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DOUBLE SEQUENCE TRANSFORMATIONS THAT GUARANTEE A GIVEN RATE OF P-CONVERGENCE

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Abstract

In this paper the following sequence space is presented. Let $[t]$ be a positive double sequence and define the sequence space $\Omega^{''}(t) = \{$complex sequences $x : x_{k,l} = O(t_{k,l})\}$. The set of geometrically dominated double sequences is defined as $G^{''} = \cup_{r,s\in(0,1)}G(r,s)$ where $G(r,s) = \{$complex sequences $x : x_{k,l} = O(r^k s^l)\}$ for each r, s in the interval $(0,1)$. Using this definition, four dimensional matrix characterizations of $l^{\infty,\infty}$, $c^{''}$, and $c_0^{''}$ into $G^{''}$ and into $\Omega^{''}(t)$ are presented. In addition to these definitions and characterizations it should be noted that this ensure a rate of converges of at least as fast as $[t]$. Other natural implications will also be presented.

1 Introduction

This paper considers four dimensional factorable transformations of the form

$$(Ax)_{m,n} = \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}x_{k,l} = \sum_{k=0}^{\infty} a_{m,k}x_k \sum_{l=0}^{\infty} a_{n,l}x_l,$$

via factorable double sequences $[x_{k,l}] = [x_k][x_l]$, and the Pringsheim notion of convergence. For the sake of completeness, we shall state Pringsheim's definition for convergence. A double sequence $[x]$ is said to be convergent in the Pringsheim sense to L provided that, given $\epsilon > 0$ there exists an $N \in \mathbf{N}$ such that $|x_{k,l} - L| < \epsilon$ whenever $k, l > N$, (see, [2]). Using this definition and the transformation stated above we will examine the set of geometrically dominated double sequences

$$G^{''} = \cup_{r,s\in(0,1)}G(r,s).$$

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In addition we will also characterize the type of four dimensional summability matrices that maps $l^{\infty,\infty}$, $c^{''}$, and $c_0^{''}$ into $G^{''}$ and into $\Omega^{''}(t)$ where, $l^{\infty,\infty}$, the space of all bounded double sequences, $c^{''}$, the space of bounded double Pringsheim convergence sequences; $c_0^{''}$, the space of bounded double Pringsheim null sequences, respectively.

2 Main Result

Definition 1. *For each r and s in the interval $(0,1)$, let*

$$G(r,s) = \{\textit{complex sequences } x : x_{k,l} = O(r^k s^l)\}$$

and define the set of **geometrically dominated double sequences** *as*

$$G^{''} = \cup_{r,s\in(0,1)} G(r,s).$$

Without lost of generality we can replace the double geometric sequence $[r^k s^l]$ with a nonnegative double sequence $[t]$ and define $\Omega^{''}(t)$ as $\{x : x_{k,l} = O(t_{k,l})\}$. Also we will let T denote a double sequence $\{t^{m,n}\}_{m,n=0,0}^{\infty,\infty}$ of nonnegative number sequences such that $t^{m,n} \in \Omega^{''}(t^{m,n})$ for each m, n. Thus

$$\Omega^{''}(t^{m,n}) \subseteq \Omega^{''}(t^{m,n+1}) \subseteq \Omega^{''}(t^{m+1,n+1}),$$

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and define $D(T)$ as $\cup_{m,n=0,0}^{\infty,\infty}\Omega^{''}(t^{m,n})$. We will utilize the following notation. Given a four dimensional summability matrix A we will denote μ and σ as the double sequences $\sup_{k,l}|a_{m,n,k,l}|$ and $\sum_{k,l=0,0}^{\infty,\infty}|a_{m,n,k,l}|$, respectively. The following example is similar to Example 1 presented by Fricke and Fridy in [1].

Example 1. *If for each pair (m,n), $t_{k,l}^{m,n}$ is the geometric sequence $\{r_m^k s_n^l\}$, where $0 < r_m < 1, 0 < s_n < 1$ and both increases to 1 then $D(T) = G^{"}$*

Let us present our first theorem. Observe that this and the remaining results are the multidimensional analog of Fricke and Fridy in [1].

