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CONVERGENCE THEOREMS OF FINITE-STEP ITERATION WITH ERRORS FOR NON-SELF ASYMPTOTICALLY NONEXPANSIVE IN THE INTERMEDIATE SENSE MAPPINGS

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Abstract

Let K be a nonempty closed convex nonexpansive retract of a uniformly convex Banach space E with P as a nonexpansive retraction. Let $T\colon K \to E$ be non-self asymptotically nonexpansive in the intermediate sense mapping with $F(T) \neq \phi$. Let $\{\alpha_n^{(i)}\}$, $\{\beta_n^{(i)}\}$ and $\{\gamma_n^{(i)}\}$ are sequences in $[0,1]$ with $\alpha_n^{(i)} + \beta_n^{(i)} + \gamma_n^{(i)} = 1$ for all $i = 1, 2, \ldots, N$. From arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ iteratively by (8), where $\{u_n^{(i)}\}$ for all $i = 1, 2, \ldots, N$ are bounded sequences in K with $\sum_{n=1}^{\infty} u_n^{(i)} < \infty$. (i) If the dual E^* of E has the *Kadec-Klee* property, then $\{x_n\}$ converges weakly to a fixed point of T; (ii) if T satisfies condition (A), then $\{x_n\}$ converges strongly to a fixed point of T. The results presented in this paper extend and improve the corresponding results of Rhoades [1], Chidume et al. [4, 6], Schu [11, 12], Osilike and Aniagbosor [18], Tan and Xu [17], Plubtieng and Wangkeeree [21], Su and Qin [31] and many others.

1 Introduction and Preliminaries

Let K be a nonempty closed convex subset of a Banach space E. A self mapping $T\colon K \to K$ is called asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$; $k_n \to 1$ as $n \to \infty$ such that for all $x, y \in K$, the following inequality holds:

$$\|T^n x - T^n y\| \;\; \leq \;\; k_n \|x - y\|, \quad \forall n \geq 1 \tag{1}$$

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T is called uniformly L-Lipschitzian if there exists a constant $L > 0$ such that for all $x, y \in K$,

$$\|T^n x - T^n y\| \quad \leq \quad L\,\|x - y\|\,, \quad \forall n \geq 1. \tag{2}$$

T is called asymptotically nonexpansive type [28] if the following inequality holds:

$$\limsup_{n\to\infty} \sup_{y\in K} \Big(\|T^n x - T^n y\| - \|x - y\| \Big) \quad \leq \quad 0. \tag{3}$$

for every $x \in K$, and that T^N be continuous for some $N \geq 1$.

T is called asymptotically nonexpansive in the intermediate sense [20] if T is uniformly continuous and

$$\limsup_{n\to\infty} \sup_{x,y\in K} \Big(\|T^n x - T^n y\| - \|x - y\| \Big) \quad \leq \quad 0. \tag{4}$$

The class of asymptotically nonexpansive maps was introduced by Goebel and Kirk [13] as an important generalization of the class of nonexpansive maps (i.e., mappings $T\colon K \to K$ such that $\|Tx - Ty\| \leq \|x - y\|$, $\forall x, y \in K$) who proved that if K is a nonempty closed convex subset of a real uniformly convex Banach space and T is an asymptotically nonexpansive self-mapping of K, then T has a fixed point.

Iterative techniques for approximating fixed points of nonexpansive mappings and asymptotically nonexpansive mappings have been studied by various authors (see e.g., [26], [2, 3], [5], [24], [18], [7], [25], [1], [11, 12], [8], [14, 15, 16, 17]) using the Mann iteration method (see e.g., [30]) or the Ishikawa iteration method (see e.g., [23]).

In 1978, Bose [22] proved that if K is a bounded closed convex nonempty subset of a uniformly convex Banach space E satisfying Opial's [32] condition and $T\colon K \to K$ is an asymptotically nonexpansive mapping, then the sequence $\{T^n x\}$ converges weakly to a fixed point of T provided T is asymptotically regular at $x \in K$, i.e., $\lim_{n\to\infty} \left\|T^n x - T^{n+1} x\right\| = 0$. Passty [7] and also Xu [9] proved that the requirement that E satisfies Opial's condition can be replaced by the condition that E has a Fréchet differentiable norm. Furthermore, Tan and Xu [14, 15] later proved that the asymptotic regularity of T can be weakened to the weakly asymptotic regularity of T at x, i.e., $\omega - \lim_{n\to\infty}(T^n x - T^{n+1}x) = 0$.

In [11, 12], Schu introduced a modified Mann process to approximate fixed points of asymptotically nonexpansive self-maps defined on nonempty closed convex and bounded subsets of a Hilbert space H.

In 1994, Rhoades [1] extended the Schu's result to uniformly convex Banach space using a modified Ishikawa iteration method.

In all the above results, the operator T remains a self-mapping of a nonempty closed convex subset K of a uniformly convex Banach space. If, however, the domain of T, $D(T)$ is a proper subset of E (and this is the case in several applications), and T maps $D(T)$ into E, then the iteration processes of Mann and Ishikawa studied by these authors; and their modifications introduced by Schu may fail to be well defined.

In 2003, Chidume et al [4] studied the iterative scheme defined by

$$x_1 \in K,$$

$$x_{n+1} = P((1-\alpha_n)x_n + \alpha_n T(PT)^{n-1}x_n), \quad n \geq 1, \tag{5}$$

in the framework of uniformly convex Banach space, where K is a closed convex nonexpansive retract of a real uniformly convex Banach space E with P as a nonexpansive retract. $T\colon K \to E$ is an asymptotically nonexpansive non-self map with sequence $\{k_n\} \subset [1,\infty)$, $k_n \to 1$ as $n \to \infty$. $\{\alpha_n\}_{n=1}^{\infty}$ is a real sequence in $[0,1]$ satisfying the condition $\epsilon \leq \alpha_n \leq 1-\epsilon$ for all $n \geq 1$ and for some $\epsilon > 0$. They proved strong and weak convergence theorems for asymptotically nonexpansive non-self maps.

In 2005, Shahzad [19] studied the sequence $\{x_n\}$ defined by

$$x_1 \in K,$$

$$x_{n+1} = P((1-\alpha_n)x_n + \alpha_n TP[(1-\beta_n)x_n + \beta_n Tx_n]), \tag{6}$$

where K is a nonempty closed convex nonexpansive retract of a real uniformly convex Banach space E with P as a nonexpansive retraction. He proved weak and strong convergence theorems for non-self nonexpansive mappings in Banach spaces.

Recently, Su and Qin [31] studied the sequence $\{x_n\}$ defined by

$$x_1 \in K,$$

$$\begin{aligned} z_n &= P(\alpha''_n T(PT)^{n-1} x_n + (1-\alpha''_n)x_n), \\ y_n &= P(\alpha'_n T(PT)^{n-1} z_n + (1-\alpha'_n)x_n), \\ x_{n+1} &= P(\alpha_n T(PT)^{n-1} y_n + (1-\alpha_n)x_n), \end{aligned} \tag{7}$$

where $\{\alpha_n\}$, $\{\alpha'_n\}$ and $\{\alpha''_n\}$ are real sequences in $(0,1)$ and K is a nonempty closed convex nonexpansive retract of a uniformly convex Banach space E with P as a nonexpansive retraction. They proved weak and strong convergence theorems for asymptotically nonexpansive non-self mappings in uniformly convex Banach space.

