ , whose unique set-fixed point  $A_N$  is named the attractor of the considered IFS.

Now, assume that  $(X, d)$  is a compact metric space and we consider a countable system of contractions  $(\omega_n)_n$  on  $X$  into itself with contractivity factors, respectively  $r_n$ ,  $n = 1, 2, \dots$ . We say that  $(\omega_n)_n$  is a countable iterated function system (abbreviated CIFS) if  $\sup_n r_n < 1$ . The associated set function  $\mathcal{S} : \mathcal{K}(X) \rightarrow \mathcal{K}(X)$  given by

$$\mathcal{S}(B) = \overline{\bigcup_{n \geq 1} \omega_n(B)},$$

for any  $B \in \mathcal{K}(X)$ , is a contraction having the contractivity factor  $r = \sup_n r_n$ . According to the Banach contraction principle, there is a unique  $A \in \mathcal{K}(X)$  such that  $\mathcal{S}(A) = A$ , namely the attractor of the considered CIFS.

The attractor of CIFS  $(\omega_n)_n$  can be approximated in the Hausdorff metric by the attractors of partial IFSs,  $N \geq 1$ ,  $(\omega_n)_{n=1}^N$  ([6, Th.2.3]). Also, concerning the matter of the attractor  $A$ , one has the following result ([6, Cor.2.1]):

**LEMMA 2.4.** *The attractor of CIFS  $(\omega_n)_n$  represents the adherence of the set of fixed points  $e_{i_1 \dots i_p}$  of all contractions  $\omega_{i_1 \dots i_p}$ ,  $p \geq 1$  and  $i_j \geq 1$ , where  $\omega_{i_1 \dots i_p} := \omega_{i_1} \circ \dots \circ \omega_{i_p}$ . In symbols,*

$$A = \overline{\{e_{i_1 \dots i_p}; p, i_j = 1, 2, \dots\}}.$$

We us consider further a metric space  $(T, d_T)$  and a sequences of mappings  $\omega_n : T \times X \rightarrow X$  and  $r_n : T \rightarrow [0, 1)$ ,  $n = 1, 2, \dots$ , obeying the following three properties:

- (i) for each  $t \in T$ ,  $d(\omega_n(t, x), \omega_n(t, y)) \leq r_n(t)d(x, y)$ , for any  $x, y \in X$ ,  $n \geq 1$ ;
- (ii) there is  $C > 0$  such that  $d_T(\omega_n(t, x), \omega_n(s, x)) \leq C d_T(t, s)$ , for all  $x \in X$ ,  $t, s \in T$ ,  $n \geq 1$ ;
- (iii)  $\sup_{n, t} r_n(t) < 1$ .

We define  $\mathcal{S} : T \times \mathcal{K}(X) \rightarrow \mathcal{K}(X)$ ,  $\mathcal{S}(t, B) = \overline{\bigcup_{n \geq 1} \omega_n(t, B)}$ , for any  $t \in T$  and any  $B \in \mathcal{K}(X)$ . It follows that, for each  $t \in T$ ,  $\mathcal{S}(t, \cdot)$  is a contraction map on  $\mathcal{K}(X)$  with the contraction ratio  $r(t) = \sup_n r_n(t) < 1$ .

The following theorem tell us that the attractor of a CIFS depends continuously on the parameter  $t \in T$  ([8, Th.6]).

**THEOREM 2.2.** *Under the above conditions, the function  $t \mapsto A(t)$  is continuous from  $T$  into  $\mathcal{K}(X)$ , where, for  $t \in T$ ,  $A(t)$  means the attractor of the CIFS  $(\omega_n(t, \cdot))_{n \geq 1}$ .*

### 3 Generalized Countable Iterated Function Systems

Throughout in this section  $(X, d)$  will be a compact metric space and we consider the metric

$$d_2((x_1, y_1), (x_2, y_2)) = \max\{d(x_1, x_2), d(y_1, y_2)\}$$

on  $X \times X$ . Then  $(X \times X, d_2)$  is a compact metric space.

