Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Filomat **25:2** (2011), 137–149 DOI: 10.2298/FIL1102137D

COUPLED FIXED POINT THEOREMS IN PARTIALLY ORDERED CONE METRIC SPACES

Hui-Sheng Ding and Lu Li

Abstract

This paper is concerned with mixed monotone mappings in partially ordered cone metric spaces. We establish several fixed point theorems, which generalize and complement some known results. Especially, even in a partially ordered metric space, our main results are generalizations of the fixed point theorems due to Bhaskar and Lakshmikantham [T. Grana Bhaskar, V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Anal. TMA 65 (2006) 1379–1393].

1 Introduction

The existence of fixed points in partially ordered metric spaces was initiated in [1], where some applications to matrix equations are studied. Since then, such problems have been of great interest for many mathematicians. Especially, Bhaskar and Lakshmikantham [2] established a fixed point theorem for mixed monotone mappings in partially ordered metric spaces, i.e.,

Theorem 1.1. *Let $F : X \times X \to X$ be a continuous mapping with the mixed monotone property on X. Assume that there exists a $k \in [0, 1)$ with*

$$d(F(x, y), F(u, v)) \leq \frac{k}{2}[d(x, u) + d(y, v)], \quad \forall x \geq u, y \leq v.$$

2010 *Mathematics Subject Classifications.* 47H10, 54H25.
Key words and Phrases. Coupled fixed point, Mixed monotone mapping, Partially ordered, Cone metric space.
Received: May 1, 2010.
Communicated by Vladimir Rakočević.
The work was supported by the NSF of China, the Key Project of Chinese Ministry of Education, the NSF of Jiangxi Province of China, and the Youth Foundation of Jiangxi Provincial Education Department(GJJ09456).

If there exist $x_0, y_0 \in X$ *such that*

$$x_0 \leq F(x_0, y_0), \quad y_0 \geq F(y_0, x_0).$$

Then, there exist $x, y \in X$ *such that*

$$x = F(x, y), \quad y = F(y, x).$$

Just as noted in [2], Theorem 1.1 can be used to investigate a large class of problems. So it is an interesting and meaningful work to study the existence of fixed points for mixed monotone mappings in partially ordered metric spaces. For more related works about mixed monotone mappings in partially ordered metric spaces, we refer the reader to [3, 4] and references therein.

On the other hand, recently, fixed point theorems in cone metric spaces and ordered cone metric spaces were investigated by many authors (see, e.g., [5–14] and references therein). Especially, the study on the existence of fixed points in partially ordered cone metric spaces has attracted more and more attention.

The aim of this paper is to extend Theorem 1.1 in a partially ordered cone metric space. As one will see, even in a partially ordered metric space, our main results are generalizations of Theorem 1.1.

Next, let us recall some basic definitions and notations about cone, cone metric, and mixed monotone mapping. For more details, we refer the reader to [2, 14].

Let E be a real Banach space. A closed convex set P in E is called a cone if the following conditions are satisfied:

(i) if $x \in P$, then $\lambda x \in P$ for any $\lambda \geq 0$,

(ii) if $x \in P$ and $-x \in P$, then $x = 0$.

A cone P induces a partial ordering $\leq$ in E by

$$x \leq y \quad \text{if and only if} \quad y - x \in P.$$

In addition, $x \ll y$ stands for $y - x \in P^o$, where P^o is the interior of P. A cone P is called normal if there exists a constant $k > 0$ such that

$$0 \leq x \leq y \quad \text{implies that} \quad \|x\| \leq k\|y\|,$$

where $\|\cdot\|$ is the norm on E.

Definition 1.2 ([14])**.** *Let* X *be a nonempty set and* P *be a cone in a Banach space* E*. Suppose that a mapping* $d : X \times X \to E$ *satisfies:*

(d1) $\theta \leq d(x, y)$ *for all* $x, y \in X$ *and* $d(x, y) = \theta$ *if and only if* $x = y$ *,where* θ *is the zero element of* P*;*

(d2) $d(x, y) = d(y, x)$ *for all* $x, y \in X$*;*

(d3) $d(x, y) \leq d(x, z) + d(z, y)$ *for all* $x, y, z \in X$*.*

Then d is called a cone metric on X and (X,d) is called a cone metric space.