Motivated by Su and Qin [31] and some others, the purpose of this paper is to construct a finite-step iterative scheme with errors for approximating fixed point of non-self asymptotically nonexpansive in the intermediate sense mappings (when such a fixed point exists) and to prove weak and strong convergence theorems for such maps.

Let K be a nonempty closed convex subset of a uniformly convex Banach space E and $T\colon K \to E$ is asymptotically nonexpansive non-self mappings in the intermediate sense. In this paper, the following iteration scheme is studied: For a given $x_1 \in K$, and a fixed $N \in \mathbf{N}$ ($\mathbf{N}$ denote the set of all positive integers), compute the sequence $\{x_n\}$ by

$$\begin{aligned} x_{n+1} = x_n^{(N)} &= P(\alpha_n^{(N)} T(PT)^{n-1} x_n^{(N-1)} + \beta_n^{(N)} x_n + \gamma_n^{(N)} u_n^{(N)}) \\ x_n^{(N-1)} &= P(\alpha_n^{(N-1)} T(PT)^{n-1} x_n^{(N-2)} + \beta_n^{(N-1)} x_n + \gamma_n^{(N-1)} u_n^{(N-1)}) \\ \ldots &= \ldots \\ \ldots &= \ldots \\ x_n^{(3)} &= P(\alpha_n^{(3)} T(PT)^{n-1} x_n^{(2)} + \beta_n^{(3)} x_n + \gamma_n^{(3)} u_n^{(3)}) \\ x_n^{(2)} &= P(\alpha_n^{(2)} T(PT)^{n-1} x_n^{(1)} + \beta_n^{(2)} x_n + \gamma_n^{(2)} u_n^{(2)}) \\ x_n^{(1)} &= P(\alpha_n^{(1)} T(PT)^{n-1} x_n + \beta_n^{(1)} x_n + \gamma_n^{(1)} u_n^{(1)}) \end{aligned} \tag{8}$$

where $\{u_n^{(1)}\}$, $\{u_n^{(2)}\}$, $\ldots$, $\{u_n^{(N)}\}$ are bounded sequences in K and $\{\alpha_n^{(i)}\}$, $\{\beta_n^{(i)}\}$, $\{\gamma_n^{(i)}\}$ are appropriate real sequences in $[0,1]$ such that $\alpha_n^{(i)} + \beta_n^{(i)} + \gamma_n^{(i)} = 1$ for each $i \in \{1, 2, \ldots, N\}$.

Our theorems improve and generalize some previous results. Our weak convergence result applies not only to L^p-spaces with $1 < p < \infty$ but also to other spaces which do not satisfy Opial's condition or have a Fréchet differentiable norm. More precisely, we prove weak convergence of the above defined iteration scheme with errors (8) in a uniformly convex Banach space whose dual has the *Kadec-Klee* property. It is worth mentioning that there are uniformly convex Banach spaces, which have neither a Fréchet differentiable norm nor Opial's property; however their dual does have the *Kadec-Klee* property (see, e.g., [10, 27]).

Let E be a real Banach space. A subset K of E is said to be a retract of E if there exists a continuous map $P\colon E \to E$ such that $Px = x$ for all $x \in K$. A map $P\colon E \to E$ is said to be a retraction if $P^2 = P$. It follows that if a map P is a retraction, then $Py = y$ for all y in the range of P. A set K is optimal if each point outside K can be moved to be closure to all points of K. It is well known (see, e.g., [29]) that

(i) if E is a separable, strictly convex, smooth, reflexive Banach space, and if $K \subset E$ is an optimal set with interior, then K is a nonexpansive retract of E;

(ii) a subset of ℓ^p, with $1 < p < \infty$, is a nonexpansive retract if and only if it is optimal.

Note that every nonexpansive retract is optimal. In strictly convex Banach spaces, optimal sets are closed and convex. However, every closed convex subset of a Hilbert space is optimal and also a nonexpansive retract.

A mapping T with domain $D(T)$ and range $R(T)$ in E is said to be demiclosed at p if whenever $\{x_n\}$ is a sequence in $D(T)$ such that $\{x_n\}$ converges weakly to $x^* \in D(T)$ and $\{Tx_n\}$ converges strongly to p, then $Tx^* = p$.

A Banach space E is said to have the *Kadec-Klee* property if for every sequence $\{x_n\}$ in E, $x_n \to x$ weakly and $\|x_n\| \to \|x\|$ strongly together imply $\|x_n - x\| \to 0$.

We recall the following definition:

A mapping $T\colon K \to K$ with $F(T) \neq \phi$ is said to satisfy condition (A) [8] on K if there exists a nondecreasing function $f\colon [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$ such that for all $x \in K$, $\|x - Tx\| \geq f(d(x,F))$ where $d(x,F(T)) = \inf\{\|x - p\| : p \in F(T)\}$.

In order to prove our main results, we will make use of the following lemmas:

Lemma 1.1.(see [16]) Let $\{s_n\}$ and $\{t_n\}$ be sequences of nonnegative real numbers satisfying the inequality

$$s_{n+1} \leq s_n + t_n \qquad \forall n \geq 1.$$

If $\sum_{n=1}^{\infty} t_n < \infty$, then $\lim_{n\to\infty} s_n$ exists. Moreover, if there exists a subsequence

$\{s_{n_j}\}$ of $\{s_n\}$ such that $s_{n_j} \to 0$ as $j \to \infty$, then $s_n \to 0$ as $n \to \infty$.

Lemma 1.2. (Schu [12]) Let E be a uniformly convex Banach space and $0 < a \leq t_n \leq b < 1$ for all $n \geq 1$. Suppose that $\{x_n\}$ and $\{y_n\}$ are sequences in E satisfying

$$\limsup_{n\to\infty} \|x_n\| \leq r, \qquad \limsup_{n\to\infty} \|y_n\| \leq r,$$

$$\lim_{n\to\infty} \|t_n x_n + (1-t_n) y_n\| = r,$$

for some $r \geq 0$. Then

$$\lim_{n\to\infty} \|x_n - y_n\| = 0.$$

Lemma 1.3. (demiclosed principle for non-self map [6]) Let E be a uniformly convex Banach space and K a nonempty closed convex subset of E. Let $T\colon K \to E$ be a mapping which is asymptotically nonexpansive in the intermediate sense. If $\{x_n\}$ is a sequence in K converging weakly to x^* and if

$$\lim_{j\to\infty} \Big(\limsup_{n\to\infty} \big\|x_n - T(PT)^{j-1} x_n\big\| \Big) = 0,$$

then $Tx^* = x^*$.

