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SOME COMMON FIXED POINT THEOREMS IN VECTOR METRIC SPACES

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Abstract

In this paper we give some theorems on point of coincidence and common fixed points for two self mappings satisfying some general contractive conditions in vector metric spaces. Our results generalize some well-known recent results.

1 Introduction

Vector metric space, which is introduced in [5] by motivated the paper [6], is generalization of metric space, where the metric is Riesz space valued. Actually, in both of them, the metric map is vector space valued. One of the differences between our metric definition and Huang-Zhang's metric definition is that there exists a cone due to the natural existence of ordering on Riesz space. The other difference is that our definition eliminates the requirement for the vector space to have a topological structure.

A Riesz space (or a vector lattice) is an ordered vector space and a lattice. Let E be a Riesz space with the positive cone $E_+ = \{x \in E : x \geq 0\}$. If (a_n) is a decreasing sequence in E such that $\inf a_n = a$, we write $a_n \downarrow a$.

Definition 1. *The Riesz space E is said to be Archimedean if $\frac{1}{n}a \downarrow 0$ holds for every $a \in E_+$.*

Definition 2. *A sequence (b_n) is said to order convergent (or o-convergent) to b if there is a sequence (a_n) in E satisfying $a_n \downarrow 0$ and $|b_n - b| \leq a_n$ for all n, and written $b_n \xrightarrow{o} b$ or $o\text{-}\lim b_n = b$, where $|a| = \sup\{a, -a\}$ for any $a \in E$.*

Definition 3. *A sequence (b_n) is said to be order-Cauchy (or o-Cauchy) if there exists a sequence (a_n) in E such that $a_n \downarrow 0$ and $|b_n - b_{n+p}| \leq a_n$ holds for all n and p.*

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Definition 4. *The Riesz space E is said to be o-Cauchy complete if every o-Cauchy sequence is o-convergent.*

For notations and other facts regarding Riesz spaces we refer to [2].

In Section 2, we recall some basic concepts of vector metric spaces, then in Section 3, we give the main results of this work.

2 Vector Metric Spaces

We can find the following concepts and properties in [5].

Definition 5. *Let X be a non-empty set and E be a Riesz space. The function $d : X \times X \to E$ is said to be a vector metric (or E-metric) if it is satisfying the following properties:*
(vm1) $d(x, y) = 0$ if and only if $x = y$,
(vm2) $d(x, y) \leq d(x, z) + d(y, z)$
for all $x, y, z \in X$. Also the triple (X, d, E) (briefly X with the default parameters omitted) is said to be vector metric space.

For arbitrary elements x, y, z, w of a vector metric space, the following statements are satisfied.
(i) $0 \leq d(x, y)$; (ii) $d(x, y) = d(y, x)$; (iii) $|d(x, z) - d(y, z)| \leq d(x, y)$;
(iv) $|d(x, z) - d(y, w)| \leq d(x, y) + d(z, w)$.

Now we give some examples of vector metric spaces.

Example 1. *(a) A Riesz space E is a vector metric space with $d : E \times E \to E$ defined by $d(x, y) = |x - y|$. This vector metric is called to be absolute valued metric on E.*

It is well known that $\mathbb{R}^2$ is a Riesz space with coordinatwise ordering defined by

$$(x_1, y_1) \leq (x_2, y_2) \Longleftrightarrow x_1 \leq x_2 \text{ and } y_1 \leq y_2$$

for $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$. Again $\mathbb{R}^2$ is a Riesz space with lexicographical ordering defined by

$$(x_1, y_1) \leq (x_2, y_2) \Longleftrightarrow x_1 < x_2 \text{ or } x_1 = x_2, y_1 \leq y_2.$$

Note that $\mathbb{R}^2$ is Archimedean with coordinatwise ordering but not with lexicographical ordering.

Example 2. *(a) Let $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ defined by*

$$d((x_1, y_1), (x_2, y_2)) = (\alpha |x_1 - x_2|, \beta |y_1 - y_2|)$$

is a vector metric, where α, β are positive real numbers.
(b) Let $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2$ defined by

$$d(x, y) = (\alpha |x - y|, \beta |x - y|)$$

is a vector metric, where $\alpha, \beta \geq 0$ and $\alpha + \beta > 0$.

