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# ON QUASI-$M$-HYPONORMAL OPERATORS

**Young Min Han and Ju Hee Son**

**Abstract**

An operator $T$ is called *quasi-$M$-hyponormal* if there exists a positive real number $M$ such that $T^*(M^2(T-\lambda)^*(T-\lambda))T \geq T^*(T-\lambda)(T-\lambda)^*T$ for all $\lambda \in \mathbb{C}$, which is a generalization of $M$-hyponormality. In this paper, we consider the local spectral properties for quasi-$M$-hyponormal operators and Weyl type theorems for algebraically quasi-$M$-hyponormal operators, respectively. It is also proved that if $T$ is an algebraically quasi-$M$-hyponormal operator, then the spectral mapping theorem holds for the Weyl spectrum and for the essential approximate point spectrum.

## 1 Properties for quasi-$M$-hyponormal operators

Throughout this note we assume that $\mathcal{H}$ is an infinite dimensional separable Hilbert space. Let $B(\mathcal{H})$ denote the algebra of bounded linear operators acting on $\mathcal{H}$. If $T \in B(\mathcal{H})$ we shall write $N(T)$ and $R(T)$ for the null space and range of $T$. A closed subspace $\mathcal{M}$ of $\mathcal{H}$ is $T$-*invariant* (or $\mathcal{M}$ is *invariant under* $T$) if $T(\mathcal{M}) \subseteq \mathcal{M}$. We say that an operator $T$ has a nontrivial invariant closed subspace $\mathcal{M}$ of $\mathcal{H}$ if $\mathcal{M}$ is closed, $T$-invariant, and $\{0\} \neq \mathcal{M} \neq \mathcal{H}$. A closed subspace $\mathcal{M}$ of $\mathcal{H}$ is called $T$-*hyperinvariant* if $S(\mathcal{M}) \subseteq \mathcal{M}$ for every $S \in B(\mathcal{H})$ such that $TS = ST$.

**Definition 1.1.** Let $T \in B(\mathcal{H})$.

(1) An operator $T$ is said to be *hyponormal* if

$$T^*T \geq TT^*.$$

(2) An operator $T$ is said to be *$M$-hyponormal* if there exists a positive real number $M$ such that

$$M^2(T-\lambda)^*(T-\lambda) \geq (T-\lambda)(T-\lambda)^* \quad \text{for all } \lambda \in \mathbb{C}.$$

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(3) An operator $T$ is called *quasi-$M$-hyponormal* if there exists a positive real number $M$ such that

$$T^*(M^2(T-\lambda)^*(T-\lambda))T \geq T^*(T-\lambda)(T-\lambda)^*T \quad \text{for all } \lambda \in \mathbb{C}.$$

(4) An operator $T$ is called *algebraically quasi-$M$-hyponormal* if there exists a non-constant complex polynomial $p$ such that $p(T)$ is a quasi-$M$-hyponormal operator.

In general, the following implication holds:

$$\text{Hyponormal} \Longrightarrow M\text{-hyponormal} \Longrightarrow \text{quasi-}M\text{-hyponormal}$$
$$\Longrightarrow \text{algebraically quasi-}M\text{-hyponormal}.$$

The following facts follow from the above definition and some well known facts about $M$-hyponormal operators.

(i) If $T$ is $M$-hyponormal, then so is $T-\lambda$ for each $\lambda \in \mathbb{C}$.

(ii) If $T$ is $M$-hyponormal and $\mathcal{M} \subseteq \mathcal{H}$ is a closed $T$-invariant subspace, then $T|\mathcal{M}$ is $M$-hyponormal.

(iii) If $T$ is $M$-hyponormal, then $N(T-\lambda) \subseteq N(T-\lambda)^*$ for every $\lambda \in \mathbb{C}$.

(iv) Every quasinilpotent $M$-hyponormal operator is a zero operator.

(v) $T$ is $M$-hyponormal if and only if there exists a positive real number $M$ such that $M||(T-\lambda)x|| \geq ||(T-\lambda)^*x||$ for all $x \in \mathcal{H}$ and for all $\lambda \in \mathbb{C}$.

We begin with the following proposition.

**Proposition 1.2.** $T$ is quasi-$M$-hyponormal if and only if there exists a positive real number $M$ such that

$$M||(T-\lambda)Tx|| \geq ||(T-\lambda)^*Tx|| \quad \text{for all } x \in \mathcal{H} \text{ and for all } \lambda \in \mathbb{C}.$$

*Proof.* Suppose $T$ is quasi-$M$-hyponormal. Then there exists a positive real number $M$ such that

$$T^*(M^2(T-\lambda)^*(T-\lambda))T \geq T^*(T-\lambda)(T-\lambda)^*T \quad \text{for all } \lambda \in \mathbb{C}.$$

Let $x \in \mathcal{H}$ be arbitrary. Then we have

$$\begin{aligned} M^2||(T-\lambda)Tx||^2 &= < M^2T^*(T-\lambda)^*(T-\lambda)Tx, x > \\ &\geq < T^*(T-\lambda)(T-\lambda)^*Tx, x > \\ &= < (T-\lambda)^*Tx, (T-\lambda)^*Tx > \\ &= ||(T-\lambda)^*Tx||^2. \end{aligned}$$

Therefore $M||(T-\lambda)Tx|| \geq ||(T-\lambda)^*Tx||$ for all $x \in \mathcal{H}$ and for all $\lambda \in \mathbb{C}$.

Conversely, suppose that there exists a positive real number $M$ such that $M||(T-\lambda)Tx|| \geq ||(T-\lambda)^*Tx||$ for all $x \in \mathcal{H}$ and for all $\lambda \in \mathbb{C}$. Then we have

$$M^2||(T-\lambda)Tx||^2 \geq ||(T-\lambda)^*Tx||^2 \text{ for all } x \in \mathcal{H} \text{ and for all } \lambda \in \mathbb{C}.$$

Therefore $T^*(M^2(T-\lambda)^*(T-\lambda))T \geq T^*(T-\lambda)(T-\lambda)^*T$ for all $\lambda \in \mathbb{C}$, and hence $T$ is quasi-$M$-hyponormal. □

We have the following interesting corollaries from this proposition.