Theorem 1. *If A is a four dimensional summability matrix and T, $D(T)$, and μ are as given above, then the following are equivalent:*

1. *$\mu \in D(T)$;*
2. *there exists a $[t^{(m,n)}]$ in T such that $\mu \in \Omega^{''}(t^{m,n})$;*
3. *there exists a $[t^{(m,n)}]$ in T such that $A : l^{''} \to \Omega^{''}(t^{m,n})$;*
4. *$A : l^{''} \to D(T)$,*
   where $l^{''} = \{x_{k,l} : \sum_{k,l=1,1}^{\infty,\infty}|x_{k,l}| < \infty\}$.

Proof. Note (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are straight forward implications and as such the proofs are omitted. We will focus our attention on establishing that (4) $\Rightarrow$ (1), thus completing the proof. Since $A : l^{''} \to D(T)$ we are granted that

$$\mu_{m,n} < \infty \text{ for each } (m,n). \tag{1}$$

In addition

$$\text{each pairwise column of } A \text{ is in } D(T) \tag{2}$$

It is clear from (2) that the sum of any finite number of pairwise columns of double sequence of A is in $D(T)$. Therefore there exists a double sequence $[t^{p,q}]$ in T and a positive double number sequence $[B_{p,q}]$ that increases to ∞ in the Pringsheim sense, such that

$$\sum_{k,l\leq p,q} |a_{m,n,k,l}| \leq B_{p,q} t^{p,q}_{m,n} \text{ for each } (m,n). \tag{3}$$

We will establish our theorem in the following manner. First we will assume (1), (2), and (3), but $\mu \notin D(T)$. Then we will construct a double sequence $x \in l^{''}$ such that $Ax \notin D(T)$. We begin by observing that for each pairwise row of A we can select an entry such that

$$\left|a_{m,n,k_p,l_q}\right| \geq \frac{1}{4}\mu_{m,n}.$$

Since $\mu \notin D(T)$, (2) grants us a double subsequence $\{a_{\alpha_i,\beta_j,\phi_i,\psi_j}\}_{i,j=0,0}^{\infty,\infty}$ such that α_i, β_j, ϕ_i, and ψ_j are increasing single dimensional sequences on i and j. In addition, since $\mu \notin D(T)$ we can choose α_i and β_j such that

$$\sup_{i,j}\left\{\frac{\mu_{\alpha_i,\beta_j}}{t^{p,q}_{\alpha_i,\beta_j}}\right\} = \infty \text{ for all } (p,q). \tag{4}$$

Thus for each (i,j) we have the following :

$$\left|a_{\alpha_i,\beta_j,\phi_i,\psi_j}\right| \geq \frac{1}{4}\mu_{\alpha_i,\beta_j} = \frac{1}{4}\sup_{k,l}\left\{\left|a_{\alpha_i,\beta_j,k,l}\right|\right\}. \tag{5}$$

Now we choose a double subsequence of the double subsequence such that for each (ρ,ϱ)

$$\begin{aligned} \left|a_{\alpha_{i_\rho},\beta_{j_\varrho},\phi_{i_\rho},\psi_{j_\varrho}}\right| &> 144 B_{\phi_{i_\rho},\psi_{j_\varrho}} t^{(\phi_{i_\rho},\psi_{j_\varrho})}_{\alpha_{i_\rho},\beta_{j_\varrho}} 4^{\rho+\varrho} \\ &= 12 B_{\phi_{i_\rho}} t^{(\phi_{i_\rho})}_{\alpha_{i_\rho}} 4^{\rho} \cdot 12 B_{\psi_{j_\varrho}} t^{(\psi_{j_\varrho})}_{\beta_{j_\varrho}} 4^{\varrho}. \end{aligned} \tag{6}$$