Lemma 1.4. (see [10]) Let E be a real reflexive Banach space such that its dual E^* has the *Kadec-Klee* property. Let $\{x_n\}$ be a bounded sequence in E and $x^*, y^* \in w_w(x_n)$; here $w_w(x_n)$ denotes the weak w-limit set of $\{x_n\}$. Suppose $\lim_{n\to\infty} \|t x_n + (1-t) x^* - y^*\|$ exists for all $t \in [0,1]$. Then $x^* = y^*$.

2 Main Results

Definition 2.1. (see [4]) Let E be a real normed linear space, K a nonempty subset of E. Let $P\colon E \to K$ be the nonexpansive retraction of E onto K. A map $T\colon K \to E$ is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1,\infty)$; $k_n \to 1$ as $n \to \infty$ such that for all $x, y \in K$, the following inequality holds:

$$\big\|T(PT)^{n-1}x - T(PT)^{n-1}y\big\| \quad \leq \quad k_n \|x-y\|, \quad \forall n \geq 1. \tag{9}$$

T is called uniformly L-Lipschitzian if there exists a constant $L > 0$ such that for all $x, y \in K$

$$\left\|T(PT)^{n-1}x - T(PT)^{n-1}y\right\| \leq L\|x-y\|, \quad \forall n \geq 1. \tag{10}$$

T is called asymptotically nonexpansive type if the following inequality holds:

$$\limsup_{n\to\infty}\sup_{y\in K}\left(\left\|T(PT)^{n-1}x - T(PT)^{n-1}y\right\| - \|x-y\|\right) \leq 0, \tag{11}$$

for every $x \in K$, and that T^N be continuous for some $N \geq 1$.

T is called asymptotically nonexpansive in the intermediate sense (Chidume et al. [6]) if T is uniformly continuous and

$$\limsup_{n\to\infty}\sup_{x,y\in K}\left(\left\|T(PT)^{n-1}x - T(PT)^{n-1}y\right\| - \|x-y\|\right) \leq 0. \tag{12}$$

Lemma 2.2. Let E be a uniformly convex Banach space and K be a nonempty closed convex subset which is also a nonexpansive retract of E. Let $T\colon K \to E$ be asymptotically nonexpansive non-self mapping in the intermediate sense with $F(T) \neq \emptyset$. Put

$$G_n = \max\left\{\sup_{x,y\in K}\left(\left\|T(PT)^{n-1}x - T(PT)^{n-1}y\right\| - \|x-y\|\right),\ 0\right\}, \quad \forall n \geq 1,$$

such that $\sum_{n=1}^{\infty} G_n < \infty$. Let $\{\alpha_n^{(i)}\}$, $\{\beta_n^{(i)}\}$ and $\{\gamma_n^{(i)}\}$ are sequences in $[0,1]$ with $\alpha_n^{(i)} + \beta_n^{(i)} + \gamma_n^{(i)} = 1$ for all $i = 1, 2, \ldots, N$. From arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ iteratively by (8), where $\{u_n^{(i)}\}$ are bounded sequences in K for all $i = 1, 2, \ldots, N$ with $\sum_{n=1}^{\infty} u_n^{(i)} < \infty$. Then for any $x^* \in F(T)$, $\lim_{n\to\infty}\|x_n - x^*\|$ exists.

Proof. For any $x^* \in F(T)$, we note that

$$\begin{aligned}
\left\|x_n^{(1)}-x^*\right\| &= \left\|P(\alpha_n^{(1)}T(PT)^{n-1}x_n+\beta_n^{(1)}x_n+\gamma_n^{(1)}u_n^{(1)})-Px^*\right\| \\
&= \left\|\alpha_n^{(1)}T(PT)^{n-1}x_n+\beta_n^{(1)}x_n+\gamma_n^{(1)}u_n^{(1)}-x^*\right\| \\
&\leq \alpha_n^{(1)}\left\|T(PT)^{n-1}x_n-x^*\right\|+\beta_n^{(1)}\left\|x_n-x^*\right\|+\gamma_n^{(1)}\left\|u_n^{(1)}-x^*\right\| \\
&\leq \alpha_n^{(1)}\left\|x_n-x^*\right\|+G_n+\beta_n^{(1)}\left\|x_n-x^*\right\|+\gamma_n^{(1)}\left\|u_n^{(1)}-x^*\right\| \\
&\leq \left(\alpha_n^{(1)}+\beta_n^{(1)}\right)\left\|x_n-x^*\right\|+G_n+\gamma_n^{(1)}\left\|u_n^{(1)}-x^*\right\| \\
&\leq \left\|x_n-x^*\right\|+\sigma_n^{(1)} \qquad (13)
\end{aligned}$$

where $\sigma_n^{(1)} = G_n+\gamma_n^{(1)}\left\|u_n^{(1)}-x^*\right\|$. Since $\sum_{n=1}^{\infty}\gamma_n^{(1)} < \infty$, $\sum_{n=1}^{\infty}G_n < \infty$ and $\{u_n^{(1)}\}$ is bounded, we can see that $\sum_{n=1}^{\infty}\sigma_n^{(1)} < \infty$. It follows from (13) that

$$\begin{aligned}
\left\|x_n^{(2)}-x^*\right\| &= \left\|P(\alpha_n^{(2)}T(PT)^{n-1}x_n^{(1)}+\beta_n^{(2)}x_n+\gamma_n^{(2)}u_n^{(2)})-Px^*\right\| \\
&= \left\|\alpha_n^{(2)}T(PT)^{n-1}x_n^{(1)}+\beta_n^{(2)}x_n+\gamma_n^{(2)}u_n^{(2)}-x^*\right\| \\
&\leq \alpha_n^{(2)}\left\|T(PT)^{n-1}x_n^{(1)}-x^*\right\|+\beta_n^{(2)}\left\|x_n-x^*\right\|+\gamma_n^{(2)}\left\|u_n^{(2)}-x^*\right\| \\
&\leq \alpha_n^{(2)}\left\|x_n^{(1)}-x^*\right\|+G_n+\beta_n^{(2)}\left\|x_n-x^*\right\|+\gamma_n^{(2)}\left\|u_n^{(2)}-x^*\right\| \\
&\leq \alpha_n^{(2)}\Big(\left\|x_n-x^*\right\|+\sigma_n^{(1)}\Big)+G_n+\beta_n^{(2)}\left\|x_n-x^*\right\| \\
&\quad +\gamma_n^{(2)}\left\|u_n^{(2)}-x^*\right\| \\
&\leq \left(\alpha_n^{(2)}+\beta_n^{(2)}\right)\left\|x_n-x^*\right\|+G_n+\alpha_n^{(2)}\sigma_n^{(1)}+\gamma_n^{(2)}\left\|u_n^{(2)}-x^*\right\| \\
&\leq \left\|x_n-x^*\right\|+\sigma_n^{(2)} \qquad (14)
\end{aligned}$$