**Corollary 1.3.** Suppose $T$ is quasi-$M$-hyponormal and it has dense range. Then $T$ is $M$-hyponormal.

*Proof.* Since $T$ has dense range, $\overline{T(\mathcal{H})} = \mathcal{H}$. Let $y \in \mathcal{H}$. Then there exists a sequence $(x_n)$ in $\mathcal{H}$ such that $T(x_n) \longrightarrow y$ as $n \longrightarrow \infty$. Since $T$ is quasi-$M$-hyponormal, it follows from Proposition 1.2 that there exists a positive real number $M$ such that

$$M||(T-\lambda)Tx|| \geq ||(T-\lambda)^*Tx|| \text{ for all } x \in \mathcal{H} \text{ and for all } \lambda \in \mathbb{C}.$$

In particular,

$$M||(T-\lambda)Tx_n|| \geq ||(T-\lambda)^*Tx_n|| \text{ for all } n \in \mathcal{H} \text{ and for all } \lambda \in \mathbb{C}.$$

Therefore

$$\begin{aligned} M||(T-\lambda)y|| &= M||\lim_{n\to\infty}(T-\lambda)T(x_n)|| = \lim_{n\to\infty} M||(T-\lambda)T(x_n)|| \\ &\geq \lim_{n\to\infty}||(T-\lambda)^*T(x_n)|| = ||\lim_{n\to\infty}(T-\lambda)^*T(x_n)|| = ||(T-\lambda)^*y||. \end{aligned}$$

Therefore $T$ is $M$-hyponormal. □

**Corollary 1.4.** Suppose $T$ is quasi-$M$-hyponormal but not $M$-hyponormal. Then $T$ is not invertible.

**Corollary 1.5.** Suppose $T$ is such that $\alpha T + \beta$ is quasi-$M$-hyponormal for all scalars $\alpha$ and $\beta$. Then $T$ is $M$-hyponormal.

*Proof.* Since the spectrum of every operator is a nonempty compact subset of $\mathbb{C}$, we can find scalars $\alpha \neq 0$ and $\beta$ such that $S := \alpha T + \beta$ is invertible quasi-$M$-hyponormal. It follows from Corollary 1.3 that $S$ is $M$-hyponormal. Therefore $T = \frac{1}{\alpha}(S-\beta)$ is also $M$-hyponormal. □

**Corollary 1.6.** Suppose $T$ is nonzero quasi-$M$-hyponormal and it has no nontrivial $T$-invariant closed subspace. Then $T$ is $M$-hyponormal.

*Proof.* Since $T$ has no nontrivial invariant closed subspace, it has no nontrivial hyperinvariant subspace. But $N(T)$ and $\overline{R(T)}$ are hyperinvariant subspaces, and $T \neq 0$, hence $N(T) \neq \mathcal{H}$ and $\overline{R(T)} \neq \{0\}$. Therefore $N(T) = \{0\}$ and $\overline{R(T)} = \mathcal{H}$. In particular, $T$ has dense range. It follows from Corollary 1.3 that $T$ is $M$-hyponormal. □

It is well known that if $T$ is $M$-hyponormal and a closed subspace $\mathcal{M}$ of $\mathcal{H}$ is $T$-invariant, then $T|\mathcal{M}$ is $M$-hyponormal. We obtain a similar result for a quasi-$M$-hyponormal operator.

**Proposition 1.7.** The restriction $T|\mathcal{M}$ of a quasi-$M$-hyponormal operator $T$ to a $T$-invariant closed subspace $\mathcal{M}$ of $\mathcal{H}$ is quasi-$M$-hyponormal.

*Proof.* Let

$$T = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \quad \text{on } \mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp}.$$

Since $T$ is quasi-$M$-hyponormal, there is a positive real number $M$ such that

$$T^*(M^2(T-\lambda)^*(T-\lambda))T \geq T^*(T-\lambda)(T-\lambda)^*T \quad \text{for all } \lambda \in \mathbb{C}.$$

Therefore

$$M^2 \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}^* \begin{pmatrix} A-\lambda & C \\ 0 & B-\lambda \end{pmatrix}^* \begin{pmatrix} A-\lambda & C \\ 0 & B-\lambda \end{pmatrix} \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \geq$$
$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}^* \begin{pmatrix} A-\lambda & C \\ 0 & B-\lambda \end{pmatrix} \begin{pmatrix} A-\lambda & C \\ 0 & B-\lambda \end{pmatrix}^* \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}.$$

Hence

$$\begin{pmatrix} M^2A^*(A-\lambda)^*(A-\lambda)A - A^*(A-\lambda)(A-\lambda)^*A - A^*CC^*A & D \\ E & F \end{pmatrix} \geq 0,$$

for some operators $D$, $E$ and $F$. Therefore we have

$$M^2A^*(A-\lambda)^*(A-\lambda)A - A^*(A-\lambda)(A-\lambda)^*A - A^*CC^*A \geq 0.$$

But $A^*CC^*A = (C^*A)^*(C^*A) \geq 0$, hence

$$M^2A^*(A-\lambda)^*(A-\lambda)A \geq A^*(A-\lambda)(A-\lambda)^*A.$$

This shows that $A = T|\mathcal{M}$ is quasi-$M$-hyponormal. □

We give a structure theorem for quasi-$M$-hyponormal operators.