Definition 6. *A sequence* (x_n) *in a vector metric space* (X, d, E) *vectorial converges (or* E*-converges) to some* $x \in E$*, written* $x_n \overset{d,E}{\to} x$*, if there is a sequence* (a_n) *in* E *satisfying* $a_n \downarrow 0$ *and* $d(x_n, x) \leq a_n$ *for all* n*.*

Definition 7. *A sequence* (x_n) *is called* E*-Cauchy sequence whenever there exists a sequence* (a_n) *in* E *such that* $a_n \downarrow 0$ *and* $d(x_n, x_{n+p}) \leq a_n$ *holds for all* n *and* p*.*

Definition 8. *A vector metric space* X *is called* E*-complete if each* E*-Cauchy sequence in* X E*-converges to a limit in* X*.*

Using the above definitions, we have the following properties.

If $x_n \overset{d,E}{\to} x$, then (i) The limit x is unique, (ii) Every subsequence of (x_n) E-converges to x, (iii) If also $y_n \overset{d,E}{\to} y$, then $d(x_n, y_n) \overset{o}{\to} d(x, y)$.

When $E = \mathbb{R}$, the concepts of vectorial convergence and convergence in metric are the same. When also $X = E$ and d is the concepts of absolute valued vector metric, vectorial convergence and convergence in order are the same. When $E = \mathbb{R}$, the concepts of E-Cauchy sequence and Cauchy sequence are the same.

Remark 1. *If* E *is a Riesz space and* $a \leq ka$ *where* $a \in E_+, k \in [0, 1)$*, then* $a = 0$*.*

Proof. The condition $a \leq ka$ means that $-(1-k)a = ka - a \in E_+$. Since $a \in E_+$ and $1 - k > 0$, then also $(1-k)a \in E_+$. Thus we have $(1-k)a = 0$ and $a = 0$. □

3 Main Results

Recently, many authors have studied on common fixed point theorems for weakly compatible pairs (see [1], [3], [4], [7] and [8]). Let T and S be self maps of a set X. If $y = Tx = Sx$ for some $x \in X$, then y is said to be a point of coincidence and x is said to be a coincidence point of T and S. If T and S are weakly compatible, that is, they are commuting at their coincidence point on X, then the point of coincidence y is the unique common fixed point of these maps [1].

Theorem 1. *Let* X *be an vector metric space with* E *is Archimedean. Suppose the mappings* $S, T : X \to X$ *satisfies the following conditions*

(i) for all $x, y \in X$*,*

$$d(Tx, Ty) \leq ku(x, y) \tag{1}$$

where $k \in [0, 1)$ *is a constant and*

$$u(x, y) \in \left\{ d(Sx, Sy), d(Sx, Tx), d(Sy, Ty), \frac{1}{2}[d(Sx, Ty) + d(Sy, Tx)] \right\},$$

(ii) $T(X) \subseteq S(X)$*,*

(iii) $S(X)$ *or* $T(X)$ *is* E*-complete subspace of* X*.*

Then T *and* S *have a unique point of coincidence in* X*. Moreover, if* S *and* T *are weakly compatible, then they have a unique common fixed point in* X*.*

Proof. Let $x_0, x_1 \in X$. Define the sequence (x_n) by $Sx_{n+1} = Tx_n = y_n$ for $n \in \mathbb{N}$.

We first show that

$$d(y_n, y_{n+1}) \leq kd(y_{n-1}, y_n), \tag{2}$$

for all n. We have that

$$d(y_n, y_{n+1}) = d(Tx_n, Tx_{n+1}) \leq ku(x_n, x_{n+1}),$$

for all n. Now we have to consider the following there cases:

If $u(x_n, x_{n+1}) = d(y_{n-1}, y_n)$ then clearly (2) holds. If $u(x_n, x_{n+1}) = d(y_n, y_{n+1})$ then according to Remark 1 $d(y_n, y_{n+1}) = 0$,and (2) is immediate. Finally, suppose that $u(x_n, x_{n+1}) = \frac{1}{2}d(y_{n-1}, y_{n+1})$. Then,

$$d(y_n, y_{n+1}) \leq \frac{k}{2}d(y_{n-1}, y_{n+1}) \leq \frac{k}{2}d(y_{n-1}, y_n) + \frac{1}{2}d(y_n, y_{n+1})$$

holds, and we prove (2).