Definition 1.3 ([14])**.** *Let (X,d) be a cone metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. If $\forall c \gg \theta$, there exists $N \in \mathbb{N}$ such that for all $n > N$, $d(x_n,x) \ll c$, then we call that $\{x_n\}$ converges to x, and we denote it by $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x$, $n \to \infty$. If $\forall c \gg \theta$, there exists $N \in \mathbb{N}$ such that for all $n,m > N$, $d(x_n,x_m) \ll c$, then $\{x_n\}$ is called a Cauchy sequence in X. In addition, (X,d) is called complete cone metric space if every Cauchy sequence is convergent.*

Remark 1.4. *It is easy to see that $\{x_n\}$ is a Cauchy sequence whenever $\{x_n\}$ is convergent in a cone metric space X.*

Definition 1.5 ([2])**.** *Let $(X,\sqsubseteq)$ be a partially ordered set and $F : X \times X \to X$. The mapping is said to has the mixed monotone property if $F(x_1,y_1) \sqsubseteq F(x_2,y_2)$ for all $x_1,x_2,y_1,y_2 \in X$ with $x_1 \sqsubseteq x_2$ and $y_2 \sqsubseteq y_1$.*

2 Main results

Throughout the rest of this paper, we denote by $\mathbb{N}$ the set of positive integers, by E a Banach space, by P a cone in E with $P^o \neq \emptyset$, by θ the zero element of P, and by $\leq$ the partial order induced by P. In addition, we denote by $(X,\sqsubseteq,d)$ an ordered cone metric space, i.e., $\sqsubseteq$ is a partial order on the set X, and d is a cone metric on X with the underlying cone P. Moreover, we call a mapping $F : X \times X \to X$ is continuous provided that $F(x_n,y_n) \to F(x,y)$ whenever $x_n \to x$ and $y_n \to y$, where $x,y,x_n,y_n \in X$, $\forall n \in \mathbb{N}$.

First, we prove a lemma, which will be used in the proof of our main results.

Lemma 2.1. *Let (X,d) be a cone metric space with the underlying cone P. Assume that*

$$x_n \to x_0,\ x_n \to y_0,\ n \to \infty.$$

Then $x_0 = y_0$.

Proof. Let $h \in P^o$. Then $\frac{h}{k} \in P^o$ for each $k \in \mathbb{N}$. So for each $k \in \mathbb{N}$, there exists $N_k \in \mathbb{N}$ such that

$$d(x_{N_k},x_0) \ll \frac{h}{k},\ d(x_{N_k},y_0) \ll \frac{h}{k}.$$

Thus

$$d(x_0,y_0) \leq d(x_{N_k},x_0) + d(x_{N_k},y_0) \leq \frac{2h}{k}, \quad \forall k \in \mathbb{N},$$

which means that $\frac{2h}{k} - d(x_0,y_0) \in P$ for each $k \in \mathbb{N}$. Letting $k \to \infty$, it follows that $-d(x_0,y_0) \in P$, i.e., $d(x_0,y_0) = \theta$. Thus, $x_0 = y_0$. $\square$

In fact, Lemma 2.1 is a slight generalization of [14, Lemma 2], where P is a normal cone.

Now, let us present one of our main results.

Theorem 2.2. *Let $(X, \sqsubseteq, d)$ be a complete ordered cone metric space, and $F : X \times X \to X$ be a continuous mapping with the mixed monotone property on X. Suppose that the following assumptions hold:*

(A1) there exist $\alpha, \beta, \gamma \geq 0$ with $2\alpha + 3\beta + 3\gamma < 2$ such that

$$\begin{aligned} d(F(x,y), F(u,v)) \quad &\leq \quad \alpha \cdot \frac{d(x,u) + d(y,v)}{2} + \beta \cdot \frac{d(x, F(x,y)) + d(u, F(u,v)) + d(y,v)}{2} \\ &\quad + \gamma \cdot \frac{d(x, F(u,v)) + d(u, F(x,y)) + d(y,v)}{2} \end{aligned}$$

for all $u \sqsubseteq x, y \sqsubseteq v$;

(A2) there exist $x_0, y_0 \in X$ such that $x_0 \sqsubseteq F(x_0, y_0)$ and $F(y_0, x_0) \sqsubseteq y_0$.