#### 3.1 Definition. Continuity with respect to a parameter

**DEFINITION 3.1.** *A sequence of contractions  $\omega_n : X \times X \rightarrow X$  with  $\sup_n \text{Lip}(\omega_n) < 1$  is said to be a generalized countable iterated function system of order two on  $X$ , abbreviated GCIFS.*

*If  $N \geq 1$  is an integer, then the finite family of functions  $(\omega_n)_{n=1}^N$  is called the partial generalized iterated function system (GIFS) of  $(\omega_n)_n$ .*

By corollary 2.1, it follows immediately that  $\mathcal{S} : \mathcal{K}(X) \times \mathcal{K}(X) \rightarrow \mathcal{K}(X)$  given by (2.6) is a contraction having the contractivity factor  $r = \sup r_n$ , where  $r_n$  means the contraction ratio of  $\omega_n$ ,  $n = 1, 2, \dots$ . At the same time, the set function

$$\mathcal{S}_N : \mathcal{K}(X) \times \mathcal{K}(X) \rightarrow \mathcal{K}(X), \quad \mathcal{S}_N(B, C) := \bigcup_{n=1}^N \omega_n(B, C)$$

is a contraction with the contractivity factor  $r_N = \max_{1 \leq n \leq N} r_n$ .

**THEOREM 3.1.** [4, Th.2.1] (Banach Contraction Principle) *Let  $(X, d)$  be a complete metric space and  $f : X \times X \rightarrow X$  be a contraction with contractivity factor  $c \in [0, 1)$ . Then there exists a unique  $e \in X$  such that  $f(e, e) = e$ . Moreover, for any  $x_0, x_1 \in X$ , the sequence  $(x_k)_{k \geq 0}$  defined by  $x_{k+1} = f(x_k, x_{k-1})$ ,  $k \geq 1$ , is convergent to  $e$ .*

*Furthermore,*

$$d(x_k, e) \leq \frac{2c^{\lfloor k/2 \rfloor}}{1-c} \max\{d(x_0, x_1), d(x_1, x_2)\}.$$

We say that  $e \in X$  with  $e = f(e, e)$  is a fixed point of  $f$ .

In view of the aforesaid, one obtain:



**THEOREM 3.2.** *Let  $(X, d)$  be a compact metric space and  $(\omega_n)_n$  be a GCIFS on  $X$ . Then there is a unique  $A \in \mathcal{K}(X)$  such that  $\mathcal{S}(A, A) = A$ .*

*Moreover, if  $B_0$  and  $B_1$  be arbitrary sets in  $\mathcal{K}(X)$ , then the sequence  $(B_k)_{k \geq 0}$  given by  $B_{k+1} = \mathcal{S}(B_k, B_{k-1})$ ,  $k \geq 1$ , is converging to  $A$ .*

*Similarly, there uniquely exists a set  $A_N \in \mathcal{K}(X)$  with  $\mathcal{S}_N(A_N, A_N) = A_N$ .*

The sets  $A, A_N \in \mathcal{K}(X)$  given in the above theorem are called the **attractor** of GCIFS  $(\omega_n)_n$ , respectively of GIFS  $(\omega_n)_{n=1}^N$ .

We give now a construction of a CIFS associated to the considered GCIFS. For each  $n \geq 1$ , we put

$$\tilde{\omega}_n : X \rightarrow X, \tilde{\omega}_n(x) := \omega_n(x, x).$$

Then  $\tilde{\omega}_n$  is a contraction map having the contraction ratio less than  $\text{Lip}(\omega_n)$  and the same fixed point as  $\omega_n$ . The CIFS  $(\tilde{\omega}_n)_n$  is said to be **associated** to GCIFS. If  $\tilde{A} \in \mathcal{K}(X)$  is the attractor of the associated CIFS, then  $\tilde{A} \subset A$ .