Let us now define the double sequence $[x]$ as

$$x_{k,l} := \begin{cases} 4^{-\rho-\varrho}, & \text{if } k = \phi_{i_\rho}, l = \psi_{j_\varrho} \text{ and } a_{\alpha_{i_\rho},\beta_{j_\varrho},\phi_{i_\rho},\psi_{j_\varrho}} \neq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Thus for each (ρ,ϱ) we are granted the following:

$$
\begin{aligned}
\left|(Ax)_{\alpha_{i_\rho},\beta_{j_\varrho}}\right| &\geq \left|a_{\alpha_{i_\rho},\beta_{j_\varrho},\phi_{i_\rho},\psi_{j_\varrho}}\right|4^{-\rho-\varrho}\\
&- \sum_{\{\lambda\leq\rho;\chi<\varrho\}}\left|a_{\alpha_{i_\rho},\beta_{j_\varrho},\phi_{i_\lambda},\psi_{j_\chi}}\right|4^{-\lambda-\chi} - \sum_{\{\lambda>\rho;\chi\leq\varrho\}}\left|a_{\alpha_{i_\rho},\beta_{j_\varrho},\phi_{i_\lambda},\psi_{j_\chi}}\right|4^{-\lambda-\chi}\\
&- \sum_{\{\lambda<\rho;\chi\geq\varrho\}}\left|a_{\alpha_{i_\rho},\beta_{j_\varrho},\phi_{i_\lambda},\psi_{j_\chi}}\right|4^{-\lambda-\chi} - \sum_{\{\lambda\geq\rho;\chi>\varrho\}}\left|a_{\alpha_{i_\rho},\beta_{j_\varrho},\phi_{i_\lambda},\psi_{j_\chi}}\right|4^{-\lambda-\chi}\\
&\geq \mu_{\alpha_{i_\rho},\beta_{j_\varrho}}4^{-\rho-\varrho-1} - B_{\phi_{i_\rho},\psi_{j_\varrho}}t^{(\phi_{i_\rho},\psi_{j_\varrho})}_{\alpha_{i_\rho},\beta_{j_\varrho}}\\
&- \frac{1}{3}\mu_{\beta_{j_\varrho}}4^{-\varrho+1}\sum_{\{\lambda<\rho\}}\left|a_{\alpha_{i_\rho},\phi_{i_\lambda}}\right|4^{-\lambda} - \frac{1}{3}\mu_{\alpha_{i_\rho}}4^{-\rho}\sum_{\{\chi\leq\varrho\}}\left|a_{\beta_{j_\varrho},\psi_{j_\chi}}\right|4^{-\chi}\\
&- \frac{1}{9}\mu_{\alpha_{i_\rho},\beta_{j_\varrho}}4^{-\rho-\varrho}\\
&\geq \mu_{\alpha_{i_\rho},\beta_{j_\varrho}}4^{-\rho-\varrho-1} - B_{\phi_{i_\rho},\psi_{j_\varrho}}t^{(\phi_{i_\rho},\psi_{j_\varrho})}_{\alpha_{i_\rho},\beta_{j_\varrho}}\\
&- \frac{1}{3}\mu_{\beta_{j_\varrho}}4^{-\varrho}B_{\phi_{i_\rho}}t^{(\phi_{i_\rho})}_{\alpha_{i_\rho}} - \frac{1}{3}\mu_{\alpha_{i_\rho}}4^{-\rho}B_{\psi_{j_\varrho}}t^{(\psi_{j_\varrho})}_{\beta_{j_\varrho}}\\
&- \frac{1}{9}\mu_{\alpha_{i_\rho},\beta_{j_\varrho}}4^{-\rho-\varrho}\\
&\geq \frac{5}{36}\mu_{\alpha_{i_\rho},\beta_{j_\varrho}}4^{-\rho-\varrho} - B_{\phi_{i_\rho},\psi_{j_\varrho}}t^{(\phi_{i_\rho},\psi_{j_\varrho})}_{\alpha_{i_\rho},\beta_{j_\varrho}}\\
&- \frac{1}{3}\mu_{\beta_{j_\varrho}}4^{-\varrho}B_{\phi_{i_\rho}}t^{(\phi_{i_\rho})}_{\alpha_{i_\rho}} - \frac{1}{3}\mu_{\alpha_{i_\rho}}4^{-\rho}B_{\psi_{j_\varrho}}t^{(\psi_{j_\varrho})}_{\beta_{j_\varrho}}\\
&> \frac{5}{36}[144B_{\phi_{i_\rho},\psi_{j_\varrho}}t^{(\phi_{i_\rho},\psi_{j_\varrho})}_{\alpha_{i_\rho},\beta_{j_\varrho}}4^{\rho+\varrho}]4^{-\rho-\varrho} - B_{\phi_{i_\rho},\psi_{j_\varrho}}t^{(\phi_{i_\rho},\psi_{j_\varrho})}_{\alpha_{i_\rho},\beta_{j_\varrho}}\\
&- \frac{1}{3}\mu_{\beta_{j_\varrho}}4^{-\varrho}B_{\phi_{i_\rho}}t^{(\phi_{i_\rho})}_{\alpha_{i_\rho}} - \frac{1}{3}\mu_{\alpha_{i_\rho}}4^{-\rho}B_{\psi_{j_\varrho}}t^{(\psi_{j_\varrho})}_{\beta_{j_\varrho}}\\
&> 19B_{\phi_{i_\rho},\psi_{j_\varrho}}t^{(\phi_{i_\rho},\psi_{j_\varrho})}_{\alpha_{i_\rho},\beta_{j_\varrho}}\\
&- 4B_{\psi_{j_\varrho}}t^{(\psi_{j_\varrho})}_{\beta_{j_\varrho}}B_{\phi_{i_\rho}}t^{(\phi_{i_\rho})}_{\alpha_{i_\rho}} - 4B_{\phi_{i_\rho}}t^{(\phi_{i_\rho})}_{\alpha_{i_\rho}}B_{\psi_{j_\varrho}}t^{(\psi_{j_\varrho})}_{\beta_{j_\varrho}}\\
&= 11B_{\phi_{i_\rho},\psi_{j_\varrho}}t^{(\phi_{i_\rho},\psi_{j_\varrho})}_{\alpha_{i_\rho},\beta_{j_\varrho}}.
\end{aligned}
$$

