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ON THE SOLUTION OF STEINHAUS FUNCTIONAL EQUATION USING WEAKLY PICARD OPERATORS

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Abstract

In this paper we obtain existence results regarding the solutions g of a Steinhaus type functional equation of the form $g(x) + g(f(x)) = F(x)$, under the significantly weaker assumption that f is a weakly Picard operator. The solutions are given in terms of sums of either convergent series or divergent series but summable by some method of summability.

1 Introduction

Let (X, d) be a metric space, Y a real Banach space and let us consider the functional equation

$$g(x) + g(f(x)) = F(x),\ x \in X, \tag{1}$$

where $f : X \to X$, $F : X \to Y$ are given mappings, while $g : X \to Y$ is the unknown mapping.

In the particular case when f, F and g are functions of a single variable x (real or complex), functional equations of the form (1) have been studied by several authors, see [1], [10], [11], [13], [15], [16], [20], [26] and references therein.

More precisely, Steinhaus [26] studied the equation $g(x) + g(x^2) = x$, while Hardy [11] considered the equation $g(x) + g(x^\alpha) = x$ where $\alpha > 0$. Kuczma [13] extended further Hardy and Steinhaus results and obtained the solutions of the equation $g(x) + g(f(x)) = F(x)$ in the form of sums of certain convergent series. All these results were obtained in the case x is a real variable. On the other hand, Racliş [20] studied the same equation $g(x) + g(f(x)) = F(x)$ but in the case x is complex and found meromorphic solutions.

Later, Bajraktarević [1] extended the results of Kuczma [13], by obtaining solutions of the equation (1) in the case when the series of functions given by Kuczma is divergent but summable by some regular method of summability. Malenica [15]

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extended further Bajraktarević's results. A more general form of the equation (1), that is,

$$g(x) + A(x)g(f(x)) = F(x), \tag{2}$$

was solved by Gercevanoff in [9], while Ghermănescu [10] solved the $(n+1)$-term functional equation

$$A_0 g(x) + A_1 g(f(x)) + A_2 g(f^2(x)) + \cdots + A_n g(f^n(x)) = F(x), \tag{3}$$

where, as usually, $f^n(x)$ stands for the n-th iterate of f, defined by $f^0(x) = x$ and $f^n(x) = f(f^{n-1}(x))$, $n \geq 1$.

For the equation (2), some results regarding its solutions, in the case when F satisfies some special conditions, were obtained by Malenica, see [16] and references therein.

In the papers [1], [15], x is considered a real variable and the function is like in the next Example.

Example 1. *Let $f : [a,b] \to [a,b]$ satisfy the following conditions:*
a) f is a continuous and strictly increasing, and
b) $f(a) = a$, $f(b) = b$ and $f(x) > x$, for all $x \in (a,b)$.

Then, for all $x \in (a,b]$, the sequence of iterates of f, $\{f^n(x)\}_{n \geq 0}$, converges to b, that is,

$$\lim_{n \to \infty} f^n(x) = b. \tag{4}$$

By observing that property (4) is characteristic to the so called *Picard operators*, introduced by Rus in 1984, see for example [21], [24], the author in [2] solved the equation (1) in the general case when X is an arbitrary metric space, Y is a real Banach space and f is a Picard operator on X.

Starting from the fact that f in Example 1, considered on the entire interval $[a,b]$, is actually a weakly Picard operator and not a Picard operator, the main aim of this paper is to show that we can solve the functional equations (1)-(3) under more general conditions, i.e., by considering that f is a *weakly Picard mapping* and by using series of functions which are convergent or divergent but summable by some regular methods of summability. Our results thus obtained extend, improve, generalize and unify many results in literature regarding this topic: [1], [2], [11], [13], [15] and [26].

The paper is organized as follows: in Section 2 we introduce the basic notions and results regarding Picard operators and weakly Picard operators and also illustrate them by some relevant examples; then, in Section 3 we present some definitions and results related to series and summability in Banach spaces, while, in the last section, we state and prove our main result in this paper, an existence theorem for the Steinhaus functional equation.

2 Picard operators and Weakly Picard operators

For the theory of Picard operators and weakly Picard operators we basically refer to Rus [24].