where $\sigma_n^{(2)} = G_n+\alpha_n^{(2)}\sigma_n^{(1)}+\gamma_n^{(2)}\left\|u_n^{(2)}-x^*\right\|$. Since $\sum_{n=1}^{\infty}G_n < \infty$, $\sum_{n=1}^{\infty}\gamma_n^{(2)} < \infty$, $\sum_{n=1}^{\infty}\sigma_n^{(1)} < \infty$ and $\{u_n^{(2)}\}$ is bounded, we can see that $\sum_{n=1}^{\infty}\sigma_n^{(2)} < \infty$. Similarly, we see that

$$\begin{aligned}\left\|x_n^{(3)}-x^*\right\| &\le \alpha_n^{(3)}\left\|T(PT)^{n-1}x_n^{(2)}-x^*\right\|+\beta_n^{(3)}\|x_n-x^*\|+\gamma_n^{(3)}\left\|u_n^{(3)}-x^*\right\| \\ &\le \alpha_n^{(3)}\left\|x_n^{(2)}-x^*\right\|+G_n+\beta_n^{(3)}\|x_n-x^*\|+\gamma_n^{(3)}\left\|u_n^{(3)}-x^*\right\| \\ &\le \alpha_n^{(3)}\Big(\|x_n-x^*\|+\sigma_n^{(2)}\Big)+G_n+\beta_n^{(3)}\|x_n-x^*\| \\ &\quad +\gamma_n^{(3)}\left\|u_n^{(3)}-x^*\right\| \\ &\le \Big(\alpha_n^{(3)}+\beta_n^{(3)}\Big)\|x_n-x^*\|+G_n+\alpha_n^{(3)}\sigma_n^{(2)}+\gamma_n^{(3)}\left\|u_n^{(3)}-x^*\right\| \\ &\le \|x_n-x^*\|+\sigma_n^{(3)} \qquad (15)\end{aligned}$$

where $\sigma_n^{(3)}=G_n+\alpha_n^{(3)}\sigma_n^{(2)}+\gamma_n^{(3)}\left\|u_n^{(3)}-x^*\right\|$. Since $\sum_{n=1}^{\infty}G_n<\infty$, $\sum_{n=1}^{\infty}\gamma_n^{(3)}<\infty$, $\sum_{n=1}^{\infty}\sigma_n^{(2)}<\infty$ and $\{u_n^{(3)}\}$ is bounded, we can see that $\sum_{n=1}^{\infty}\sigma_n^{(3)}<\infty$. Continuing the above process, we get

$$\begin{aligned}\|x_{n+1}-x^*\| &= \left\|x_n^{(N)}-x^*\right\| \\ &\le \|x_n-x^*\|+\sigma_n^{(N)} \qquad (16)\end{aligned}$$

where $\{\sigma_n^{(N)}\}$ is nonnegative real sequence such that $\sum_{n=1}^{\infty}\sigma_n^{(N)}<\infty$. By Lemma 1.1, we have $\lim_{n\to\infty}\|x_n-x^*\|$ exists. This completes the proof.

Lemma 2.3. Let E be a uniformly convex Banach space and K be a nonempty closed convex subset which is also a nonexpansive retract of E. Let $T\colon K\to E$ be asymptotically nonexpansive non-self mapping in the intermediate sense with $F(T)\neq\emptyset$. Put

$$G_n=\max\left\{\sup_{x,y\in K}\Big(\left\|T(PT)^{n-1}x-T(PT)^{n-1}y\right\|-\|x-y\|\Big),\ 0\right\},\qquad \forall n\ge 1,$$

such that $\sum_{n=1}^{\infty}G_n<\infty$. Let $\{\alpha_n^{(i)}\}$, $\{\beta_n^{(i)}\}$ and $\{\gamma_n^{(i)}\}$ are sequences in $[0,1]$ with $\alpha_n^{(i)}+\beta_{(n)}^i+\gamma_{(n)}^i=1$ for all $i=1,2,\ldots,N$. From arbitrary $x_1\in K$, let $\{x_n\}$ be the sequence defined by (8)and some $\alpha,\beta\in(0,1)$ with the following restrictions:

(i) $0<\alpha\le\alpha_n^{(i)}\le\beta<1$, $1\le i\le N$, $\forall n\ge n_0$ for some $n_0\in\mathbf{N}$;

(ii) $\sum_{n=1}^{\infty}\gamma_n^{(i)}<\infty$, $1\le i\le N$.

Then $\lim_{n\to\infty}\|x_n-Tx_n\|=0$.

Proof. For any $x^* \in F(T)$, it follows from Lemma 2.2 that $\lim_{n\to\infty} \|x_n - x^*\|$ exists. Let $\lim_{n\to\infty} \|x_n - x^*\| = a$ for some $a \geq 0$. we note that

$$\left\|x_n^{(N-1)} - x^*\right\| \leq \|x_n - x^*\| + \sigma_n^{(N-1)}, \quad \forall n \geq 1,$$

where $\{\sigma_n^{(N-1)}\}$ is nonnegative real sequence such that $\sum_{n=1}^{\infty} \sigma_n^{(N-1)} < \infty$. It follows that

$$\limsup_{n\to\infty} \left\|x_n^{(N-1)} - x^*\right\| \leq \limsup_{n\to\infty} \Big(\|x_n - x^*\| + \sigma_n^{(N-1)} \Big) = \lim_{n\to\infty} \|x_n - x^*\| = a$$

and so

$$\begin{aligned} \limsup_{n\to\infty} \left\|T(PT)^{n-1}x_n^{(N-1)} - x^*\right\| &\leq \limsup_{n\to\infty} \Big(\left\|x_n^{(N-1)} - x^*\right\| + G_n \Big) \\ &= \limsup_{n\to\infty} \left\|x_n^{(N-1)} - x^*\right\| \leq a. \end{aligned}$$

Next, consider

$$\left\|T(PT)^{n-1}x_n^{(N-1)} - x^* + \gamma_n^{(N)}(u_n^{(N)} - x_n)\right\| \leq \left\|T(PT)^{n-1}x_n^{(N-1)} - x^*\right\| + \gamma_n^{(N)} \left\|u_n^{(N)} - x_n\right\|.$$

Thus,

$$\limsup_{n\to\infty} \left\|T(PT)^{n-1}x_n^{(N-1)} - x^* + \gamma_n^{(N)}(u_n^{(N)} - x_n)\right\| \quad \leq \quad a. \tag{17}$$