We have

$$d(y_n, y_{n+1}) \leq k^n d(y_0, y_1).$$

Thus for all n and p,

$$\begin{aligned} d(y_n, y_{n+p}) &\leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \cdots + d(y_{n+p-1}, y_{n+p}) \\ &\leq (k^n + k^{n+1} + \cdots + k^{n+p-1})d(y_0, y_1) \\ &\leq \frac{k^n}{1-k}d(y_0, y_1) \end{aligned}$$

holds. Now, since E is Archimedean then (y_n) is an E-Cauchy sequence. Since the range of S contains the range of T and the range of at least one is E-complete, there exists a $z \in S(X)$ such that $Sx_n \overset{d,E}{\to} z$. Hence there exists a sequence (a_n) in E such that $a_n \downarrow 0$ and $d(Sx_n, z) \leq a_n$. On the other hand, we can find $w \in X$ such that $Sw = z$.

Let us show that $Tw = z$. We have

$$d(Tw, z) \leq d(Tw, Tx_n) + d(Tx_n, z) \leq ku(x_n, w) + a_{n+1}$$

for all n. Since

$$u(x_n, w) \in \left\{ d(Sx_n, Sw), d(Sx_n, Tx_n), d(Sw, Tw), \frac{1}{2}[d(Sx_n, Tw) + d(Sw, Tx_n)] \right\},$$

at least one of the following four cases holds for all n.

Case 1:

$$d(Tw, z) \leq d(Sx_n, Sw) + a_{n+1} \leq a_n + a_{n+1} \leq 2a_n.$$

Case 2:

$$d(Tw, z) \leq d(Sx_n, Tx_n) + a_{n+1} \leq d(Sx_n, z) + 2a_{n+1} \leq 3a_n.$$

Case 3:

$$d(Tw,z) \leq kd(Sw,Tw) + a_{n+1} \leq kd(Tw,z) + a_n,$$

that is, $d(Tw,z) \leq \frac{1}{1-k}a_n$.

Case 4:

$$\begin{aligned} d(Tw,z) &\leq \frac{1}{2}[d(Sx_n,Tw) + d(Sw,Tx_n)] + a_{n+1} \\ &\leq \frac{1}{2}d(Sx_n,Tw) + \frac{3}{2}a_{n+1} \\ &\leq \frac{1}{2}d(Sx_n,z) + \frac{1}{2}d(Tw,z) + \frac{3}{2}a_n \\ &\leq \frac{1}{2}d(Tw,z) + 2a_n, \end{aligned}$$

that is, $d(Tw,z) \leq 4a_n$.

Since the infimum of sequences on the right side of last inequality are zero, then $d(Tw,z) = 0$, i.e. $Tw = z$. Therefore, z is a point of coincidence of T and S. If z_1 is another point of coincidence then there is $w_1 \in X$ with $z_1 = Tw_1 = Sw_1$. Now from (1), it follows that

$$d(z,z_1) = d(Tw,Tw_1) \leq ku(w,w_1),$$

where

$$\begin{aligned} u(w,w_1) &\in \left\{d(Sw,Sw_1), d(Sw,Tw), d(Sw_1,Tw_1), \frac{1}{2}[d(Sw,Tw_1) + d(Sw_1,Tw)]\right\} \\ &= \{0, d(z,z_1)\}. \end{aligned}$$

Hence, $d(z,z_1) = 0$,that is, $z = z_1$.

If S and T are weakly compatible, then it is obvious that z is unique common fixed point of T and S by [1]. □

Theorem 2. *Let X be an vector metric space with E is Archimedean. Suppose the mappings $S,T : X \to X$ satisfies the following conditions*

(i) for all $x,y \in X$,

$$d(Tx,Ty) \leq ku(x,y) \tag{3}$$

where $k \in [0,1)$ is a constant and

$$u(x,y) \in \left\{d(Sx,Sy), \frac{1}{2}[d(Sx,Tx) + d(Sy,Ty)], \frac{1}{2}[d(Sx,Ty) + d(Sy,Tx)]\right\},$$

(ii) $T(X) \subseteq S(X)$,

(iii) $S(X)$ or $T(X)$ is E-complete subspace of X.