Then F has a coupled fixed point, i.e., there exist $x_*, y_* \in X$ such that $F(x_*, y_*) = x_*$ and $F(y_*, x_*) = y_*$.

Proof. Let

$$x_n = F(x_{n-1}, y_{n-1}), \; y_n = F(y_{n-1}, x_{n-1}), \quad n = 1, 2, \ldots.$$

Since F has the mixed monotone property on X, by (A2), we get

$$x_0 \sqsubseteq x_1 \sqsubseteq \cdots \sqsubseteq x_n \sqsubseteq x_{n+1} \sqsubseteq \cdots$$

and

$$\cdots \sqsubseteq y_{n+1} \sqsubseteq y_n \sqsubseteq \cdots \sqsubseteq y_1 \sqsubseteq y_0.$$

Now, let

$$e = \frac{d(x_1, x_0) + d(y_1, y_0)}{2}, \quad \lambda = \frac{2(\alpha + \beta + \gamma)}{2 - \beta - \gamma}.$$

Then, by (A1), we have

$$\begin{aligned} d(x_2, x_1) \quad &= \quad d(F(x_1, y_1), F(x_0, y_0)) \\ &\leq \quad \alpha \cdot \frac{d(x_1, x_0) + d(y_1, y_0)}{2} + \beta \cdot \frac{d(x_1, F(x_1, y_1)) + d(x_0, F(x_0, y_0)) + d(y_1, y_0)}{2} \\ &\quad + \gamma \cdot \frac{d(x_1, F(x_0, y_0)) + d(x_0, F(x_1, y_1)) + d(y_1, y_0)}{2} \\ &= \quad \alpha e + \beta \cdot \frac{d(x_1, x_2) + d(x_0, x_1) + d(y_1, y_0)}{2} + \gamma \cdot \frac{d(x_1, x_1) + d(x_0, x_2) + d(y_1, y_0)}{2} \\ &\leq \quad \alpha e + \beta e + \frac{\beta}{2} \cdot d(x_1, x_2) + \gamma \cdot \frac{d(x_0, x_1) + d(x_1, x_2) + d(y_1, y_0)}{2} \\ &= \quad (\alpha + \beta + \gamma) e + \frac{\beta + \gamma}{2} d(x_1, x_2). \end{aligned}$$

Thus, we obtain

$$d(x_2, x_1) \le \frac{2(\alpha+\beta+\gamma)}{2-\beta-\gamma} \cdot e = \lambda e,$$

where $0 \le \lambda < 1$ since $2\alpha + 3\beta + 3\gamma < 2$.

Also, one can get

$$\begin{aligned} d(y_1, y_2) &= d(F(y_0, x_0), F(y_1, x_1)) \\ &\le \alpha \cdot \frac{d(y_0, y_1) + d(x_0, x_1)}{2} + \beta \cdot \frac{d(y_0, y_1) + d(y_1, y_2) + d(x_0, x_1)}{2} \\ &\quad + \gamma \cdot \frac{d(y_0, y_2) + d(y_1, y_1) + d(x_0, x_1)}{2} \\ &\le (\alpha+\beta+\gamma)e + \frac{\beta+\gamma}{2} d(y_1, y_2), \end{aligned}$$

which gives

$$d(y_1, y_2) \le \frac{2(\alpha+\beta+\gamma)}{2-\beta-\gamma} \cdot e = \lambda e.$$