**Theorem 1.8.** Suppose $T$ is quasi-$M$-hyponormal and it does not have dense range. Then

$$T = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \text{ on } \mathcal{H} = \overline{T(\mathcal{H})} \oplus N(T^*),$$

where $A := T|\overline{T(\mathcal{H})}$ is an $M$-hyponormal operator.

*Proof.* Since $T$ does not have dense range, we can represent $T$ as the $2 \times 2$ operator matrix as follows:

$$T = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \text{ on } \mathcal{H} = \overline{T(\mathcal{H})} \oplus N(T^*).$$

We shall show that $A$ is an $M$-hyponormal operator. Let $\lambda \in \mathbb{C}$ be arbitrary. Since $T$ is quasi-$M$-hyponormal, there exists a positive real number $M$ such that $T^*(M^2(T-\lambda)^*(T-\lambda))T \geq T^*(T-\lambda)(T-\lambda)^*T$. Therefore $T^*(M^2(T-\lambda)^*(T-\lambda)-(T-\lambda)(T-\lambda)^*)T \geq 0$, and hence $< T^*(M^2(T-\lambda)^*(T-\lambda)-(T-\lambda)(T-\lambda)^*)Tx, x > \geq 0$ for all $x \in \mathcal{H}$. So $< (M^2(T-\lambda)^*(T-\lambda)-(T-\lambda)(T-\lambda)^*)Tx, Tx > \geq 0$ for all $x \in \mathcal{H}$. Therefore $A$ is an $M$-hyponormal operator. □

For an $M$-hyponormal operator $T$ we have $N(T-\lambda) \subseteq N(T-\lambda)^*$ for every $\lambda \in \mathbb{C}$. We have a similar result for quasi-$M$-hyponormal operators under restricted condition on $\lambda$ as follows.

**Proposition 1.9.** Suppose $T$ is quasi-$M$-hyponormal. Then $N(T-\alpha) \subseteq N(T-\alpha)^*$ for each $\alpha \neq 0$.

*Proof.* Suppose $x \in N(T-\alpha)$. Then $Tx = \alpha x$. Since $T$ is quasi-$M$-hyponormal, it follows from Proposition 1.2 that there exists a positive real number $M$ such that $M||(T-\lambda)Ty|| \geq ||(T-\lambda)^*Ty||$ for all $y \in \mathcal{H}$ and for all $\lambda \in \mathbb{C}$. In particular, $M||(T-\alpha)Tx|| \geq ||(T-\alpha)^*Tx||$. Since $Tx = \alpha x$, $0 = M|\alpha|||(T-\alpha)x|| = M||(T-\alpha)\alpha x|| \geq ||(T-\alpha)^*\alpha x|| = |\alpha|||(T-\alpha)^*x||$. Since $|\alpha| \neq 0$, $||(T-\alpha)^*x|| = 0$. Therefore $x \in N(T-\alpha)^*$, and hence $N(T-\alpha) \subseteq N(T-\alpha)^*$ for each $\alpha \neq 0$. □

From this proposition we obtain several corollaries.

**Corollary 1.10.** Suppose $T$ is quasi-$M$-hyponormal and $\alpha, \beta \in \sigma_p(T) \setminus \{0\}$ with $\alpha \neq \beta$. Then $N(T-\alpha) \perp N(T-\beta)$.

*Proof.* Let $x \in N(T-\alpha)$ and $y \in N(T-\beta)$. Then $Tx = \alpha x$ and $Ty = \beta y$. Therefore

$$\alpha < x, y > = < \alpha x, y > = < Tx, y > = < x, T^*y > = < x, \overline{\beta} y > = \beta < x, y > .$$

Hence $\alpha < x, y > = \beta < x, y >$, and so $(\alpha - \beta) < x, y > = 0$. But $\alpha \neq \beta$, hence $< x, y > = 0$. Therefore $N(T-\alpha) \perp N(T-\beta)$. □

For $T \in B(\mathcal{H})$, the smallest nonnegative integer $p$ such that $N(T^p) = N(T^{p+1})$ is called the *ascent* of $T$ and denoted by $p(T)$. If no such integer exists, we set $p(T) = \infty$. The smallest nonnegative integer $q$ such that $R(T^q) = R(T^{q+1})$ is called the *descent* of $T$ and denoted by $q(T)$. If no such integer exists, we set $q(T) = \infty$.

It is well known that if $T$ is an $M$-hyponormal operator, then $T-\lambda$ has finite ascent for each $\lambda \in \mathbb{C}$. The following corollary shows that this property holds for every quasi-$M$-hyponormal operator.

**Corollary 1.11.** Suppose $T$ is quasi-$M$-hyponormal. Then $T-\lambda$ has finite ascent for each $\lambda \in \mathbb{C}$.

*Proof.* Suppose $T$ is quasi-$M$-hyponormal. We consider two cases:
Case I: Suppose $x \in N(T^2)$. Then $||T^2x|| = 0$. Since $T$ is quasi-$M$-hyponormal, it follows from Proposition 1.2 that there exists a positive real number $M$ such that $M||T^2x|| \geq ||T^*Tx||$. Therefore $||T^*Tx|| = 0$, and hence $x \in N(T^*T)$. But $N(T^*T) = N(T)$, hence $x \in N(T)$. Therefore $N(T^2) \subseteq N(T)$, and so $N(T^2) = N(T)$.
Case II: Suppose $\lambda \neq 0$. It follows from Proposition 1.9 that $N(T-\lambda) \subseteq N(T^* - \bar{\lambda})$. Therefore we can represent $T-\lambda$ as the following $2\times 2$ operator matrix with respect to the decomposition $N(T-\lambda) \oplus N(T-\lambda)^{\perp}$:

$$T-\lambda = \begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix}.$$

Let $x \in N(T-\lambda)^2$. Write $x = y + z$, where $y \in N(T-\lambda)$ and $z \in N(T-\lambda)^{\perp}$. Then $0 = (T-\lambda)^2 x = (T-\lambda)^2 z$, so that $(T-\lambda)z \in N(T-\lambda) \cap N(T-\lambda)^{\perp} = \{0\}$, which implies that $z \in N(T-\lambda)$, and hence $x \in N(T-\lambda)$. Therefore $N(T-\lambda)^2 \subseteq N(T-\lambda)$, and hence $N(T-\lambda) = N(T-\lambda)^2$. □