Then T and S have a unique point of coincidence in X. Moreover, if S and T are weakly compatible, then they have a unique common fixed point in X.

Proof. Let us define the sequences (x_n) and (y_n) as in the proof of Theorem 1. We first show that

$$d(y_n, y_{n+1}) \leq kd(y_{n-1}, y_n), \tag{4}$$

for all n. Notice that

$$d(y_n, y_{n+1}) = d(Tx_n, Tx_{n+1}) \leq ku(x_n, x_{n+1}),$$

for all n.

As in Theorem 1, we have to consider three cases: $u(x_n, x_{n+1}) = d(y_{n-1}, y_n)$, $u(x_n, x_{n+1}) = \frac{1}{2}[d(y_{n-1}, y_n) + d(y_n, y_{n+1})]$ and $u(x_n, x_{n+1}) = \frac{1}{2}d(y_{n-1}, y_{n+1})$. First and third have been shown in the proof of Theorem 1. Consider only the second case. If $u(x_n, x_{n+1}) = \frac{1}{2}[d(y_{n-1}, y_n) + d(y_n, y_{n+1})]$, then from (3) we have

$$d(y_n, y_{n+1}) \leq \frac{1}{2}[d(y_{n-1}, y_n) + d(y_n, y_{n+1})] \leq \frac{k}{2}d(y_{n-1}, y_n) + \frac{1}{2}d(y_n, y_{n+1}).$$

Hence, (4) holds.

In the proof of Theorem 1 we illustrate that (y_n) is an E-Cauchy sequence. Then there exist $z \in S(X)$, $w \in X$ and (a_n) in E such that $Sw = z$, $d(Sx_n, z) \leq a_n$ and $a_n \downarrow 0$.

Now, we have to show that $Tw = z$. We have

$$d(Tw, z) \leq d(Tw, Tx_n) + d(Tx_n, z) \leq u(x_n, w) + a_{n+1}$$

for all n. Since

$$u(x_n, w) \in \left\{ d(Sx_n, Sw), \frac{1}{2}[d(Sx_n, Tx_n) + d(Sw, Tw)], \frac{1}{2}[d(Sx_n, Tw) + d(Sw, Tx_n)] \right\},$$

at least one of the three cases holds for all n. Consider only the case of $u(x_n, w) = \frac{1}{2}[d(Sx_n, Tx_n) + d(Sw, Tw)]$ because the other two cases have shown that the proof of Theorem 1. It is satisfied that

$$\begin{aligned} d(Tw, z) &\leq \frac{1}{2}[d(Sx_n, Tx_n) + d(Sw, Tw)] + a_{n+1} \\ &\leq \frac{1}{2}d(Sx_n, z) + \frac{1}{2}d(Tx_n, z) + \frac{1}{2}d(Tw, z) + a_{n+1} \\ &\leq \frac{1}{2}a_n + \frac{1}{2}d(Tw, z) + \frac{3}{2}a_{n+1} \\ &\leq \frac{1}{2}d(Tw, z) + 2a_n, \end{aligned}$$

that is, $d(Tw, z) \leq 4a_n$. Since $4a_n \downarrow 0$, then $Tw = z$. Hence, z is a point of coincidence of T and S. The uniqueness of z as in the proof of Theorem 1. Also, if S and T are weakly compatible, then it is obvious that z is unique common fixed point of T and S by [1]. □

Theorem 3. *Let X be an vector metric space with E is Archimedean. Suppose the mappings $S,T : X \to X$ satisfies the following conditions*
(i) for all $x, y \in X$,

$$d(Tx,Ty) \leq bd(Sx,Tx) + cd(Sy,Ty) + ed(Sx,Ty) + fd(Sy,Tx) + gd(Sx,Sy) \quad (5)$$

where a, b, c, e and f are nonnegative and $b + c + e + f + g < 1$,
(ii) $T(X) \subseteq S(X)$,
(iii) $S(X)$ or $T(X)$ is E-complete subspace of X.
Then T and S have a unique point of coincidence in X. Moreover, if S and T are weakly compatible, then they have a unique common fixed point in X.