Similar to the above proof, one can show that

$$d(x_{n+1}, x_n) \le \frac{2(\alpha+\beta+\gamma)}{2-\beta-\gamma} \cdot \frac{d(x_n, x_{n-1}) + d(y_n, y_{n-1})}{2} = \lambda \cdot \frac{d(x_n, x_{n-1}) + d(y_n, y_{n-1})}{2}$$

and

$$d(y_n, y_{n+1}) \le \frac{2(\alpha+\beta+\gamma)}{2-\beta-\gamma} \cdot \frac{d(x_n, x_{n-1}) + d(y_n, y_{n-1})}{2} = \lambda \cdot \frac{d(x_n, x_{n-1}) + d(y_n, y_{n-1})}{2},$$

where $n = 1, 2, \ldots$. Thus,

$$d(x_3, x_2) \le \lambda \cdot \frac{d(x_2, x_1) + d(y_2, y_1)}{2} \le \lambda^2 e,$$

$$\vdots$$

$$d(x_{n+1}, x_n) \le \lambda \cdot \frac{d(x_n, x_{n-1}) + d(y_n, y_{n-1})}{2} \le \lambda^n e.$$

$$\vdots$$

Also, by induction, we deduce that

$$d(y_n, y_{n+1}) \le \lambda^n e, \quad n = 1, 2, \ldots.$$

Next, let us prove that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. In fact, for $m > n$, we have

$$d(x_n, x_m) \le d(x_n, x_{n+1}) + \cdots d(x_{m-1}, x_m) \le (\lambda^n + \cdots \lambda^{m-1})e \le \frac{\lambda^n}{1-\lambda} e.$$

Then, for every $c \gg \theta$, there exists $N \in \mathbb{N}$ such that

$$d(x_n, x_m) \le \frac{\lambda^n}{1-\lambda} e \ll c, \quad m > n > N.$$

Thus, $\{x_n\}$ is a Cauchy sequence. Similarly, one can also show that $\{y_n\}$ is a Cauchy sequence.

Now, since the cone metric space $(X, \sqsubseteq, d)$ is complete, there exist $x_*, y_* \in X$ such that

$$x_n \to x_*, \; y_n \to y_*, \quad n \to \infty.$$

Then, by the continuity of F, the constructions of $\{x_n\}$, $\{y_n\}$, and Lemma 2.1, we can conclude that $F(x_*, y_*) = x_*$ and $F(y_*, x_*) = y_*$. □

In the case that F is not continuous, one can use the following theorem:

Theorem 2.3. *Suppose all the assumptions of Theorem 2.2 except for the continuity of F are satisfied. Moreover, assume that X has the following properties:*

(a) if an increasing sequence $\{x_n\}$ converges to x in X, then $x_n \sqsubseteq x$ for all $n \in \mathbb{N}$;

(b) if an decreasing sequence $\{y_n\}$ converges to y in X, then $y \sqsubseteq y_n$ for all $n \in \mathbb{N}$.

Then the conclusions of Theorem 2.2 also hold.

Proof. Let $\{x_n\}, \{y_n\}, x_*, y_*$ be as in Theorem 2.2. It remains to prove that $F(x_*, y_*) = x_*$ and $F(y_*, x_*) = y_*$.

By the assumptions (a) and (b), $x_n \sqsubseteq x_*$ and $y_* \sqsubseteq y_n$ for all $n \in \mathbb{N}$. Then, using (A1), we obtain

$$\begin{aligned}
& d(F(x_*, y_*), x_n) \\
= \;& d(F(x_*, y_*), F(x_{n-1}, y_{n-1})) \\
\le \;& \alpha \cdot \frac{d(x_*, x_{n-1}) + d(y_*, y_{n-1})}{2} + \beta \cdot \frac{d(x_*, F(x_*, y_*)) + d(x_{n-1}, x_n) + d(y_*, y_{n-1})}{2} \\
& + \gamma \cdot \frac{d(x_*, x_n) + d(x_{n-1}, F(x_*, y_*)) + d(y_*, y_{n-1})}{2} \\
\le \;& \alpha \cdot \frac{d(x_*, x_{n-1}) + d(y_*, y_{n-1})}{2} + \beta \cdot \frac{d(x_*, x_n) + d(x_{n-1}, x_n) + d(y_*, y_{n-1})}{2} \\
& + \gamma \cdot \frac{d(x_*, x_n) + d(x_{n-1}, x_n) + d(y_*, y_{n-1})}{2} + \frac{\beta + \gamma}{2} \cdot d(x_n, F(x_*, y_*)).
\end{aligned}$$