Let us define  $\tilde{\mathcal{S}} : \mathcal{K}(X) \rightarrow \mathcal{K}(X)$ ,  $\tilde{\mathcal{S}}(B) := \mathcal{S}(B, B)$ . Then  $\tilde{\mathcal{S}}$  is also a contraction (see Corollary 2.1 and Lemma 2.3) and its unique set-fixed point is  $A$ . Further, according to the Banach contraction principle, for every  $C \in \mathcal{K}(X)$ , the sequence  $(\tilde{\mathcal{S}}^k(C))_k$  converges to  $A$ . More precisely, one has

**LEMMA 3.1.** *Let us consider a set  $C_0 \in \mathcal{K}(X)$ . Then the sequence  $(C_k)_k$  given by  $C_k := \mathcal{S}(C_{k-1}, C_{k-1})$ ,  $k \geq 1$ , is converging in the Hausdorff metric to the attractor  $A$  of the considered GCIFS.*

*Furthermore, we have*

$$h(A, C_k) \leq \frac{r^{k+1}}{1-r} h(C_0, \mathcal{S}(C_0, C_0)).$$

In view of the aforesaid and Lemma 2.4, one can observe that the attractor of the GCIFS contain the fixed points of its contractions.

**PROPOSITION 3.1.** *The attractor  $A$  of a GCIFS  $(\omega_n)_n$  contains the fixed points of all  $\omega_n$ ,  $n = 1, 2, \dots$ . Furthermore, one has*

$$A \supset \overline{\{e_{i_1 \dots i_p}; p, i_j = 1, 2, \dots\}},$$

where  $e_{i_1 \dots i_p}$  denotes the unique fixed point of the contraction  $\tilde{\omega}_{i_1} \circ \dots \circ \tilde{\omega}_{i_p}$ .

**REMARK 3.1.** *Every CIFS can be seen as a GCIFS. Indeed, if  $(\omega_n)_n$  constitutes a CIFS on  $X$ , then the sequence of mappings  $\bar{\omega}_n : X \times X \rightarrow X$  defined by  $\bar{\omega}_n(x, y) := \omega_n(x)$  is a GCIFS having the same attractor. Thence, the GCIFS represents an improvement of CIFS.*

Next, we will prove that, if the contractions of a GCIFS is Lipschitz maps with respect to a parameter and the supremum of the Lipschitz constants is finite, then the attractor depends continuously with respect to the respective parameter.

**THEOREM 3.3.** *Let us consider a metric space  $(T, d_T)$  and the sequences of maps  $\omega_n : T \times X \times X \rightarrow X$  and  $r_n : T \rightarrow [0, 1]$ ,  $n = 1, 2, \dots$ , satisfying the following requirements:*

(i) *for each  $t \in T$ , we have*

$$d(\omega_n(t, x_1, y_1), \omega_n(t, x_2, y_2)) \leq r_n(t) d_2((x_1, y_1), (x_2, y_2)),$$

*for any  $x_1, x_2, y_1, y_2 \in X$  and any  $n \geq 1$ ;*

(ii) *there is  $C > 0$  such that*

$$d(\omega_n(t, x, y), \omega_n(s, x, y)) \leq C d_T(t, s),$$

*for all  $x, y \in X$ ,  $t, s \in T$ ,  $n \geq 1$ ;*

(iii)  *$r := \sup_{n,t} r_n(t) < 1$ .*

*Then, if  $A(t)$  denotes the attractor of the GCIFS  $(\omega_n(t, \cdot, \cdot))_n$ , then the mapping  $t \mapsto A(t)$  has Lipschitz constant  $\frac{C}{1-r}$ , hence it is uniform continuous.*

*Proof.* Let us define  $\mathcal{S} : T \times \mathcal{K}(X) \times \mathcal{K}(X) \longrightarrow \mathcal{K}(X)$ ,

$$\mathcal{S}(t, B, C) = \overline{\bigcup_{n \geq 1} \omega_n(t, B, C)},$$

for any  $t \in T$ ,  $B, C \in \mathcal{K}(X)$ . It follows that, for each  $t \in T$ ,  $\mathcal{S}(t, \cdot, \cdot)$  is a contraction mapping on  $\mathcal{K}(X) \times \mathcal{K}(X)$  with the contraction ratio  $\sup_n r_n(t) < 1$ .