Thus $Ax \notin \Omega^{''}(t^{(\phi_{i_\rho},\psi_{j_\varrho})})$, and thus $Ax \notin D(T)$. □

Listed below are immediate corollaries of Theorem 2.1. By taking T to be the one defined in Example 2.1, the proof of Corollary 2.1 clearly follows. For Corollary 2.2 we simply fix $[t]$ and the result follows.

Corollary 1. *If A is a four dimensional summability matrix and $\mu_{m,n} = \sup_{k,l}|a_{m,n,k,l}|$, then $A : l^{''} \to G^{''}$ if and only if $\mu \in G^{''}$*

Corollary 2. *If A is a four dimensional summability matrix and $[t]$ is a nonnegative double number sequence, then $A : l^{''} \to G^{''}$ if and only if $\mu \in \Omega^{''}(t)$.*

Theorem 2. *If A is a four dimensional summability matrix and T, $D(T)$, and σ are given as above, then the following are equivalent:*

1. *$\sigma \in D(T)$;*
2. *there exists a $[t^{(m,n)}]$ in T such that $\sigma \in \Omega''(t^{(m,n)})$;*
3. *there exists a $[t^{(m,n)}]$ in T such that $A : l^{\infty,\infty} \to \Omega''(t^{(m,n)})$;*
4. *there exists a $[t^{(m,n)}]$ in T such that $A : c'' \to \Omega''(t^{(m,n)})$;*
5. *there exists a $[t^{(m,n)}]$ in T such that $A : c_0'' \to \Omega''(t^{(m,n)})$;*
6. *$A : c_0'' \to D(T)$.*

Proof. Similar to the previous theorem it is clear that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6) and as such the proofs are omitted. Therefore we only need to prove that (6) $\Rightarrow$ (1). Since $A : c_0'' \to D(T)$ it is clear that each pairwise row of A is in l'', that is $\sigma_{m,n} = \sum_{k,l=0,0}^{\infty,\infty} |a_{m,n,k,l}| < \infty$. Also each pairwise column of A is in $\Omega''(t^{(m,n)})$. Thus, similar to the analysis in Theorem 2.1 proof, the sum of any finite number of pairwise columns of double sequence of A is in $D(T)$. Let us now select a double sequence $[t^{p,q}]$ with a constant $B_{p,q} > 0$ such that for each (p,q)

$$\sum_{\{k\leq p, l\leq q\}} |a_{m,n,k,l}| \leq B_{p,q} t^{(p,q)}_{m,n}. \tag{7}$$