Then, it follows that

$$\begin{aligned}
& \frac{2 - \beta - \gamma}{2} d(F(x_*, y_*), x_n) \\
\le \;& \alpha \cdot \frac{d(x_*, x_{n-1}) + d(y_*, y_{n-1})}{2} + \beta \cdot \frac{d(x_*, x_n) + d(x_{n-1}, x_n) + d(y_*, y_{n-1})}{2} \\
& + \gamma \cdot \frac{d(x_*, x_n) + d(x_{n-1}, x_n) + d(y_*, y_{n-1})}{2}.
\end{aligned}$$

On the other hand, for every $c \gg \theta$, there exists $N \in \mathbb{N}$ such that for all $n > N$, there hold

$$d(x_*, x_{n-1}) \ll c,\ d(x_*, x_n) \ll c,\ d(y_*, y_{n-1}) \ll c,\ d(x_{n-1}, x_n) \ll c.$$

Then, we have

$$\frac{2-\beta-\gamma}{2} d(F(x_*, y_*), x_n) \ll \frac{2\alpha+3\beta+3\gamma}{2} \cdot c \leq c,$$

which yields that $x_n \to F(x_*, y_*),\ n \to \infty$. Recalling that $x_n \to x_*,\ n \to \infty$, it follows from Lemma 2.1 that $x_* = F(x_*, y_*)$. By a similar argument, one can also show that $F(y_*, x_*) = y_*$.

□

Remark 2.4. Theorem 2.2 and Theorem 2.3 are generalizations and complements of some known results. For example, letting $E = \mathbb{R}$, $P = [0, +\infty)$, and $\beta = \gamma = 0$, one can get [2, Thereom 2.1] and [2, Thereom 2.2] from Theorem 2.2 and Theorem 2.3. In addition, in the case of F being independent of the second argument, one can deduce a similar result to [12, Theroem 12].

In some cases, one can show that the coupled fixed point is the same. For example, we have the following result:

Theorem 2.5. *Suppose all the assumptions of Theorem 2.2 (or Theorem 2.3) are satisfied. Moreover, assume that x_0, y_0 are comparable, and $2\alpha + \beta + 3\gamma < 2$. Then $x_* = y_*$.*

Proof. Without loss of generality, one can assume that $x_0 \sqsubseteq y_0$. Then, by the mixed monotone property of F, $x_n \sqsubseteq y_n$ for all $n \in \mathbb{N}$. Hence, we have

$$\begin{aligned}
d(y_n, x_n) &= d(F(y_{n-1}, x_{n-1}), F(x_{n-1}, y_{n-1})) \\
&\leq \alpha \cdot d(x_{n-1}, y_{n-1}) + \beta \cdot \frac{d(x_{n-1}, x_n) + d(y_{n-1}, y_n) + d(x_{n-1}, y_{n-1})}{2} \\
&\quad + \gamma \cdot \frac{d(x_{n-1}, y_n) + d(y_{n-1}, x_n) + d(x_{n-1}, y_{n-1})}{2} \\
&\leq (\alpha + \frac{\beta+\gamma}{2}) \cdot d(x_{n-1}, y_{n-1}) + \beta \cdot \frac{d(x_{n-1}, x_n) + d(y_{n-1}, y_n)}{2} \\
&\quad + \gamma \cdot \frac{d(x_{n-1}, y_n) + d(y_{n-1}, x_n)}{2} \\
&\leq \beta \cdot \frac{d(x_{n-1}, x_n) + d(y_{n-1}, y_n)}{2} + (\alpha + \frac{\beta+3\gamma}{2}) \cdot d(x_*, y_*) \\
&\quad + (\alpha + \frac{\beta+\gamma}{2}) \cdot [d(x_{n-1}, x_*) + d(y_*, y_{n-1})] \\
&\quad + \gamma \cdot \frac{d(x_{n-1}, x_*) + d(y_*, y_n) + d(y_{n-1}, y_*) + d(x_*, x_n)}{2} \\
&\leq \beta \cdot \frac{d(x_{n-1}, x_n) + d(y_{n-1}, y_n)}{2} + (\alpha + \frac{\beta+3\gamma}{2}) \cdot d(x_*, y_*)
\end{aligned}$$