We say that $T \in B(\mathcal{H})$ has the single valued extension property at $\lambda_0 \in \mathbb{C}$, abbreviated $T$ has SVEP at $\lambda_0$, if for every open neighborhood $U$ of $\lambda_0$ the only analytic function $f : U \longrightarrow \mathcal{H}$ which satisfies the equation

$$(T-\lambda)f(\lambda) = 0$$

is the constant function $f \equiv 0$ on $U$. The operator $T$ is said to have SVEP if $T$ has SVEP at every $\lambda \in \mathbb{C}$. It is well known that every $M$-hyponormal operator has SVEP. The following corollary shows that this result extends to quasi-$M$-hyponormal operators.

**Corollary 1.12.** Suppose $T$ is quasi-$M$-hyponormal. Then $T$ has SVEP.

*Proof.* Since $T-\lambda$ has finite ascent for each $\lambda \in \mathbb{C}$ by Corollary 1.11, it follows from [1, Theorem 3.8] that $T$ has SVEP. □

## 2 Weyl's theorem for algebraically quasi-$M$-hyponormal operators

Let $B_0(\mathcal{H})$ denote the ideal of compact operators acting on $\mathcal{H}$. If $T \in B(\mathcal{H})$ we shall write $\alpha(T) := \dim N(T)$, $\beta(T) := \dim N(T^*)$, and let $\sigma(T)$, $\sigma_p(T)$, $\sigma_a(T)$ and $p_0(T)$ denote the spectrum, point spectrum, approximate point spectrum and the set of poles of the resolvent of $T$, respectively. An operator $T \in B(\mathcal{H})$ is called *upper semi-Fredholm* if it has closed range and finite dimensional null space, and is called *lower semi-Fredholm* if it has closed range and its range has finite co-dimension. If $T \in B(\mathcal{H})$ is either upper or lower semi-Fredholm, then $T$ is called *semi-Fredholm.* The *index* of a semi-Fredholm operator $T \in B(\mathcal{H})$ is defined by

$$i(T) := \alpha(T) - \beta(T).$$

If both $\alpha(T)$ and $\beta(T)$ are finite, then $T$ is called *Fredholm.* $T \in B(\mathcal{H})$ is called *Weyl* if it is Fredholm of index zero, and *Browder* if it is Fredholm of finite ascent and descent. The essential spectrum $\sigma_e(T)$, the Weyl spectrum $\sigma_w(T)$ and the Browder spectrum $\sigma_b(T)$ of $T \in B(\mathcal{H})$ are defined by ([13],[14])

$$\sigma_e(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Fredholm}\},$$

$$\sigma_w(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl}\},$$

$$\sigma_b(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Browder}\},$$

respectively. Evidently

$$\sigma_e(T) \subseteq \sigma_w(T) \subseteq \sigma_b(T) = \sigma_e(T) \cup \operatorname{acc} \sigma(T),$$

where we write acc $K$ for the accumulation points of $K \subseteq \mathbb{C}$.

If we write iso $K := K \setminus \operatorname{acc} K$, then we let

$$\pi_{00}(T) := \{\lambda \in \operatorname{iso} \sigma(T) : 0 < \alpha(T - \lambda) < \infty \},$$

$$\pi_{00}^a(T) := \{\lambda \in \operatorname{iso} \sigma_a(T) : 0 < \alpha(T - \lambda) < \infty \},$$

$$p_{00}(T) := \sigma(T) \setminus \sigma_b(T).$$

We consider the sets

$$\Phi_+(\mathcal{H}) := \{T \in B(\mathcal{H}) : R(T) \text{ is closed and } \alpha(T) < \infty\},$$

$$\Phi_+^-(\mathcal{H}) := \{T \in B(\mathcal{H}) : T \in \Phi_+(\mathcal{H}) \text{ and } i(T) \leq 0\}.$$

By definition,

$$\sigma_{ea}(T) := \cap\{\sigma_a(T+K) : K \in B_0(\mathcal{H})\}$$

is the essential approximate point spectrum, and

$$\sigma_{ab}(T) := \cap\{\sigma_a(T+K) : TK = KT \text{ and } K \in B_0(\mathcal{H})\}$$

is the Browder essential approximate point spectrum.

**Definition 2.1.** Let $T \in B(\mathcal{H})$.

(1) *Weyl's theorem holds for* $T$ (in symbols, $T \in \mathcal{W}$) if

$$\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T).$$

(2) *Browder's theorem holds for* $T$ (in symbols, $T \in \mathcal{B}$) if

$$\sigma(T) \setminus \sigma_w(T) = p_{00}(T).$$

(3) *a-Weyl's theorem holds for* $T$ (in symbols, $T \in a\mathcal{W}$) if

$$\sigma_a(T) \setminus \sigma_{ea}(T) = \pi_{00}^a(T).$$

(4) *a-Browder's theorem holds for* $T$ (in symbols, $T \in a\mathcal{B}$) if

$$\sigma_{ea}(T) = \sigma_{ab}(T).$$

It is known ([14],[9]) that the following implications hold:

$$\begin{array}{ccc} \text{a-Weyl's theorem} & \Longrightarrow & \text{a-Browder's theorem} \\ \Downarrow & & \Downarrow \\ \text{Weyl's theorem} & \Longrightarrow & \text{Browder's theorem} \end{array}$$

In [19], H. Weyl proved that Weyl's theorem holds for hermitian operators. Weyl's theorem has been extended from hermitian operators to hyponormal and Toeplitz operators ([5]), and to several classes of operators including seminormal operators ([3],[4]). And S.C. Arora ([2]) has shown that Weyl's theorem holds for $M$-hyponormal operators. In this section we prove that Weyl type theorems hold for algebraically quasi-$M$-hyponormal operators using the local spectral theory.