Also,

$$\left\|x_n - x^* + \gamma_n^{(N)}(u_n^{(N)} - x_n)\right\| \leq \|x_n - x^*\| + \gamma_n^{(N)} \left\|u_n^{(N)} - x_n\right\|$$

gives that

$$\limsup_{n\to\infty} \left\|x_n - x^* + \gamma_n^{(N)}(u_n^{(N)} - x_n)\right\| \quad \leq \quad a, \tag{18}$$

and we observe that

$$
\begin{aligned}
x_n^{(N)} - x^* &= \alpha_n^{(N)} T(PT)^{n-1} x_n^{(N-1)} - \alpha_n^{(N)} x^* + \alpha_n^{(N)} \gamma_n^{(N)} u_n^{(N)} - \alpha_n^{(N)} \gamma_n^{(N)} x_n \\
&\quad + (1-\alpha_n^{(N)}) x_n - (1-\alpha_n^{(N)}) x^* - \gamma_n^{(N)} x_n + \gamma_n^{(N)} u_n^{(N)} - \alpha_n^{(N)} \gamma_n^{(N)} u_n^{(N)} \\
&\quad + \alpha_n^{(N)} \gamma_n^{(N)} x_n \\
&= \alpha_n^{(N)} (T(PT)^{n-1} x_n^{(N-1)} - x^* + \gamma_n^{(N)} (u_n^{(N)} - x_n) + (1-\alpha_n^{(N)})(x_n - x^*) \\
&\quad - (1-\alpha_n^{(N)}) \gamma_n^{(N)} x_n + (1-\alpha_n^{(N)}) \gamma_n^{(N)} u_n^{(N)} \\
&= \alpha_n^{(N)} (T(PT)^{n-1} x_n^{(N-1)} - x^* + \gamma_n^{(N)} (u_n^{(N)} - x_n) + (1-\alpha_n^{(N)}) \\
&\quad (x_n - x^* + \gamma_n^{(N)} (u_n^{(N)} - x_n)).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
a &= \lim_{n\to\infty} \|x_n^{(N)} - x^*\| \\
&= \lim_{n\to\infty} \|\alpha_n^{(N)} (T(PT)^{n-1} x_n^{(N-1)} - x^* + \gamma_n^{(N)} (u_n^{(N)} - x_n)) \\
&\qquad + (1-\alpha_n^{(N)})(x_n - x^* + \gamma_n^{(N)} (u_n^{(N)} - x_n))\|.
\end{aligned}
$$

By (17), (18) and Lemma 1.2, we have

$$
\lim_{n\to\infty} \left\| T(PT)^{n-1} x_n^{(N-1)} - x_n \right\| = 0. \tag{19}
$$

Now, we shall show that $\lim_{n\to\infty} \left\| T(PT)^{n-1} x_n^{(N-2)} - x_n \right\| = 0$. For each $n \geq 1$,

$$
\begin{aligned}
\|x_n - x^*\| &\leq \left\| T(PT)^{n-1} x_n^{(N-1)} - x_n \right\| + \left\| T(PT)^{n-1} x_n^{(N-1)} - x^* \right\| \\
&\leq \left\| T(PT)^{n-1} x_n^{(N-1)} - x_n \right\| + \left\| x_n^{(N-1)} - x^* \right\| + G_n.
\end{aligned}
$$

Using (19), we have

$$
a = \lim_{n\to\infty} \|x_n - x^*\| \leq \liminf_{n\to\infty} \left\| x_n^{(N-1)} - x^* \right\|.
$$

It follows that

$$
a \leq \liminf_{n\to\infty} \left\| x_n^{(N-1)} - x^* \right\| \leq \limsup_{n\to\infty} \left\| x_n^{(N-1)} - x^* \right\| \leq a.
$$

This implies that

$$
\lim_{n\to\infty} \left\| x_n^{(N-1)} - x^* \right\| = a.
$$

On the other hand, we have

$$\left\|x_n^{(N-2)} - x^*\right\| \leq \|x_n - x^*\| + \sigma_n^{(N-2)}, \quad \forall n \geq 1,$$

where $\sum_{n=1}^{\infty} \sigma_n^{(N-2)} < \infty$. Therefore,

$$\limsup_{n\to\infty} \left\|x_n^{(N-2)} - x^*\right\| \leq \limsup_{n\to\infty} \Big(\|x_n - x^*\| + G_n\Big) \leq a.$$

Next, consider

$$\begin{aligned} \left\|T(PT)^{n-1}x_n^{(N-2)} - x^* + \gamma_n^{(N-1)}(u_n^{(N-1)} - x_n)\right\| \quad &\leq \quad \left\|T(PT)^{n-1}x_n^{(N-2)} - x^*\right\| \\ &\quad + \gamma_n^{(N-1)} \left\|u_n^{(N-1)} - x_n\right\|. \end{aligned}$$

Thus,

$$\limsup_{n\to\infty} \left\|T(PT)^{n-1}x_n^{(N-2)} - x^* + \gamma_n^{(N-1)}(u_n^{(N-1)} - x_n)\right\| \quad \leq \quad a. \tag{20}$$

Also,

$$\left\|x_n - x^* + \gamma_n^{(N-1)}(u_n^{(N-1)} - x_n)\right\| \leq \|x_n - x^*\| + \gamma_n^{(N-1)} \left\|u_n^{(N-1)} - x_n\right\|$$

gives that

$$\limsup_{n\to\infty} \left\|x_n - x^* + \gamma_n^{(N-1)}(u_n^{(N-1)} - x_n)\right\| \quad \leq \quad a, \tag{21}$$

and we observe that

$$\begin{aligned} x_n^{(N-1)} - x^* \quad &= \quad \alpha_n^{(N-1)}T(PT)^{n-1}x_n^{(N-2)} + (1 - \alpha_n^{(N-1)})x_n - \gamma_n^{(N-1)}x_n \\ &\quad + \gamma_n^{(N-1)}u_n^{(N-1)} - (1 - \alpha_n^{(N-1)})x^* - \alpha_n^{(N-1)}x^* \\ &= \quad \alpha_n^{(N-1)}(T(PT)^{n-1}x_n^{(N-2)} - x^* + \gamma_n^{(N-1)}(u_n^{(N-1)} - x_n) \\ &\quad + (1 - \alpha_n^{(N-1)})(x_n - x^* + \gamma_n^{(N-1)}(u_n^{(N-1)} - x_n)), \end{aligned}$$

and hence

$$\begin{aligned} a &= \lim_{n\to\infty} \|x_n^{(N-1)} - x^*\| \\ &= \lim_{n\to\infty} \|\alpha_n^{(N-1)}(T(PT)^{n-1}x_n^{(N-2)} - x^* + \gamma_n^{(N-1)}(u_n^{(N-1)} - x_n)) \\ &\quad + (1 - \alpha_n^{(N-1)})(x_n - x^* + \gamma_n^{(N-1)}(u_n^{(N-1)} - x_n))\|. \end{aligned}$$

By (20), (21) and Lemma 1.2, we have

$$\lim_{n\to\infty}\left\|T(PT)^{n-1}x_n^{(N-2)}-x_n\right\|=0. \tag{22}$$

Similarly, by using the same argument as in the proof above, we have

$$\lim_{n\to\infty}\left\|T(PT)^{n-1}x_n^{(N-2)}-x^*\right\|=0.$$

Continuing similar process, we have

$$\lim_{n\to\infty}\left\|T(PT)^{n-1}x_n^{(1)}-x_n\right\|=0.$$

Now,

$$\left\|T(PT)^{n-1}x_n-x^*+\gamma_n^{(1)}(u_n^{(1)}-x_n))\right\|\le\left\|T(PT)^{n-1}x_n-x^*\right\|+\gamma_n^{(1)}\left\|u_n^{(1)}-x_n\right\|.$$