Proof. Let us define the sequences (x_n) and (y_n) as in the proof of Theorem 1. We have to show that

$$d(y_n, y_{n+1}) \leq kd(y_{n-1}, y_n), \quad (6)$$

for some $k \in [0, 1)$ and all n. Consider $Sx_{n+1} = Tx_n = y_n$ for all n. Then

$$d(y_n, y_{n+1}) \leq (b + g)d(y_{n-1}, y_n) + cd(y_n, y_{n+1}) + ed(y_{n-1}, y_{n+1})$$

and

$$d(y_{n+1}, y_n) \leq bd(y_n, y_{n+1}) + (c + g)d(y_{n-1}, y_n) + fd(y_{n-1}, y_{n+1})$$

for all n. Hence,

$$d(y_n, y_{n+1}) \leq \frac{b + c + e + f + 2g}{2 - (b + c + e + f)} d(y_{n-1}, y_n).$$

If we choose $k = \frac{b+c+e+f+2g}{2-(b+c+e+f)}$, then $k \in [0, 1)$ and (6) is hold.

In the proof of Theorem 1 we illustrate that (y_n) is an E-Cauchy sequence.Then there exist $z \in S(X)$, $w \in X$ and (a_n) in E such that $Sw = z$, $d(Sx_n, z) \leq a_n$ and $a_n \downarrow 0$.

Let us show that $Tw = z$. We have

$$\begin{aligned} d(Tw, z) &\leq d(Tw, Tx_n) + d(Tx_n, z) \\ &\leq (b + f)d(Tw, z) + (c + f + g)d(Sx_n, z) + (c + e + 1)d(Tx_n, z) \\ &\leq (b + f)d(Tw, z) + (c + f + g)a_n + (c + e + 1)a_{n+1} \\ &\leq (b + f)d(Tw, z) + (2c + e + f + g + 1)a_n, \end{aligned}$$

that is, $d(Tw, z) \leq \frac{2c+e+f+g+1}{1-(b+f)} a_n$ for all n. Then $d(Tw, z) = 0$, i.e. $Tw = z$. Hence, z is a point of coincidence of T and S. The uniqueness of z is easily seen. Also, if S and T are weakly compatible, then it is obvious that z is unique common fixed point of T and S by [1]. □

Corollary 1. *Let X be an vector metric space with E is Archimedean. Suppose the mappings $S,T : X \to X$ satisfies the following conditions*
(i) for all $x, y \in X$,

$$d(Tx, Ty) \leq kd(Sx, Sy) \quad (7)$$

where $k < 1$,

(ii) $T(X) \subseteq S(X)$,

(iii) $S(X)$ *or* $T(X)$ *is* E*-complete subspace of* X.

Then T *and* S *have a unique point of coincidence in* X. *Moreover, if* S *and* T *are weakly compatible, then they have a unique common fixed point in* X.

Now we give an illustrative example.

Example 3. *Let* $E = \mathbb{R}^2$ *with coordinatwise ordering (since* $\mathbb{R}^2$ *is not Archimedean with lexicographical ordering, then we can not use this ordering),* $X = \mathbb{R}$, $d(x, y) = (|x - y|, \alpha |x - y|)$, $\alpha > 0$, $Tx = x^2 + 1$ *and* $Sx = 2x^2$. *Then, for all* $x, y \in X$ *we have*

$$d(Tx, Ty) = \frac{1}{2} d(Sx, Sy) \leq kd(Sx, Sy)$$

for $k \in [\frac{1}{2}, 1)$,

$$T(X) = [1, \infty) \subseteq [0, \infty) = S(X)$$

and $T(X)$ *is* E*-complete subspace of* X. *Therefore all conditions of Corollary 1 are satisfied. Thus* T *and* S *have a unique point of coincidence in* X.

Remark 2. *Note that in Example 3,* $z = 2 \in X$ *is unique point of coincidence and* $x = 1$ *and* $y = -1$ *are coincidence points of* T *and* S,

$$2 = T(1) = S(1) = T(-1) = S(-1).$$

Also, note that T *and* S *have not a common fixed point, because they are not weakly compatible, since*

$$TS(1) = T(2) = 5 \neq 8 = S(2) = ST(1).$$

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