We begin with the following lemma.

**Lemma 2.2.** Suppose $T$ is quasinilpotent algebraically quasi-$M$-hyponormal. Then $T$ is nilpotent.

*Proof.* We first assume that $T$ is quasi-$M$-hyponormal. We consider two cases:
Case I: Suppose $T$ has dense range. It follows from Corollary 1.3 that $T$ is $M$-hyponormal. But every quasinilpotent $M$-hyponormal operator is the zero operator, hence $T$ is nilpotent.
Case II: Suppose $T$ does not have dense range. Then by Theorem 1.8 we can represent $T$ as the $2 \times 2$ operator matrix:

$$T = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \text{ on } \mathcal{H} = \overline{T(\mathcal{H})} \oplus N(T^*) \text{ and } A \text{ is } M\text{-hyponormal.}$$

Since $T$ is quasinilpotent, $\sigma(T) = \{0\}$. But $\sigma(T) = \sigma(A) \cup \{0\}$, hence $\sigma(A) = \{0\}$. Since $A$ is $M$-hyponormal, $A = 0$. Therefore $T$ is nilpotent.
Now we suppose that $T$ is algebraically quasi-$M$-hyponormal. Then there exists a nonconstant polynomial $p$ such that $p(T)$ is quasi-$M$-hyponormal. If $p(T)$ has dense range, then $p(T)$ is $M$-hyponormal. Therefore $T$ is an algebraically $M$-hyponormal operator, and hence it is nilpotent by [6, Lemma 8]. If $p(T)$ does not have dense range, then by Theorem 1.8 we can represent $p(T)$ as the upper triangular matrix

$$p(T) = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \text{ on } \mathcal{H} = \overline{p(T)(\mathcal{H})} \oplus N(p(T)^*),$$

where $A := p(T)|\overline{p(T)(\mathcal{H})}$ is $M$-hyponormal. Since $T$ is quasinilpotent, $\sigma(p(T)) = p(\sigma(T)) = \{p(0)\}$. But $\sigma(p(T)) = \sigma(A) \cup \{0\}$, hence $\sigma(A) \cup \{0\} = \{p(0)\}$. So $p(0) = 0$, and hence $p(T)$ is quasinilpotent. Since $p(T)$ is quasi-$M$-hyponormal, by the previous argument $p(T)$ is nilpotent. On the other hand, since $p(0) = 0$, $p(z) = cz^m(z - \lambda_1)(z - \lambda_2) \cdots (z - \lambda_n)$ for some natural number $m$. Therefore $p(T) = cT^m(T - \lambda_1)(T - \lambda_2) \cdots (T - \lambda_n)$. Since $p(T)$ is nilpotent, $T$ is nilpotent. This completes the proof. □

An operator $T \in B(\mathcal{H})$ is called *isoloid* if every isolated point of $\sigma(T)$ is an eigenvalue of $T$ and an operator $T \in B(\mathcal{H})$ is called *polaroid* if iso $\sigma(T) \subseteq p_0(T)$. In general, if $T$ is polaroid, then it is isoloid. However, the converse is not true. Consider the following example: let $T \in B(\ell_2)$ be defined by

$$T(x_1, x_2, x_3, \cdots) = (\frac{1}{2}x_2, \frac{1}{3}x_3, \cdots).$$

Then $T$ is a compact quasinilpotent operator with $\alpha(T) = 1$, and so $T$ is isoloid. However, since $p(T) = \infty$, $T$ is not polaroid.

In [6], it was shown that every algebraically $M$-hyponormal operator is isoloid. We can prove more:

**Lemma 2.3.** Suppose $T$ is algebraically quasi-$M$-hyponormal. Then $T$ is polaroid.

*Proof.* Suppose $T$ is algebraically quasi-$M$-hyponormal. Then $p(T)$ is quasi-$M$-hyponormal for some nonconstant polynomial $p$. Let $\lambda \in \text{iso}\ \sigma(T)$. Using the spectral projection $P := \frac{1}{2\pi i}\int_{\partial D}(\mu - T)^{-1}d\mu$, where $D$ is a closed disk of center $\lambda$ which contains no other points of $\sigma(T)$, we can represent $T$ as the direct sum

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}, \text{ where } \sigma(T_1) = \{\lambda\} \text{ and } \sigma(T_2) = \sigma(T) \setminus \{\lambda\}.$$

Since $T_1$ is algebraically quasi-$M$-hyponormal, $T_1 - \lambda$ is also algebraically quasi-$M$-hyponormal. But $\sigma(T_1 - \lambda) = \{0\}$, it follows from Lemma 2.2 that $T_1 - \lambda$ is nilpotent. Therefore $T_1 - \lambda$ has finite ascent and descent. On the other hand, since $T_2 - \lambda$ is invertible, clearly it has finite ascent and descent. Therefore $T - \lambda$ has finite ascent and descent, and hence $\lambda$ is a pole of the resolvent of $T$. Thus $\lambda \in \text{iso}\ \sigma(T)$ implies $\lambda \in p_0(T)$, and so $\text{iso}\ \sigma(T) \subseteq p_0(T)$. Hence $T$ is polaroid. $\square$

In the following theorem, recall that $H(\sigma(T))$ is the space of functions analytic in an open neighborhood of $\sigma(T)$.

**Theorem 2.4.** Suppose $T$ or $T^*$ is algebraically quasi-$M$-hyponormal. Then $f(T) \in \mathcal{W}$ for every $f \in H(\sigma(T))$.