We will first show that

$$h(\omega_n(t, M, M), \omega_n(s, M, M)) \leq C d_T(t, s), \quad \forall M \subset X, \quad \forall t, s \in T. \quad (3.7)$$

By symmetry, it is enough to prove

$$D(\omega_n(t, M, M), \omega_n(s, M, M)) \leq C d_T(t, s). \quad (3.8)$$

Let be  $t, s \in T$ . Choose  $w \in \omega_n(t, M, M)$ . Then, there are  $x, y \in M$  such that  $w = \omega_n(t, x, y)$ . Let be  $z = \omega_n(s, x, y) \in \omega_n(s, M, M)$ . By (ii) we deduce that  $d(w, z) \leq C d_T(t, s)$ , hence  $\sup_{w \in \omega_n(t, M, M)} \inf_{z \in \omega_n(s, M, M)} d(w, z) \leq C d_T(t, s)$  which proves (3.8).

Next, by using Theorem 2.1(i) and Lemma 2.3, one has

$$h(\omega_n(s, M, M), \omega_n(s, N, N)) \leq r_n(s) h(M, N), \quad \forall M, N \subset X, s \in T, n \geq 1. \quad (3.9)$$

Now, for every  $t, s \in T$ , taking respectively  $A(t)$ ,  $A(s)$  in the place of  $M$  and  $N$  in (3.7) and (3.9), we obtain

$$\begin{aligned}
h(A(t), A(s)) &= h\left(\overline{\bigcup_{n \geq 1} \omega_n(t, A(t), A(t))}, \overline{\bigcup_{n \geq 1} \omega_n(s, A(s), A(s))}\right) \\
&\leq \sup_n h(\omega_n(t, A(t), A(t)), \omega_n(s, A(s), A(s))) \\
&\leq \sup_n h(\omega_n(t, A(t), A(t)), \omega_n(s, A(t), A(t))) \\
&\quad + \sup_n h(\omega_n(s, A(t), A(t)), \omega_n(s, A(s), A(s))) \\
&\leq Cd_T(t, s) + rh(A(t), A(s)).
\end{aligned}$$

It follows  $h(A(t), A(s)) \leq \frac{C}{1-r} d_T(t, s)$  which implies the Lipschitz property of  $A(t)$  completing the proof.  $\square$

Under the preceding hypothesis, the associated CIFS  $\tilde{\omega}_n(t, x) = \omega_n(t, x, x)$ ,  $n \geq 1$ , obeys the conditions of Theorem 2.2. So, the attractor  $\tilde{A}(t)$  of the CIFS  $(\tilde{\omega}_n(t, \cdot))_n$ , depends continuously on the parameter  $t$ .

### 3.2 Approximation of the attractor of a GCIFS

LEMMA 3.2. *Under the conditions of Theorem 3.2, we have*

$$A_N \xrightarrow[N]{} A,$$

with respect to the Hausdorff metric.

*Proof.* Let be  $\varepsilon > 0$ . By applying Lemma 2.2 (a) to the increasing sequence  $\left(\bigcup_{n=1}^N \omega_n(A, A)\right)_N$ , we can find  $N_\varepsilon \geq 1$  such that, for any  $N \geq N_\varepsilon$ , we have

$$h\left(\bigcup_{n=1}^N \omega_n(A, A), \overline{\bigcup_{n \geq 1} \omega_n(A, A)}\right) < \varepsilon(1 - \lambda), \quad (3.10)$$

where  $\lambda = \sup_n r_n$ . Thereinafter, for every  $N \geq N_\varepsilon$ ,

$$\begin{aligned}
h(A_N, A) &= h(\mathcal{S}_N(A_N, A_N), \mathcal{S}(A, A)) = h\left(\bigcup_{n=1}^N \omega_n(A_N, A_N), \overline{\bigcup_{n \geq 1} \omega_n(A, A)}\right) \\
&\leq h\left(\bigcup_{n=1}^N \omega_n(A_N, A_N), \bigcup_{n=1}^N \omega_n(A, A)\right) + h\left(\bigcup_{n=1}^N \omega_n(A, A), \overline{\bigcup_{n \geq 1} \omega_n(A, A)}\right) \\
&\leq \sup_{1 \leq n \leq N} h(\omega_n(A_N, A_N), \omega_n(A, A)) + \varepsilon(1 - \lambda) \leq \lambda h(A_N, A) + \varepsilon(1 - \lambda).
\end{aligned}$$

Consequently, by using (3.10), Lemma 2.3 and Theorem 2.1, one obtain  $h(A_N, A) < \varepsilon$ , completing the proof.  $\square$