Definition 1. (I.A. Rus, [24]). *Let (X, d) be a metric space. An operator $f : X \to X$ is called a Picard operator (briefly PO) if:*
(i) $Fix\,(f) := \{x \in X : f(x) = x\} = \{x^*\}$,
(ii) $\{f^n(x)\}_{n \geq 0}$ converges to x^* as $n \to \infty$, for all $x \in X$.

The term of (weakly) *Picard operator*, adopted by I.A. Rus in [24] and in subsequent papers, is motivated by the fact that Émile Picard has been the first mathematician who, based probably on ideas of Cauchy and Liouville, developed systematically the method of successive approximations in a series of papers on the existence of solutions of initial value problems for ordinary differential equations, see [19] (and also [4], for some bibliographical and historical comments).

In terms of the Picard operators theory, several fundamental results in metrical fixed point theory can be restated in a simplified form (see [21] and [24] for more details), as shown by the next example.

Example 2. (Contraction Mapping Principle). *Let (X, d) be a complete metric space and let $f : X \to X$ be an a-contraction, i.e., a mapping for which there exists a constant $a \in [0, 1)$ such that*

$$d(f(x), f(y)) \leq ad(x, y), \forall x, y \in X.$$

Then f is a PO.

Definition 2. *Let (X, d) be a metric space. A map $f : X \to X$ is called a weakly Picard operator (briefly WPO) if the sequence $\{f^n(x)\}_{n \geq 0}$ converges for all $x \in X$ and the limit (which may depend on x) is a fixed point of f.*

It is obvious that any PO is a WPO, but the reverse in not true, as shown by Examples 5 and 6.

Example 3. ([3]) *Let (X, d) be a complete metric space and let $f : X \to X$ be an almost contraction, that is, a mapping for which there exist the constants $a \in [0, 1)$ and $L \geq 0$ such that*

$$d(f(x), f(y)) \leq ad(x, y) + Ld(y, f(x)), \forall x, y \in X.$$

Then f is a WPO.

For other results and applications of PO and WPO see [3], [4]-[6], [8], [18] and [21]-[25].

The next examples illustrate how general the concepts of PO and WPO are in comparison to the functions f used in [1] and [15] to solve functional equations, where the functions f is assumed to satisfy the two very sharp conditions a) and b) in Example 1.

Example 4. *Let $X = [0, 1]$ be endowed with the usual norm and let $f : [0, 1] \to [0, 1]$ be defined by $f(x) = \frac{2}{5}$, if $x \in \left[0, \frac{2}{3}\right)$ and $f(x) = \frac{1}{5}$, if $x \in \left[\frac{2}{3}, 1\right]$. Then f is a discontinuous almost contraction that does not satisfy neither assumption a) nor b) in Example 1 but which has a unique fixed point, that is, $Fix\,(f) = \left\{\frac{2}{5}\right\}$. It is a PO, see [17].*

Example 5. *Let $[0, 1]$ be the unit interval with the usual norm and let $f : [0, 1] \to [0, 1]$ be given by $f(x) = \frac{1}{2}$ for $x \in [0, 2/3)$ and $f(x) = 1$, for $x \in [2/3, 1]$. Then f is a discontinuous almost contraction that does not satisfy neither a) nor b) in Example 1 but which has two fixed points, that is, $Fix\,(f) = \left\{\frac{1}{2}, 1\right\}$. It is a WPO, see [17].*

Example 6. *Let $[0, 1]$ be the unit interval with the usual norm and let $f : [0, 1] \to [0, 1]$ be the identity map, i.e., $f(x) = x$, for all $x \in [0, 1]$. Then f is a continuous almost contraction that does not satisfy b) in Example 1 and the set of fixed points of f is the entire interval $[0, 1]$, i.e., $Fix\,(f) = [0, 1]$. It is a WPO, see [3].*

The following lemma, which is important by itself, expresses a natural property of PO and WPO and which, in the particular case of almost contractions, has been obtained in a previous paper [7].

Lemma 1. *Let $f : X \to X$ be a WPO operator and let $b \in Fix\,(f)$ be a fixed point of f. Then f is continuous at b.*

3 Sequences and series in Banach spaces

In this section we recall some known definitions and properties concerning sequences in metric spaces and series in a real Banach space. For the sake of completeness, we also introduce some concepts and results regarding the summability of divergent series in the setting of a Banach space, similar to the well known ones in the case of numerical series [12], see also [2].