*Proof.* Suppose that $T$ is algebraically quasi-$M$-hyponormal. We first show that $T \in \mathcal{W}$. Suppose $\lambda \in \sigma(T) \setminus \sigma_w(T)$. Then $T - \lambda$ is Weyl but not invertible. We claim that $\lambda \in \partial\sigma(T)$. Assume to the contrary that $\lambda$ is an interior point of $\sigma(T)$. Then there exists a neighborhood $U$ of $\lambda$ such that $\alpha(T - \mu) > 0$ for all $\mu \in U$. It follows from [11, Theorem 10] that $T$ does not have SVEP. On the other hand, since $T$ is algebraically quasi-$M$-hyponormal, $p(T)$ is quasi-$M$-hyponormal for some nonconstant polynomial $p$. It follows from Corollary 1.12 that $p(T)$ has SVEP. Therefore $T$ has SVEP by [16, Theorem 3.3.9]. This is a contradiction. So $\lambda \in \partial\sigma(T) \setminus \sigma_w(T)$, and it follows from the punctured neighborhood theorem that $\lambda \in \pi_{00}(T)$. Conversely, suppose that $\lambda \in \pi_{00}(T)$. Using the spectral projection

$P := \frac{1}{2\pi i}\int_{\partial D}(\mu - T)^{-1}d\mu$, where $D$ is a closed disk of center $\lambda$ which contains no other points of $\sigma(T)$, we can represent $T$ as the direct sum

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}, \text{ where } \sigma(T_1) = \{\lambda\} \text{ and } \sigma(T_2) = \sigma(T) \setminus \{\lambda\}.$$

Since $\sigma(T_1) = \{\lambda\}$, $T_1 - \lambda$ is quasinilpotent. But $T$ is algebraically quasi-$M$-hyponormal, hence $T_1$ is also algebraically quasi-$M$-hyponormal. It follows from Lemma 2.2 that $T_1 - \lambda$ is nilpotent. Since $\lambda \in \pi_{00}(T)$, $T_1 - \lambda$ is a finite dimensional operator. Therefore $T_1 - \lambda$ is Weyl. Since $T_2 - \lambda$ is invertible, $T - \lambda$ is Weyl. Thus $T \in \mathcal{W}$. Now we claim that $f(\sigma_w(T)) = \sigma_w(f(T))$ for all $f \in H(\sigma(T))$. Let $f \in H(\sigma(T))$. Since $\sigma_w(f(T)) \subseteq f(\sigma_w(T))$ with no other restriction on $T$, it suffices to show that $f(\sigma_w(T)) \subseteq \sigma_w(f(T))$. Suppose that $\lambda \notin \sigma_w(f(T))$. Then $f(T) - \lambda$ is Weyl and

$$f(T) - \lambda = c(T - \alpha_1)(T - \alpha_2)\cdots(T - \alpha_n)g(T), \tag{2.4}$$

where $c, \alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{C}$ and $g(T)$ is invertible. Since the operators in the right side of (2.4) commute, every $T - \alpha_i$ is Fredholm. Since $T$ is algebraically quasi-$M$-hyponormal, $T$ has SVEP. Therefore by [1, Corollary 3.19] $i(T - \alpha_i) \leq 0$ for each $i = 1, 2, \cdots, n$. Therefore $\lambda \notin f(\sigma_w(T))$, and hence $f(\sigma_w(T)) = \sigma_w(f(T))$. Now recall ([1, Lemma 3.89]) that if $T$ is isoloid then

$$f\big(\sigma(T) \setminus \pi_{00}(T)\big) = \sigma(f(T)) \setminus \pi_{00}(f(T)) \quad \text{for every } f \in H(\sigma(T)).$$

Since $T$ is isoloid by Lemma 2.3 and $T \in \mathcal{W}$,

$$\sigma(f(T)) \setminus \pi_{00}(f(T)) = f\big(\sigma(T) \setminus \pi_{00}(T)\big) = f(\sigma_w(T)) = \sigma_w(f(T)),$$

which implies that $f(T) \in \mathcal{W}$.

Now suppose that $T^*$ is algebraically quasi-$M$-hyponormal. We first show that $T \in \mathcal{W}$. Suppose that $\lambda \in \sigma(T) \setminus \sigma_w(T)$. Observe that $\sigma(T^*) = \overline{\sigma(T)}$ and $\sigma_w(T^*) = \overline{\sigma_w(T)}$. So $\overline{\lambda} \in \sigma(T^*) \setminus \sigma_w(T^*)$. Since $T^* \in \mathcal{W}$, $\overline{\lambda} \in \pi_{00}(T^*)$. Therefore $\lambda$ is an isolated point of $\sigma(T)$, and so $\lambda \in \pi_{00}(T)$. Conversely, suppose that $\lambda \in \pi_{00}(T)$. Then $\lambda$ is an isolated point of $\sigma(T)$ and $0 < \alpha(T - \lambda) < \infty$. Since $\overline{\lambda}$ is an isolated point of $\sigma(T^*)$ and $T^*$ is algebraically quasi-$M$-hyponormal, it follows from Lemma 2.3 that $\overline{\lambda} \in p_0(T^*)$. So $\lambda \in p_0(T)$, and hence $T - \lambda$ is Weyl. Consequently, $\lambda \in \sigma(T) \setminus \sigma_w(T)$. Thus $T \in \mathcal{W}$. Now we show that $f(\sigma_w(T)) = \sigma_w(f(T))$ for each $f \in H(\sigma(T))$. Let $f \in H(\sigma(T))$. Suppose that $\lambda \notin \sigma_w(f(T))$. Then $f(T) - \lambda$ is Weyl. Since $T^*$ is algebraically quasi-$M$-hyponormal, it has SVEP. It follows from

[1, Corollary 3.19] that $i(T-\alpha_i) \geq 0$ for each $i = 1, 2, \cdots, n$. Since

$$0 \leq \sum_{i=1}^{n} i(T-\alpha_i) = i(f(T)-\lambda) = 0,$$

$T - \alpha_i$ is Weyl for each $i = 1, 2, \cdots, n$. Hence $\lambda \notin f(\sigma_w(T))$, and so $f(\sigma_w(T)) \subseteq \sigma_w(f(T))$. Thus $f(\sigma_w(T)) = \sigma_w(f(T))$ for each $f \in H(\sigma(T))$. Since $T \in \mathcal{W}$ and $T$ is isoloid, $f(T) \in \mathcal{W}$ for every $f \in H(\sigma(T))$. This completes the proof. □

From the proof of Theorem 2.4, we obtain the following useful consequence.