$$+(\alpha+\frac{\beta}{2}+\gamma)\cdot[d(x_{n-1},x_*)+d(y_*,y_{n-1})]+\gamma\cdot\frac{d(y_*,y_n)+d(x_*,x_n)}{2}.$$

In view of $d(x_*,y_*)\leq d(x_*,x_n)+d(x_n,y_n)+d(y_n,y_*)$, one can get

$$\begin{aligned} d(x_*,y_*) &\leq (\alpha+\frac{\beta+3\gamma}{2})\cdot d(x_*,y_*)+\beta\cdot\frac{d(x_{n-1},x_n)+d(y_{n-1},y_n)}{2} \\ &\quad +(\alpha+\frac{\beta}{2}+\gamma)\cdot[d(x_{n-1},x_*)+d(y_*,y_{n-1})]+(1+\frac{\gamma}{2})\cdot[d(y_*,y_n)+d(x_*,x_n)]. \end{aligned}$$

Thus,

$$\begin{aligned} & (1-\frac{2\alpha+\beta+3\gamma}{2})d(x_*,y_*) \\ \leq\ & \beta\cdot\frac{d(x_{n-1},x_n)+d(y_{n-1},y_n)}{2}+(\alpha+\frac{\beta}{2}+\gamma)\cdot[d(x_{n-1},x_*)+d(y_*,y_{n-1})] \\ & +(1+\frac{\gamma}{2})\cdot[d(y_*,y_n)+d(x_*,x_n)]. \end{aligned}$$

On the other hand, since $x_n\to x_*$ and $y_n\to y_*$, for every $c\gg\theta$, there exists $N\in\mathbb{N}$ such that for all $n>N$,

$$d(x_{n-1},x_n),d(y_{n-1},y_n),d(x_{n-1},x_*),d(y_*,y_{n-1}),d(y_*,y_n),d(x_*,x_n)\ll c.$$

Then, it follows that

$$(1-\frac{2\alpha+\beta+3\gamma}{2})d(x_*,y_*)\leq(2\alpha+2\beta+3\gamma+2)\cdot c.$$

In view of $2\alpha+\beta+3\gamma<2$, we get

$$d(x_*,y_*)\leq\frac{4\alpha+4\beta+6\gamma+4}{2-2\alpha-\beta-3\gamma}\cdot c,\quad\forall c\gg\theta,$$

which means that $d(x_*,y_*)=\theta$. So $x_*=y_*$. □

Next, we establish a fixed point theorem for a class of quasicontraction.

Theorem 2.6. *Let $(X,\sqsubseteq,d)$ be a complete ordered cone metric space, and $F: X\times X\to X$ be a mapping with the mixed monotone property on X. Suppose that the following assumptions hold:*

(H1) there exists $\lambda\in[0,\frac{2}{3})$ such that for each $u\sqsubseteq x, y\sqsubseteq v$, there exists $z\in M_F(x,y,u,v)$ satisfying

$$d(F(x,y),F(u,v))\quad\leq\quad\lambda z,$$

where $M_F(x,y,u,v)$ is the following subset of P:

$$\left\{\frac{d(x,u)+d(y,v)}{2},\frac{d(x,F(x,y))+d(u,F(u,v))+d(y,v)}{2},\right.$$
$$\left.\frac{d(x,F(u,v))+d(u,F(x,y))+d(y,v)}{2}\right\}.$$

(H2) there exist $x_0, y_0 \in X$ *such that* $x_0 \sqsubseteq F(x_0, y_0)$ *and* $F(y_0, x_0) \sqsubseteq y_0$.