LEMMA 3.3. *Let us consider two arbitrary sets  $B_0, B_1 \in \mathcal{K}(X)$  and, for each  $k \geq 1$ ,  $B_{k+1}^N = \mathcal{S}_N(B_k^N, B_{k-1}^N)$  and, respectively  $B_{k+1} = \mathcal{S}(B_k, B_{k-1})$ . Then  $B_k^N \xrightarrow[N]{} B_k$ , for any  $k = 0, 1, \dots$ .*

*Proof.* Firstly, by using the same argument as in the proof of (3.10), for some  $k \geq 1$  and  $\varepsilon > 0$ , there is  $N_\varepsilon \geq 1$  such that, whenever  $N \geq N_\varepsilon$ , one has

$$h\left(\bigcup_{n=1}^N \omega_n(B_k, B_{k-1}), \overline{\bigcup_{n \geq 1} \omega_n(B_k, B_{k-1})}\right) < \frac{\varepsilon}{2}.$$

Next, we proceed by mathematical induction with respect to  $k$ . We suppose that  $h(B_m^N, B_m) \xrightarrow[N]{} 0$  for all  $m \leq k$ . Hence there is  $N^* \geq N_\varepsilon$  such that

$$h(B_m^N, B_m) < \frac{\varepsilon}{2 \sup r_n}$$

and withal  $h(B_{m-1}^N, B_{m-1}) < \frac{\varepsilon}{2 \sup r_n}$ , for any  $N \geq N^*$ .

In view of the aforesaid, we find

$$\begin{aligned} h(B_{k+1}^N, B_{k+1}) &= h(\mathcal{S}(B_k^N, B_{k-1}^N), \mathcal{S}(B_k, B_{k-1})) \\ &= h\left(\bigcup_{n=1}^N \omega_n(B_k^N, B_{k-1}^N), \overline{\bigcup_{n \geq 1} \omega_n(B_k, B_{k-1})}\right) \\ &\leq h\left(\bigcup_{n=1}^N \omega_n(B_k^N, B_{k-1}^N), \bigcup_{n=1}^N \omega_n(B_k, B_{k-1})\right) \\ &\quad + h\left(\bigcup_{n=1}^N \omega_n(B_k, B_{k-1}), \overline{\bigcup_{n \geq 1} \omega_n(B_k, B_{k-1})}\right) \\ &\leq \sup_n r_n \cdot \max\{h(B_k^N, B_k), h(B_{k-1}^N, B_{k-1})\} + \frac{\varepsilon}{2} < \varepsilon. \end{aligned}$$

□

According to Lemmas 3.2, 3.3 and Theorem 3.2 we deduce immediately the following result which is useful to approximate the attractor of a GCIFS.

THEOREM 3.4. *Let  $A$  be the attractor of a GCIFS  $(\omega_n)_{n \geq 1}$  and  $B_0, B_1 \in \mathcal{K}(X)$  be some arbitrary sets. Then  $A$  is approximated with respect to the Hausdorff metric by the attractors  $A_N$  of the associated partial GIFS  $(\omega_n)_{n=1}^N$  and, moreover, it is also approximated by the sequence  $(B_k)_{k \geq 0}$ , where  $B_k = \mathcal{S}(B_{k-1}, B_{k-2})$  for  $k \geq 2$ .*

*More precisely, we have the following diagram*

$$\begin{array}{ccc} B_k^N & \xrightarrow[k]{} & A_N \\ \downarrow^N & & \downarrow^N \\ B_k & \xrightarrow[k]{} & A \end{array}$$

Another way to approximate the attractor of GCIFS are described below.