Thus,

$$\limsup_{n\to\infty}\left\|T(PT)^{n-1}x_n-x^*+\gamma_n^{(1)}(u_n^{(1)}-x_n))\right\| \quad\le\quad a. \tag{23}$$

Also,

$$\left\|x_n-x^*+\gamma_n^{(1)}(u_n^{(1)}-x_n))\right\|\le\|x_n-x^*\|+\gamma_n^{(1)}\left\|u_n^{(1)}-x_n\right\|$$

gives that

$$\limsup_{n\to\infty}\left\|x_n-x^*+\gamma_n^{(1)}(u_n^{(1)}-x_n))\right\| \quad\le\quad a, \tag{24}$$

and hence

$$\begin{aligned}a&=\lim_{n\to\infty}\|x_n^{(1)}-x^*\|\\&=\lim_{n\to\infty}\|\alpha_n^{(1)}(T(PT)^{n-1}x_n-x^*+\gamma_n^{(1)}(u_n^{(1)}-x_n))\\&\qquad+(1-\alpha_n^{(1)})(x_n-x^*+\gamma_n^{(1)}(u_n^{(1)}-x_n))\|.\end{aligned}$$

By (23), (24) and Lemma 1.2, we have

$$\lim_{n\to\infty} \left\|T(PT)^{n-1}x_n - x_n\right\| = 0, \tag{25}$$

and this implies that

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq \left\|\alpha_n^{(N)}(T(PT)^{n-1}x_n^{(N-1)} + (1-\alpha_n^{(N)} - \gamma_n^{(N)})x_n + \gamma_n^{(N)}u_n^{(N)} - x_n\right\| \\ &\leq \alpha_n^{(N)} \left\|T(PT)^{n-1}x_n^{(N-1)} - x_n\right\| + \gamma_n^{(N)} \left\|u_n^{(N)} - x_n\right\| \qquad (26) \\ &\to 0 \;\text{ as }\; n \to \infty. \end{aligned}$$

Thus, we have

$$\begin{aligned} \left\|T(PT)^{n-1}x_n - x_n\right\| &\leq \left\|T(PT)^{n-1}x_n - T(PT)^{n-1}x_n^{(N-1)}\right\| + \left\|T(PT)^{n-1}x_n^{(N-1)} - x_n\right\| \\ &\leq \left\|x_n - x_n^{(N-1)}\right\| + G_n + \left\|T(PT)^{n-1}x_n^{(N-1)} - x_n\right\| \\ &\leq \alpha_n^{(N-1)} \left\|x_n - T(PT)^{n-1}x_n^{(N-2)}\right\| + \gamma_n^{(N-1)} \left\|u_n^{(N-1)} - x_n\right\| \qquad (27) \\ &\quad + G_n + \left\|T(PT)^{n-1}x_n^{(N-1)} - x_n\right\| \\ &\to 0 \;\text{ as }\; n \to \infty. \end{aligned}$$

and

$$\begin{aligned} \|x_n - Tx_n\| &\leq \|x_{n+1} - x_n\| + \|x_{n+1} - T(PT)^n x_{n+1}\| \\ &\quad + \|T(PT)^n x_{n+1} - T(PT)^n x_n\| \\ &\quad + \|T(PT)^n x_n - Tx_n\| \qquad (28) \end{aligned}$$

It follows from (26), (27), (28) and by uniform continuity of T that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

This completes the proof.

Theorem 2.4. Let E be a uniformly convex Banach space and K be a nonempty closed convex subset which is also a nonexpansive retract of E. Let $T\colon K \to E$ be asymptotically nonexpansive nonself mapping in the intermediate sense with

$F(T) \neq \emptyset$. Put

$$G_n = \max\left\{\sup_{x,y\in K}\Big(\|T(PT)^{n-1}x - T(PT)^{n-1}y\| - \|x-y\|\Big),\ 0\right\}, \qquad \forall n \geq 1,$$

such that $\sum_{n=1}^{\infty} G_n < \infty$. Let $\{\alpha_n^{(i)}\}$, $\{\beta_n^{(i)}\}$ and $\{\gamma_n^{(i)}\}$ are sequences in $[0,1]$ with $\alpha_n^{(i)} + \beta_n^{(i)} + \gamma_n^{(i)} = 1$ for all $i = 1, 2, \ldots, N$. From arbitrary $x_1 \in K$, let $\{x_n\}$ be the sequence defined by (8) and some $\alpha, \beta \in (0,1)$ with the following restrictions:

(i) $0 < \alpha \leq \alpha_n^{(i)} \leq \beta < 1$, $1 \leq i \leq N$, $\forall n \geq n_0$ for some $n_0 \in \mathbf{N}$;

(ii) $\sum_{n=1}^{\infty} \gamma_n^{(i)} < \infty$, $1 \leq i \leq N$.

Suppose T satisfies condition (A). Then $\{x_n\}$ converges strongly to a fixed point of T.

Proof. By Lemma 2.2, we see that $\lim_{n\to\infty} \|x_n - x^*\|$ exists for all $x^* \in F(T)$. Let $\lim_{n\to\infty} \|x_n - x^*\| = a$ for some $a \geq 0$. Without loss of generality, if $a = 0$, there is nothing to prove. Assume that $a > 0$, As proved in Lemma 2.2, we have

$$\|x_{n+1} - x^*\| \quad \leq \quad \|x_n - x^*\| + \sigma_n^{(N)} \tag{29}$$

where $\{\sigma_n^{(N)}\}$ is nonnegative real sequence such that $\sum_{n=1}^{\infty} \sigma_n^{(N)} < \infty$.

This gives that

$$d(x_{n+1}, F(T)) \leq d(x_n, F(T)) + \sigma_n^{(N)}.$$

Applying Lemma 1.1 to the above inequality, we obtain that $\lim_{n\to\infty} d(x_n, F(T))$ exists. Also by Lemma 2.3, $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Since T satisfies condition (A), we have $\lim_{n\to\infty} f(d(x_n, F(T))) = 0$. Since f is a nondecreasing function and $f(0) = 0$, therefore, we conclude that $\lim_{n\to\infty} d(x_n, F(T)) = 0$.