In what follows, without any other special mention, (X, d) will be a (complete) metric space, while Y will denote an arbitrary real Banach space. The convergence of series in Y is understood in norm.

Definition 3. *Let $\sum\limits_{n=0}^{\infty} x_n$ be a series in Y whose sequence of partial sums is denoted by $\{S_n\}_{n\geq 0}$. Consider also the sequence $\left\{S_n^{(k)}\right\}_{k\geq 0}$, defined by $S_n^{(0)} = S_n$ and, for $k \geq 1$, by*

$$S_n^{(k)} = S_0^{(k-1)} + S_1^{(k-1)} + \ldots + S_n^{(k-1)},\ (n = 0, 1, 2, \ldots).$$

If, for some k, the sequence $\{C_n^k\}_{n\geq 0}$, given by

$$C_n^k = \frac{1}{\binom{n+k}{k}} S_n^k,\ n \geq 0,$$

converges to S, then we shall say that the series $\sum_{n=0}^{\infty} x_n$ is Césaro-summable or C_0-summable with the sum S. (Here $\binom{n+k}{k}$ stands for the binomial coefficient.)

Definition 4. *Let $\{S_n\}_{k\geq 0}$ be the sequence of partial sums of the series $\sum_{n=0}^{\infty} x_n$ in Y. Consider $T = (a_{kn})$ an infinite matrix of real numbers and let us construct the following family of series*

$$\sum_{n=0}^{\infty} a_{kn} S_n,\ (k = 0, 1, 2, \ldots), \tag{5}$$

which are assumed to be convergent with the sums $\widehat{S}_k$, $k = 0, 1, 2, \ldots$, respectively. If the sequence $\left\{\widehat{S}_k\right\}_{k\geq 0}$ converges to $\widehat{S}$, then we say that the series $\sum_{n=0}^{\infty} x_n$ is Toeplitz-summable or T-summable with the sum $\widehat{S}$.

Definition 5. *A series $\sum_{n=0}^{\infty} x_n$ in Y is said to be summable by Abel's method of summability or A-summable, with the sum S, if the series $\sum_{n=0}^{\infty} t^n x_n$ converges for all $t \in [-r, r]$, $r \geq 1$, and there exists the limit*

$$\lim_{t\to 1-0} \left(\sum_{n=0}^{\infty} t^n x_n \right) = S.$$

We end this section by stating two results from [12], pag. 404-405, adapted to Banach spaces, like in [2]. Their proofs are completely similar to the case of the real line and are not given here.

Theorem 1. *Let $\{z_n\}_{n\geq 0}$ be a sequence in Y such that $\lim_{n\to\infty} z_n = 0 \in Y$. If $T = (a_{kn})$ is an infinite regular matrix whose entries satisfy:*

t_1) For each $n \geq 0$, $a_{kn} \to 0$ as $k \to \infty$;

t_2) There exists a constant C such that

$$|a_{k0}| + |a_{k1}| + \cdots + |a_{kn}| < C,\ (k \geq 0,\ n \geq 0),$$

then the series $\sum_{n=0}^{\infty} a_{kn} z_n$ converges for all $k \geq 0$.

Moreover, the sequence $\{\widehat{z}_n\}_{n\geq 0}$ *defined by*

$$\widehat{z}_n = \sum_{n=0}^{\infty} a_{kn}\, z_n,\ k = 0, 1, 2, \dots, \tag{6}$$

converges in Y *and* $\lim\limits_{n\to\infty} \widehat{z}_n = 0$.

Theorem 2. *If* $T = (a_{kn})$ *is an infinite regular matrix whose entries satisfy conditions* t_1), t_2) *from Theorem 1 and the following additional condition*

t_3) $\lim\limits_{k\to\infty} \left(\sum\limits_{n=0}^{\infty} a_{kn} \right) = 1$,

then, for any sequence $\{\widehat{z}_n\}_{n\geq 0}$ *convergent to some* $z \in Y$*, as* $n \to \infty$*, the sequence* $\{\widehat{z}_n\}_{n\geq 0}$ *given by (6) converges and* $\lim\limits_{k\to\infty} \widehat{z}_k = z$.