LEMMA 3.4. *Let us consider two sequences of sets  $(B_k)_k$  and  $(C_k)_k$  from  $\mathcal{K}(X)$  converging with respect to the Hausdorff metric to  $B$ , respectively to  $C$ , where  $B, C \in \mathcal{K}(X)$ . Then  $\mathcal{S}_k(B_k, C_k) \xrightarrow[k]{k} \mathcal{S}(B, C)$ .*

*Particulary, if  $B_k = C_k \xrightarrow[k]{k} A$  ( $A$  being the attractor of the GCIFS), then*

$$\mathcal{S}_k(B_k, B_k) \xrightarrow[k]{k} \mathcal{S}(A, A) = A.$$

*Proof.* Let  $\varepsilon > 0$  be arbitrary. Then, there exists  $k_\varepsilon \geq 1$  such that

$$h_2((B_k, C_k), (B, C)) = \max \{h(B_k, B), h(C_k, C)\} < \frac{\varepsilon}{2}, \quad \forall k \geq k_\varepsilon.$$

According to Theorem 2.1, one has

$$\begin{aligned} h\left(\bigcup_{n=1}^k \omega_n(B_k, C_k), \bigcup_{n=1}^k \omega_n(B, C)\right) &\leq \max_{1 \leq n \leq k} h(\omega_n(B_k, C_k), \omega_n(B, C)) \\ &\leq \sup_n r_n h_2((B_k, C_k), (B, C)) < \frac{\varepsilon}{2}, \quad \forall k \geq k_\varepsilon. \end{aligned} \quad (3.11)$$

By Lemma 2.2 (a) we can find  $K_\varepsilon \geq k_\varepsilon$  such that, for any  $k \geq K_\varepsilon$ ,

$$h\left(\bigcup_{n=1}^k \omega_n(B, C), \overline{\bigcup_{n \geq 1} \omega_n(B, C)}\right) < \frac{\varepsilon}{2}. \quad (3.12)$$

Finally, with (3.11) and (3.12), we have

$$\begin{aligned} h(\mathcal{S}_k(B_k, C_k), \mathcal{S}(B, C)) &= h\left(\bigcup_{n=1}^k \omega_n(B_k, C_k), \overline{\bigcup_{n \geq 1} \omega_n(B, C)}\right) \\ &\leq h\left(\bigcup_{n=1}^k \omega_n(B_k, C_k), \bigcup_{n=1}^k \omega_n(B, C)\right) + h\left(\bigcup_{n=1}^k \omega_n(B, C), \overline{\bigcup_{n \geq 1} \omega_n(B, C)}\right) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

which implies the assertion of statement.  $\square$

LEMMA 3.5. *We suppose that  $B_0, B_1 \in \mathcal{K}(X)$ ,  $B_0 \subset B_1 \subset \mathcal{S}_N(B_0, B_1)$ , for each  $N \geq 1$ . We consider further the sequence  $(B_k^N)_{k, N}$  where  $B_0^N = B_0$ ,  $B_1^N = B_1$  and  $B_{k+1}^N = \mathcal{S}_N(B_k^N, B_{k-1}^N)$ , for all  $k, N \geq 1$ . Then  $B_k^k \subset B_{k+1}^{k+1}$  and*

$$A = \lim_k B_k^k = \overline{\bigcup_{k \geq 1} B_k^k}.$$

*Moreover,  $(A_N)_N$  is increasing and  $A = \overline{\bigcup_{N \geq 1} A_N}$*

*Proof.* Let be  $N \geq 1$ . It is easy to establish by induction that  $B_k^N \subset B_{k+1}^N$  for all  $k$ . Then, in view of Lemmas 2.2 and 3.2, one has

$$A_N = \lim_k B_k^N = \overline{\bigcup_{k \geq 1} B_k^N}.$$

Also it is obvious that  $B_k^N \subset B_k^{N+1}$ , for any  $k, N$ , thence  $A_N \subset A_{N+1}$ . Thus

$$A = \lim_{k, N} B_k^N = \overline{\bigcup_{k \geq 1} \bigcup_{N \geq 1} B_k^N}.$$

Consequently, the diagonal sequence  $(B_k^k)_k$  is increasing and

$$A = \lim_k B_k^k = \overline{\bigcup_{k \geq 1} B_k^k}.$$

□

REMARK 3.2. According to the preceding lemma, if the sets  $B_0, B_1$  are finite, then the attractor of a GCIFS can be approximated by the finite sets  $B_k^k$ ,  $k \geq 1$ . This fact is very instrumental to represent, in certain cases, that attractor with the aim of computer.