Next we show that $\{x_n\}$ is a Cauchy sequence. For any $m, n \geq 1$ and for given $x^* \in F(T)$, we have from (2.21)

$$\begin{aligned}
\|x_{n+m} - x^*\| &\leq \|x_{n+m-1} - x^*\| + \sigma_{n+m-1}^{(N)} \\
&\leq \|x_{n+m-2} - x^*\| + \sigma_{n+m-2}^{(N)} + \sigma_{n+m-1}^{(N)} \\
&= \ldots \\
&= \ldots \\
&\leq \|x_n - x^*\| + \sum_{k=n}^{n+m-1} \sigma_k^{(N)}.
\end{aligned} \tag{30}$$

Since

$$\lim_{n\to\infty} d(x_n, F(T)) = 0, \qquad \sum_{n=1}^{\infty} \sigma_n^{(N)} < \infty \tag{31}$$

for any given $\varepsilon > 0$, there exists a positive integer n_1 such that

$$d(x_n, F(T)) < \frac{\varepsilon}{5}, \qquad \sum_{k=n}^{\infty} \sigma_k^{(N)} < \frac{\varepsilon}{2} \quad \forall n \geq n_1. \tag{32}$$

Hence, there exists $q \in F(T)$ such that

$$\|x_n - q\| < \frac{\varepsilon}{4} \quad \forall n \geq n_1. \tag{33}$$

Consequently, for any $n \geq n_1$ and $m \geq 1$, by (30), we have

$$\begin{aligned} \|x_{n+m} - x_n\| &\leq \|x_{n+m} - q\| + \|x_n - q\| \\ &\leq \Big\{ \|x_n - q\| + \sum_{k=n}^{n+m-1} \sigma_k^{(N)} \Big\} + \|x_n - q\| \\ &\leq 2\,\|x_n - q\| + \sum_{k=n}^{n+m-1} \sigma_k^{(N)} \\ &< 2.\frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned} \tag{34}$$

This implies that $\{x_n\}$ is a Cauchy sequence in E and so is convergent since E is complete. Let $\lim_{n\to\infty} x_n = q^*$. Then $q^* \in K$. It remains to show that $q^* \in F(T)$. Let $\varepsilon_1 > 0$ be given. Then there exists a natural number n_2 such that $\|x_n - q^*\| < \frac{\varepsilon_1}{4}$ for all $n \geq n_2$. Since $\lim_{n\to\infty} d(x_n, F(T)) = 0$, there exists a natural number $n_3 \geq n_2$ such that for all $n \geq n_3$ we have $d(x_n, F(T)) < \frac{\varepsilon_1}{5}$ and in particular we have $d(x_{n_3}, F(T)) \leq \frac{\varepsilon_1}{5}$. Therefore, there exists $w^* \in F(T)$ such that $\|x_{n_3} - w^*\| < \frac{\varepsilon_1}{4}$. For any $n \geq n_3$, we have

$$\begin{aligned} \|Tq^* - q^*\| &\leq \|Tq^* - w^*\| + \|w^* - q^*\| \\ &\leq 2\,\|q^* - w^*\| \\ &\leq 2\Big(\|q^* - x_{n_3}\| + \|x_{n_3} - w^*\| \Big) \\ &< 2\Big(\frac{\varepsilon_1}{4} + \frac{\varepsilon_1}{4}\Big) \\ &< \varepsilon_1. \end{aligned}$$

This implies that $Tq^* = q^*$. Hence $q^* \in F(T)$. Thus $\{x_n\}$ converges strongly to a fixed point of the mapping T. This completes the proof.

Lemma 2.5. Let E be a uniformly convex Banach space and K be a nonempty closed convex subset which is also a nonexpansive retract of E. Let $T\colon K \to E$ be asymptotically nonexpansive non-self mapping in the intermediate sense with $F(T) \neq \emptyset$. Put

$$G_n = \max\left\{\sup_{x,y\in K}\Big(\|T(PT)^{n-1}x - T(PT)^{n-1}y\| - \|x-y\|\Big),\ 0\right\}, \qquad \forall n \geq 1,$$

such that $\sum_{n=1}^{\infty} G_n < \infty$. Let $\{\alpha_n^{(i)}\}$, $\{\beta_n^{(i)}\}$ and $\{\gamma_n^{(i)}\}$ are sequences in $[0,1]$ with $\alpha_n^{(i)} + \beta_n^{(i)} + \gamma_n^{(i)} = 1$ for all $i = 1, 2, \dots, N$. From arbitrary $x_1 \in K$, let $\{x_n\}$ be the sequence defined by (8) and some $\alpha, \beta \in (0,1)$ with the following restrictions:

(i) $0 < \alpha \leq \alpha_n^{(i)} \leq \beta < 1$, $1 \leq i \leq N$, $\forall n \geq n_0$ for some $n_0 \in \mathbf{N}$;

(ii) $\sum_{n=1}^{\infty} \gamma_n^{(i)} < \infty$, $1 \leq i \leq N$.

Then for all $u, v \in F(T)$, $\lim_{n\to\infty} \|tx_n + (1-t)u - v\|$ exists for all $t \in [0,1]$.

Proof. Since $\{x_n\}$ is bounded there exists $R > 0$ such that $\{x_n\} \subset C := B_R(0) \cap K$, where $B_R(0) = \{x \in E : \|x\| \leq R\}$. Then C is a nonempty closed convex subset of E. Basically we follow the idea of [17]. Let $a_n(t) = \|tx_n + (1-t)u - v\|$. Then $\lim_{n\to\infty} a_n(0) = \|u - v\|$ and from Lemma 2.2, $\lim_{n\to\infty} a_n(1) = \lim_{n\to\infty} \|x_n - v\|$ exists. Without loss of generality, we may assume that $\lim_{n\to\infty} \|x_n - u\| = r > 0$ and $t \in (0,1)$. Define $W_n\colon C \to C$ by

$$\begin{array}{rcl}
W_n x &=& P(\alpha_n^{(N)} T(PT)^{n-1} x_n^{(N-1)} + \beta_n^{(N)} x_n + \gamma_n^{(N)} u_n^{(N)}) \\
x_n^{(N-1)} &=& P(\alpha_n^{(N-1)} T(PT)^{n-1} x_n^{(N-2)} + \beta_n^{(N-1)} x_n + \gamma_n^{(N-1)} u_n^{(N-1)}) \\
\dots &=& \dots \\
\dots &=& \dots \\
x_n^{(3)} &=& P(\alpha_n^{(3)} T(PT)^{n-1} x_n^{(2)} + \beta_n^{(3)} x_n + \gamma_n^{(3)} u_n^{(3)}) \\
x_n^{(2)} &=& P(\alpha_n^{(2)} T(PT)^{n-1} x_n^{(1)} + \beta_n^{(2)} x_n + \gamma_n^{(2)} u_n^{(2)}) \\
x_n^{(1)} &=& P(\alpha_n^{(1)} T(PT)^{n-1} x_n + \beta_n^{(1)} x_n + \gamma_n^{(1)} u_n^{(1)}), \quad x \in C
\end{array} \tag{35}$$

Then

$$\|W_n x - W_n y\| \leq \|x - y\| + G_n.$$

Set $S_{n,m} = W_{n+m-1}W_{n+m-2}\dots W_n$, $m \geq 1$, and

$$b_{n,m} = \|S_{n,m}(tx_n + (1-t)u) - (tS_{n,m}x_n + (1-t)v)\|.$$

Then

$$\begin{aligned}\|S_{n,m}x - S_{n,m}y\| &\leq \|x-y\| + G_{n+m-1} + G_{n+m-2} + \cdots + G_n \\ &\leq \|x-y\| + \sum_{j=n}^{n+m-1} G_j\end{aligned}$$