4 Solutions of the functional equation

Now having at disposal all necessary prerequisites, we can go back to the solution of the functional equation (1).

Let $f : X \to X$ be a WPO and let $x \in X$ be arbitrary. By definition, there exists $b \in Fix\,(f)$, such that

$$\lim_{n\to\infty} f^n(x) = b.$$

Similarly to the related papers [1], [2], [13] and [15], we consider the following series

$$\frac{1}{2}F(b) + \sum_{n=0}^{\infty} (-1)^n \left[F\left(f^n(x)\right) - F(b) \right], \tag{7}$$

in order to obtain sufficient conditions for the existence of a solution of the functional equation (1), by using either the convergence of the series (7) or its summability in some sense.

The main result reads as follows.

Theorem 3. *Let* (X, d) *be a complete metric space,* Y *an arbitrary real Banach space,* $f : X \to X$ *a given WPO and* $F : X \to Y$ *a given mapping which is continuous at* b*, for each* $b \in Fix\,(f)$*. Then the following are true:*

(1) If the series (7) converges on X*, then its sum,* g*, is a solution of the functional equation (1). Moreover, if* f *and* F *are continuous and the series (7) is uniformly convergent on* X*, then* g *is continuous on* X*;*

(2) If the series (7) is T*-summable with the sum* g*, where* $T = (a_{kn})$ *is a regular matrix transformation such that its entries* $a_{kn} \in \mathbb{R}$ *do satisfy conditions* t_1)-t_3)*, then* g *is a solution of the functional equation (1);*

(3) If the series (7) is C_0*-summable with the sum* g*, then its sum is a solution of the functional equation (1);*

(4) *If the series (7) is A-summable with the sum g, then its sum is a solution of the functional equation (1).*

Proof. (1)

By hypothesis we have

$$g(x) = \lim_{n\to\infty} \left[\frac{1}{2}F(b) + \sum_{i=0}^{n} (-1)^i \left[F\left(f^i(x)\right) - F(b) \right], \right]$$

and

$$g(f(x)) = \lim_{n\to\infty} \left[\frac{1}{2}F(b) + \sum_{i=0}^{n} (-1)^i \left[F\left(f^{i+1}(x)\right) - F(b) \right] \right],$$

from which, by straightforward calculations, we get

$$g(x) + g(f(x)) = F(x),$$

that is $g(x)$ is a solution of the functional equation (1).

(2)

Denote by $\{S_n\}_{n\geq 0}$ the sequence of partial sums of the series (7), i.e.,

$$S_n(x) = \frac{1}{2}F(b) + \sum_{i=0}^{n} (-1)^i \left[F\left(f^i(x)\right) - F(b) \right], \; n = 0, 1, 2, \ldots$$

and also denote

$$\widehat{S}_k(x) = \sum_{i=0}^{n} a_{kn} S_n(x), \; k = 0, 1, 2, \ldots$$

By the T-summability property of the series (7) we have that

$$g(x) = \lim_{n\to\infty} \widehat{S}_n(x).$$

On the other hand, we have

$$S_n(f(x)) = F(x) - S_n(x) + (-1)^n \left[F\left(f^{n+1}(x)\right) - F(b) \right], \; n = 0, 1, 2, \ldots$$

and consequently

$$\widehat{S}_k(f(x)) = \sum_{n=0}^{\infty} a_{kn} F(x) - \widehat{S}_k(x) - \sum_{n=0}^{\infty} (-1)^{n+1} a_{kn} \left[F\left(f^{n+1}(x)\right) - F(b) \right], \tag{8}$$

for all $k = 0, 1, 2, \ldots$

Now, by assumptions $t_1) - t_3)$ and continuity of F at b, we conclude that

$$\lim_{n\to\infty} (-1)^{n+1} \left[F\left(f^{n+1}(x)\right) - F(b) \right] = 0,$$

and hence, by Theorem 1, we get

$$\lim_{k\to\infty}\left(\sum_{n=0}^{\infty}(-1)^{n+1}\left[F\left(f^{n+1}(x)\right)-F(b)\right]\right)=0.$$

By Theorem 2 we also have

$$\lim_{k\to\infty}\left(\sum_{n=0}^{\infty}a_{kn}F(x)\right)=F(x),\forall x\in X,$$

which actually shows, by letting $n\to\infty$ in (8), that

$$g(f(x))=F(x)-g(x),$$

which shows that $g(x)$ is a solution of the functional equation (1). Since g is uniquely determined by the way it was constructed, it is the unique solution of the functional equation (1).