For every  $N \geq 1$ , we set  $F_N := \{e_1, \dots, e_N\}$  and  $B := \{e_1, e_2, \dots\}$ ,  $e_n$  being the fixed point of  $\omega_n$ . For a fixed integer  $N$  let be  $B_0^N = B_1^N = F_N$ . It is evident that  $F_N \subset \mathcal{S}_N(F_N, F_N)$ . Then  $B_{k+1}^N = \mathcal{S}_N(B_k^N, B_{k-1}^N) \nearrow_k A_N$ . Thus

$$\overline{B} = \lim_N F_N \subset \lim_N A_N = A.$$

Thereafter, we deduce that such a finite sets  $B_0 \subset B_1$  can be  $B_0 = B_1 = F_1$  which obviously obey the requirement of Lemma 3.5.

Finally, we give two examples of GCIFS on a compact subset of  $\mathbb{R}$ , respectively on  $\mathbb{R}^2$ .

EXAMPLE 3.1. Let us consider the compact metric space  $X := [0, 1] \subset \mathbb{R}$  equipped with the Euclidean metric. Let  $\alpha, p, q \in [0, 1]$  be any fixed constants with  $p+q \neq 0$  and  $(\alpha_n)_n$  be an increasing sequence of real numbers from  $[0, 1]$  converging to  $\alpha - \frac{p+q}{3}\alpha$ . From each  $n = 1, 2, \dots$ , we define the mapping  $\omega_n : [0, 1] \times [0, 1] \rightarrow [0, 1]$ , by

$$\omega_n(x, y) = \frac{n(px + qy)}{3n + 1} + \alpha_n.$$

Then  $(\omega_n)_n$  is a GCIFS whose attractor is  $[0, \alpha]$ .

*Proof.* Firstly, we make evident that  $\omega_n([0, 1] \times [0, 1]) = \left[0, \frac{n(p+q)}{3n+1} + \alpha_n\right] \subset [0, 1]$ , hence  $\omega_n$  is well defined.

Next, it is simple to see that  $\omega_n$  is a contraction having the contraction ratio  $r_n = \frac{(p+q)n}{3n+1}$ .

Moreover, since  $\frac{(p+q)n\alpha}{3n+1} + \alpha_n \nearrow \alpha$ , it follows

$$\mathcal{S}(A, A) = \overline{\bigcup_{n \geq 1} \omega_n(A, A)} = \overline{\bigcup_{n \geq 1} \left[0, \frac{(p+q)n\alpha}{3n+1} + \alpha_n\right]} = \overline{[0, \alpha]} = A,$$

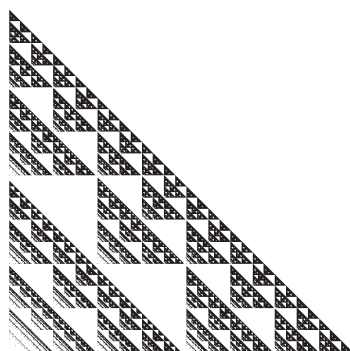
where  $A = [0, \alpha]$ . □

We present now as example a fractal of Sierpinski-infinite type as attractor of a proper GCIFS by generalizing a construction from [6].

EXAMPLE 3.2. Let  $X := \{(x, y) \in \mathbb{R}^2; 0 \leq x \leq 1, 0 \leq y \leq 1 - x\}$  be the plane surface of the closed triangle having its vertices in the points  $(0, 0)$ ,  $(0, 1)$ ,  $(1, 0)$ . Next, we consider an integer  $p \geq 2$ ,  $q \in [0, 1]$  and the contractions  $\omega_{ij} : X \times X \rightarrow X$  defined by

$$\begin{aligned} & \omega_{ij}((x_1, y_1), (x_2, y_2)) \\ &= \left( \frac{1}{p^i} (qx_1 + (1-q)x_2) + (j-1) \frac{1}{p^i}, \frac{1}{p^i} (qy_1 + (1-q)y_2) + \left( \frac{p^i - 1}{p-1} - j \right) \frac{1}{p^i} \right) \end{aligned}$$

for all  $i = 1, 2, \dots$ ,  $j = 1, 2, \dots, \frac{p^i - 1}{p-1}$ . Then  $(\omega_{i,j})_{i,j}$  constitutes a GCIFS whose attractor is given in the following figure.



The attractor associated to the considered GCIFS for  $p = 2$

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