**Corollary 2.5.** Suppose $T$ or $T^*$ is algebraically quasi-$M$-hyponormal. Then

$$\sigma_w(f(T)) = f(\sigma_w(T)) \quad \text{for every } f \in H(\sigma(T)).$$

## 3 $a$-Weyl's theorem for algebraically quasi-$M$-hyponormal operators

Let $T \in B(\mathcal{H})$. It is well known that the inclusion $\sigma_{ea}(f(T)) \subseteq f(\sigma_{ea}(T))$ holds for every $f \in H(\sigma(T))$ with no restriction on $T$ ([17, Theorem 3.3]). The next theorem shows that the spectral mapping theorem holds for the essential approximate point spectrum for algebraically quasi-$M$-hyponormal operators.

**Theorem 3.1.** Suppose $T$ or $T^*$ is algebraically quasi-$M$-hyponormal. Then

$$\sigma_{ea}(f(T)) = f(\sigma_{ea}(T)) \quad \text{for every } f \in H(\sigma(T)).$$

*Proof.* Suppose first that $T$ is algebraically quasi-$M$-hyponormal and let $f \in H(\sigma(T))$. It suffices to show that $f(\sigma_{ea}(T)) \subseteq \sigma_{ea}(f(T))$. Suppose that $\lambda \notin \sigma_{ea}(f(T))$. Then $f(T) - \lambda \in \Phi_+^-(\mathcal{H})$ and

$$f(T) - \lambda = c(T-\alpha_1)(T-\alpha_2)\cdots(T-\alpha_n)g(T), \tag{3.1}$$

where $c, \alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{C}$, and $g(T)$ is invertible. Since $T$ is algebraically quasi-$M$-hyponormal, it has SVEP. It follows from [1, Corollary 3.19] that $i(T-\alpha_i) \leq 0$ for each $i = 1, 2, \cdots, n$. Therefore $\lambda \notin f(\sigma_{ea}(T))$, and hence $\sigma_{ea}(f(T)) = f(\sigma_{ea}(T))$.

Suppose now that $T^*$ is algebraically quasi-$M$-hyponormal. Then $T^*$ has SVEP. Therefore by [1, Corollary 3.19] $i(T-\alpha_i)\geq 0$ for each $i=1,2,\cdots,n$. Since

$$0\leq \sum_{i=1}^{n} i(T-\alpha_i)=i(f(T)-\lambda)\leq 0,$$

$T-\alpha_i$ is Weyl for each $i=1,2,\cdots,n$. Hence $\lambda\notin f(\sigma_{ea}(T))$, and so $\sigma_{ea}(f(T))=f(\sigma_{ea}(T))$. This completes the proof. □

$X\in B(\mathcal{H})$ is called a *quasiaffinity* if it has trivial kernel and dense range. $S\in B(\mathcal{H})$ is said to be a *quasiaffine transform* of $T\in B(\mathcal{H})$ (notation: $S\prec T$) if there is a quasiaffinity $X\in B(\mathcal{H})$ such that $XS=TX$. If both $S\prec T$ and $T\prec S$, then we say that $S$ and $T$ are *quasisimilar*. In general, we cannot expect that Weyl's theorem holds for operators having SVEP. Consider the following example: let $T\in B(\ell_2)$ be defined by

$$T(x_1,x_2,x_3,\cdots)=(\frac{1}{2}x_2,\frac{1}{3}x_3,\cdots).$$

Then $T$ is quasinilpotent, and so $T$ has SVEP. But $\sigma(T)=\sigma_w(T)=\{0\}$ and $\pi_{00}(T)=\{0\}$, hence $T\notin\mathcal{W}$. However, we have the following theorem.

**Theorem 3.2.** Suppose $T$ is algebraically quasi-$M$-hyponormal and $S\prec T$. Then $f(S)\in a\mathcal{B}$ for every $f\in H(\sigma(S))$.

*Proof.* Suppose $T$ is algebraically quasi-$M$-hyponormal and $S\prec T$. We first show that $S$ has SVEP. Let $U$ be any open set and let $f:U\longrightarrow\mathcal{H}$ be any analytic function such that $(S-\lambda)f(\lambda)=0$ for all $\lambda\in U$. Since $S\prec T$, there exists a quasiaffinity $X$ such that $XS=TX$. So $X(S-\lambda)=(T-\lambda)X$ for all $\lambda\in U$. Since $(S-\lambda)f(\lambda)=0$ for all $\lambda\in U$, $0=X(S-\lambda)f(\lambda)=(T-\lambda)Xf(\lambda)$ for all $\lambda\in U$. But $T$ is algebraically quasi-$M$-hyponormal, hence $T$ has SVEP. Therefore $Xf(\lambda)=0$ for all $\lambda\in U$. Since $X$ is a quasiaffinity, $f(\lambda)=0$ for all $\lambda\in U$. Therefore $S$ has SVEP. Now we show that $S\in a\mathcal{B}$. It is well known that $\sigma_{ea}(S)\subseteq\sigma_{ab}(S)$. Conversely, suppose that $\lambda\in\sigma_a(S)\setminus\sigma_{ea}(S)$. Then $S-\lambda\in\Phi_+^-(\mathcal{H})$ and $S-\lambda$ is not bounded below. Since $S$ has SVEP and $S-\lambda\in\Phi_+^-(\mathcal{H})$, it follows from [1, Theorem 3.16] that $p(S-\lambda)<\infty$. Therefore by [17, Theorem 2.1], $\lambda\in\sigma_a(S)\setminus\sigma_{ab}(S)$. Thus $S\in a\mathcal{B}$. Let $f\in H(\sigma(S))$ be arbitrary. Since $S$ has SVEP, it follows from the proof of Theorem 3.1 that $\sigma_{ea}(f(S))=f(\sigma_{ea}(S))$. Therefore