(3)

The proof is similar to that of the third part of Theorem 1 in [1] and so, we'll omit here the unnecessary details. Using the same notations as in the previous case, by elementary calculations we found out that

$$C_n^k(x)=\frac{n}{n+k}C_{n-1}^k(x)+\frac{1}{\binom{n+k}{k}}S_n^{k-1}(x),$$

and so

$$C_n^k(x)+C_n^k(f(x))=F(x)-\frac{n}{2(n+k)}F(b)+\frac{1}{\binom{n+k}{k}}S_n^{k-1}(f(x)). \qquad (9)$$

Now, in view of the C_0-summability of the series (7), by letting $n\to\infty$ in (9) and using the hypothesis, that is, that

$$\lim_{n\to\infty}C_n^k(x)=g(x),$$

we finally obtain

$$g(x)+g(f(x))=F(x),$$

that is, $g(x)$ is a solution of the functional equation (1).

(4)

This follows similarly to the proof of Theorem 1, d) in [1]. □

Remark 1. *In the particular case when f is a Picard operator, by Theorem 3 we obtain as a corollary, Theorem 4 in [2].*

Remark 2. *As shown by the case (1) of Theorem 3, it is possible to ensure the continuity of the solution of the functional equation but under rather strong assumptions. If, for example, f and F are not both continuous, then g needs not be continuous, as shown by the next two examples.*

Example 7. *Let* $[0,1]$ *be the unit interval with the usual norm and let* f, $F : [0,1] \to [0,1]$ *be defined by* $f(x) = \frac{x}{2}$, *for all* $x \in [0,1)$ *and* $f(1) = \frac{1}{3}$; $F(x) = \frac{x}{2}$, *for all* $x \in [0,)]$ *and* $F(1) = \frac{1}{3}$, *respectively. It is easy to check that* f *is a Picard operator,* $f^n(x) = \frac{x}{2^n}$, *for all* $x \in [0,1)$ *and* $f^n(1) = \frac{1}{3 \cdot 2^{n-1}}$, *for all* $n \geq 1$ *and therefore:*

$$\lim_{n\to\infty} f^n(x) = 0, \forall x \in [0,1].$$

Hence $b = 0$ *and, in view of the fact that* $F(0) = 0$, *the series* (7),

$$\frac{1}{2}F(b) + \sum_{n=0}^{\infty}(-1)^n \left[F\left(f^n(x)\right) - F(b)\right],$$

becomes

$$\sum_{n=0}^{\infty}(-1)^n F\left(f^n(x)\right),$$

which is convergent, for all $x \in [0,1]$, *and its sum could be easily evaluated by using the well known formula*

$$1 - x + x^2 - x^3 + \ldots + (-1)^n x^n + \cdots = \frac{1}{1+x}, \; x \in (-1,1).$$

Theorem 3 (1) applies and the unique (discontinuous) solution of the functional equation is given by $g(x) = \frac{x}{3}$, *for* $x \in [0,1)$ *and* $g(1) = \frac{2}{9}$.

Example 8. *Let* $[0,1]$ *be the unit interval with the usual norm and let* f, $F : [0,1] \to [0,1]$ *be defined by* $f(x) = \frac{x}{2}$, *for all* $x \in [0,1]$, *and by* $F(0) = 0$, *for all* $x \in [0,1)$ *and* $F(1) = \frac{1}{3}$, *respectively. Here, the series* (7) *converges for* $x \in [0,1)$, *so* $g(x) = 0$, *for all* $x \in [0,1)$ *but does not converge for* $x = 1$. *Instead, for* $x = 1$ *the series* (7) *is A-summable and its sum is* $g(1) = \frac{1}{2}$. *We thus obtain, by Theorem 3 (4), that* g *is the unique (discontinuous) solution of the functional equation* $g(x) + g(f(x)) = F(x)$, *which can also be checked directly.*

Remark 3. *All results obtained in this section are given for the Steinhaus type functional equation* (1), *but similar considerations could be done for the more general case of the functional equations* (2) *and* (2). *These will be treated in a forthcoming paper.*

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