It is easy to see that $S_{n,m}x_n = x_{n+m}$ and

$$\|S_{n,m}x^* - x^*\| \leq \sum_{j=n}^{n+m-1} G_j,$$

since $\sum_{n=1}^{\infty} G_n < \infty$, we have $\|S_{n,m}x^* - x^*\| \to 0$ as $n \to \infty$ and hence $S_{n,m}x^* = x^*$. Observe that

$$\begin{aligned}a_{n+m}(t) &= \|tx_{n+m} + (1-t)u - v\| \\ &\leq b_{n,m} + \|S_{n,m}(tx_n + (1-t)u) - v\| \\ &\leq b_{n,m} + \|tx_n + (1-t)u - v\| + \sum_{j=n}^{n+m-1} G_j \\ &\leq b_{n,m} + a_n(t) + \sum_{j=n}^{n+m-1} G_j.\end{aligned}$$

By using [[10], Theorem 2.3], we have

$$\begin{aligned}b_{n,m} &\leq \varphi^{-1}(\|x_n - u\| - \|S_{n,m}x_n - S_{n,m}u\|) \\ &\leq \varphi^{-1}(\|x_n - u\| - \|x_{n+m} - u + u - S_{n,m}u\|) \\ &\leq \varphi^{-1}(\|x_n - u\| - (\|x_{n+m} - u\| - \|S_{n,m}u - u\|))\end{aligned}$$

and so the sequence $\{b_{n,m}\}$ converges uniformly to 0, i.e., $b_{n,m} \to 0$ as $n \to \infty$.

Since $\sum_{n=1}^{\infty} G_n < \infty$, therefore we have

$$\limsup_{n\to\infty} a_n(t) \leq \lim_{n,m\to\infty} b_{n,m} + \liminf_{n\to\infty} a_n(t) + 0 = \liminf_{n\to\infty} a_n(t).$$

This shows that $\lim_{n\to\infty} a_n(t)$ exists, that is,

$$\lim_{n\to\infty} \|tx_n + (1-t)u - v\|$$

exists for all $t \in [0,1]$. This completes the proof.

Theorem 2.6. Let E be a uniformly convex Banach space such that its dual E^* has the *Kadec-Klee* property and K be a nonempty closed convex subset which is also a nonexpansive retract of E. Let $T\colon K \to E$ be asymptotically nonexpansive non-self mapping in the intermediate sense with $F(T) \neq \emptyset$. Put

$$G_n = \max\left\{ \sup_{x,y\in K} \Big(\left\|T(PT)^{n-1}x - T(PT)^{n-1}y\right\| - \|x-y\| \Big),\ 0 \right\}, \qquad \forall n \geq 1,$$

such that $\sum_{n=1}^{\infty} G_n < \infty$. Let $\{\alpha_n^{(i)}\}$, $\{\beta_n^{(i)}\}$ and $\{\gamma_n^{(i)}\}$ are sequences in $[0,1]$ with $\alpha_n^{(i)} + \beta_n^{(i)} + \gamma_n^{(i)} = 1$ for all $i = 1, 2, \ldots, N$. From arbitrary $x_1 \in K$, let $\{x_n\}$ be the sequence defined by (8) and some $\alpha, \beta \in (0,1)$ with the following restrictions:

(i) $0 < \alpha \leq \alpha_n^{(i)} \leq \beta < 1$, $1 \leq i \leq N$, $\forall n \geq n_0$ for some $n_0 \in \mathbf{N}$;

(ii) $\sum_{n=1}^{\infty} \gamma_n^{(i)} < \infty$, $1 \leq i \leq N$.

Then $\{x_n\}$ converges weakly to some fixed point of T.

Proof. By Lemma 2.2, we have $\lim_{n\to\infty} \|x_n - x^*\|$ exists for all $x^* \in F(T)$. This implies that $\{x_n\}$ is a bounded sequence in K. Since E is reflexive, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ converges weakly to some $x^* \in K$. By Lemma 2.3, we have $\lim_{j\to\infty} \left\|x_{n_j} - Tx_{n_j}\right\| = 0$. Since T is uniformly continuous, we can get that

$$\lim_{m\to\infty} \Big(\limsup_{j} \left\|T(PT)^{m-1}x_{n_j} - x_{n_j}\right\| \Big) = 0.$$

Now Lemma 1.3 guarantees that $Tx^* = x^*$, hence this means that $x^* \in F(T)$. It remains to show that $\{x_n\}$ converges weakly to x^*. Suppose $\{x_{n_i}\}$ is another subsequence of $\{x_n\}$ converges weakly to some y^*. Then $y^* \in K$ and so $x^*, y^* \in \omega_w(x_n) \cap F(T)$. By Lemma 2.5, the limit

$$\lim_{n\to\infty} \|tx_n + (1-t)x^* - y^*\|$$

exists for all $t \in [0,1]$. By Lemma 1.4, we have $x^* = y^*$. As a result, $\omega_w(x_n) \cap F(T)$ is a singleton, and so $\{x_n\}$ converges weakly to a fixed point of T. This completes the proof.

Remark 2.7. Our results extend the corresponding results of Su and Qin [31] to the case of finite-step iterative sequences with errors for more general class of asymptotically nonexpansive nonself mappings. Also our iteration scheme generalizes the scheme of [31].

Remark 2.8. Our results also extend the corresponding results of Chidume et al [6] to the case of finite-step iterative sequences with errors.

Remark 2.9. Our results also extend the corresponding results of Plubtieng and Wangkeeree [21] to the case of non-self mappings and finite-step iteration.

Remark 2.10. Our results also extend the corresponding results of Chidume et al [4] to the case of finite-step iterative sequences with errors and more general class of asymptotically nonexpansive nonself mappings.

Remark 2.11.(see [6]) It is well known that duals of reflexive Banach spaces with Fréchet differentiable norm have the *Kadec-Klee* property. However, it is worth mentioning that there exist uniformly convex Banach spaces which have neither a Fréchet differentiable norm nor satisfy Opial's condition but their duals do have the *Kadec-Klee* property.

Remark 2.12. Theorem 2.4 extends Theorem 1.5 of Schu [11] and the corresponding result of Rhoades [1], and Osilike and Aniagbosor [18] to the case of more general class of non-self mappings and finite-step iteration scheme with errors. Furthermore, no boundedness condition is imposed on K. Under the additional hypothesis that the dual E^* of E has the *Kadec-Klee* property, Theorem 2.6 generalizes Theorem 2.1 of Schu [12] to the case of non-self maps and finite-step iteration scheme with errors in Banach spaces that includes L_p spaces $(1 < p < \infty)$, with Opial's condition and boundedness of K dispensed with. Since duals of reflexive Banach spaces with Fréchet differentiable norms have the *Kadec-Klee* property, Theorem 2.6 extends Theorem 3.1 of Tan and Xu [17] to the case of non-self maps which are asymptotically nonexpansive in the intermediate sense and finite-step iteration scheme with errors, with boundedness of K dispensed with.

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