$$\sigma_{ab}(f(S))=f(\sigma_{ab}(S))=f(\sigma_{ea}(S))=\sigma_{ea}(f(S)),$$

and hence $f(S)\in a\mathcal{B}$. □

An operator $T \in B(\mathcal{H})$ is called *a-isoloid* if every isolated point of $\sigma_a(T)$ is an eigenvalue of $T$. Clearly, if $T$ is $a$-isoloid, then it is isoloid. However, the converse is not true. Consider the following example: let $T = U \oplus Q$, where $U$ is the unilateral forward shift on $\ell_2$ and $Q$ is an injective quasinilpotent operator on $\ell_2$. Then $\sigma(T) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$ and $\sigma_a(T) = \{\lambda \in \mathbb{C} : |\lambda| = 1\} \cup \{0\}$. Therefore $T$ is isoloid but not $a$-isoloid.

Suppose that $T^*$ is algebraically quasi-$M$-hyponormal. Then we can prove more:

**Theorem 3.3.** Suppose $T^*$ is algebraically quasi-$M$-hyponormal. Then $f(T) \in a\mathcal{W}$ for every $f \in H(\sigma(T))$.

*Proof.* Suppose $T^*$ is algebraically quasi-$M$-hyponormal. We first show that $T \in a\mathcal{W}$. Suppose that $\lambda \in \sigma_a(T) \setminus \sigma_{ea}(T)$. Then $T - \lambda$ is upper semi-Fredholm and $i(T - \lambda) \leq 0$. Since $T^*$ is algebraically quasi-$M$-hyponormal, $T^*$ has SVEP. Therefore by [1, Corollary 3.19] $i(T - \lambda) \geq 0$, and hence $T - \lambda$ is Weyl. Since $T^*$ has SVEP, it follows from [11, Corollary 7] that $\sigma(T) = \sigma_a(T)$. Also, since $T \in \mathcal{W}$ by Theorem 2.4, $\lambda \in \pi_{00}^a(T)$. Conversely, suppose that $\lambda \in \pi_{00}^a(T)$. Since $T^*$ has SVEP, $\sigma(T) = \sigma_a(T)$. Therefore $\lambda$ is an isolated point of $\sigma(T)$, and hence $\overline{\lambda}$ is an isolated point of $\sigma(T^*)$. But $T^*$ is algebraically quasi-$M$-hyponormal, hence by Lemma 2.3 that $\overline{\lambda} \in p_0(T^*)$. Therefore $\lambda \in p_0(T)$, and hence $T - \lambda$ is Weyl. So $\lambda \in \sigma_a(T) \setminus \sigma_{ea}(T)$. Thus $T \in a\mathcal{W}$. Now we show that $T$ is $a$-isoloid. Let $\lambda$ be an isolated point of $\sigma_a(T)$. Since $T^*$ has SVEP, $\lambda$ is an isolated point of $\sigma(T)$. But $T^*$ is polaroid, hence $T$ is also polaroid. Therefore it is isoloid, and hence $\lambda \in \sigma_p(T)$. Thus $T$ is $a$-isoloid. Finally, we shall show that $f(T) \in a\mathcal{W}$ for every $f \in H(\sigma(T))$. Let $f \in H(\sigma(T))$. Since $T \in a\mathcal{W}$, $\sigma_{ea}(T) = \sigma_{ab}(T)$. It follows from Theorem 3.1 that

$$\sigma_{ab}(f(T)) = f(\sigma_{ab}(T)) = f(\sigma_{ea}(T)) = \sigma_{ea}(f(T)),$$

and hence $f(T) \in a\mathcal{B}$. So $\sigma_a(f(T)) \setminus \sigma_{ea}(f(T)) \subseteq \pi_{00}^a(f(T))$.
Conversely, suppose that $\lambda \in \pi_{00}^a(f(T))$. Then $\lambda$ is an isolated point of $\sigma_a(f(T))$ and $0 < \alpha(f(T) - \lambda) < \infty$. Since $\lambda$ is an isolated point of $f(\sigma_a(T))$, if $\alpha_i \in \sigma_a(T)$, then $\alpha_i$ is an isolated point of $\sigma_a(T)$ by (3.1). Since $T$ is $a$-isoloid, $0 < \alpha(T - \alpha_i) < \infty$ for each $i = 1, 2, \cdots, n$. Since $T \in a\mathcal{W}$, $T - \alpha_i$ is upper semi-Fredholm and $i(T - \alpha_i) \leq 0$ for each $i = 1, 2, \cdots, n$. Therefore $f(T) - \lambda$ is upper semi-Fredholm and $i(f(T) - \lambda) = \sum_{i=1}^{n} i(T - \alpha_i) \leq 0$. Hence $\lambda \in \sigma_a(f(T)) \setminus \sigma_{ea}(f(T))$, and so $f(T) \in a\mathcal{W}$ for each $f \in H(\sigma(T))$. This completes the proof. $\square$

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Department of Mathematics, College of Sciences, Kyung Hee University, Seoul 130-701, Republic of Korea

*E-mail*: `ymhan2004@khu.ac.kr`

*E-mail*: `jhson37@khu.ac.kr`