Suppose that $\sigma \notin D(T)$. We can now choose index sequences m_p, n_q, k_p, and l_q. After selecting k_p, l_q, m_{p-1}, and n_{q-1} we then choose $m_p > m_{p-1}$ and $n_q > n_{q-1}$ such that

$$\sigma_{m_p,n_q} \geq \left[pq(2B_{k_p,l_q} + pq) + 2B_{k_p,l_q}\right] t^{k_p,l_q}_{m_p,n_q}. \tag{8}$$

Next we choose $k_{p+1} > k_p$ and $l_{q+1} > l_q$ such that

$$\sum_{\{(k,l):k>k_{p+1} \text{ or } l>l_{q+1}\}} \left|a_{m_p,n_q,k,l}\right| < B_{k_p,l_q} t^{k_p,l_q}_{m_p,n_q}. \tag{9}$$

The inequalities above grant us the following

$$\sum_{\{(k,l):k_p<k\leq k_{p+1} \text{ or } l_q<l\leq l_{q+1}\}} \left|a_{m_p,n_q,k,l}\right| \geq pq(2B_{k_p,l_q} + pq) + 2B_{k_p,l_q}. \tag{10}$$

We now define the double sequence $[x]$ as

$$x_{k,l} := \begin{cases} \frac{\bar{a}_{m_p,n_q,k,l}}{|a_{m_p,n_q,k,l}|pq}, & \text{if } \{k_p < k \leq k_{p+1} \text{ or } l_q < l \leq l_{q+1}\} \text{ and } a_{m_p,n_q,k,l} \neq 0 \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that $x \in c_0^{''}$, ;however for each (p,q) we have

$$\begin{aligned}
\left|(Ax)_{m_p,n_q}\right| &\geq \left|\sum_{\{(k,l):k_p<k\leq k_{p+1} \text{ or } l_q<l\leq l_{q+1}\}} a_{m_p,n_q,k,l}x_{k,l}\right| \\
&- \sum_{\{(k,l):k>k_{p+1} \text{ or } l>l_{q+1}\}} \left|a_{m_p,n_q,k,l}\right| \\
&- \sum_{\{(k,l):k\leq k_p \text{ or } l\leq l_{q+1}\}} \left|a_{m_p,n_q,k,l}\right| \\
&\geq -2B_{k_p,l_q}t^{k_p,l_q}_{m_p,n_q} + \sum_{\{(k,l):k_p<k\leq k_{p+1} \text{ or } l_q<l\leq l_{q+1}\}} \left|a_{m_p,n_q,k,l}\right| \frac{1}{pq} \\
&\geq pqt^{k_p,l_q}_{m_p,n_q}.
\end{aligned}$$

Thus, $Ax \notin \Omega^{''}(t^{(k_p,l_q)})$ for $p,q = 0,1,2,3,\ldots$, and therefore $Ax \notin D(T)$. $\square$

These corollaries follow in manner similar to the first two corollaries, using Example 2.1 and fixing $[t]$.

Corollary 3. *If A is a four dimensional summability matrix and*

$$\mu_{m,n} = \sum_{k,l=0,0}^{\infty,\infty} |a_{m,n,k,l}|,$$

then A maps $l^{\infty,\infty}$, $c^{''}$, and $c_0^{''}$ into $G^{''}$ if and only if $\sigma \in G^{''}$

Corollary 4. *If A is a four dimensional summability matrix and $[t]$ is a nonnegative double number sequence then A maps $l^{\infty,\infty}$, $c^{''}$, and $c_0^{''}$ into $\Omega^{''}(t)$ if and only if $\sigma \in \Omega^{''}(t)$.*

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