(H3) F *is continuous or* X *has the following properties:*

(a) if an increasing sequence $\{x_n\}$ *converges to* x *in* X*, then* $x_n \sqsubseteq x$ *for all* $n \in \mathbb{N}$*;*

(b) if an decreasing sequence $\{y_n\}$ *converges to* y *in* X*, then* $y \sqsubseteq y_n$ *for all* $n \in \mathbb{N}$.

Then F *has a coupled fixed point, i.e., there exist* $x_*, y_* \in X$ *such that* $F(x_*, y_*) = x_*$ *and* $F(y_*, x_*) = y_*$.

Proof. Let

$$x_n = F(x_{n-1}, y_{n-1}),\ y_n = F(y_{n-1}, x_{n-1}), \quad n = 1, 2, \ldots.$$

Then

$$x_0 \sqsubseteq x_1 \sqsubseteq \cdots \sqsubseteq x_n \sqsubseteq x_{n+1} \sqsubseteq \cdots, \quad \cdots \sqsubseteq y_{n+1} \sqsubseteq y_n \sqsubseteq \cdots \sqsubseteq y_1 \sqsubseteq y_0.$$

By (H1), for each $n \in \mathbb{N}$, there exists

$$z_n \in \left\{ \frac{d(x_n, x_{n-1}) + d(y_n, y_{n-1})}{2}, \frac{d(x_n, x_{n+1}) + d(x_n, x_{n-1}) + d(y_n, y_{n-1})}{2}, \frac{d(x_{n-1}, x_{n+1}) + d(y_n, y_{n-1})}{2} \right\}$$

such that

$$d(x_{n+1}, x_n) = d(F(x_n, y_n), F(x_{n-1}, y_{n-1})) \leq \lambda z_n.$$

Now, we consider three cases:

1. If $z_n = \frac{d(x_n, x_{n-1}) + d(y_n, y_{n-1})}{2}$, then

$$d(x_{n+1}, x_n) \leq \lambda \cdot \frac{d(x_n, x_{n-1}) + d(y_n, y_{n-1})}{2} \leq \frac{2\lambda}{2 - \lambda} \cdot \frac{d(x_n, x_{n-1}) + d(y_n, y_{n-1})}{2};$$

2. If $z_n = \frac{d(x_n, x_{n+1}) + d(x_n, x_{n-1}) + d(y_n, y_{n-1})}{2}$, then

$$d(x_{n+1}, x_n) \leq \frac{\lambda}{2} d(x_{n+1}, x_n) + \lambda \cdot \frac{d(x_n, x_{n-1}) + d(y_n, y_{n-1})}{2},$$

which gives that

$$d(x_{n+1}, x_n) \leq \frac{2\lambda}{2 - \lambda} \cdot \frac{d(x_n, x_{n-1}) + d(y_n, y_{n-1})}{2};$$

3. If $z_n = \frac{d(x_{n-1}, x_{n+1}) + d(y_n, y_{n-1})}{2}$, then by case 2, we also have

$$d(x_{n+1}, x_n) \leq \frac{2\lambda}{2 - \lambda} \cdot \frac{d(x_n, x_{n-1}) + d(y_n, y_{n-1})}{2}$$

since $d(x_{n-1},x_{n+1}) \le d(x_{n-1},x_n) + d(x_n,x_{n+1})$.

Thus, we have

$$d(x_{n+1},x_n) \le \frac{2\lambda}{2-\lambda} \cdot \frac{d(x_n,x_{n-1}) + d(y_n,y_{n-1})}{2}, \quad \forall n \in \mathbb{N}.$$

By a similar proof, one can also show that

$$d(y_n,y_{n+1}) \le \frac{2\lambda}{2-\lambda} \cdot \frac{d(x_n,x_{n-1}) + d(y_n,y_{n-1})}{2}, \quad \forall n \in \mathbb{N}.$$

Then, we conclude that

$$d(x_{n+1},x_n) \le \left(\frac{2\lambda}{2-\lambda}\right)^n \cdot \frac{d(x_1,x_0) + d(y_1,y_0)}{2}, \quad \forall n \in \mathbb{N},$$

and

$$d(y_n,y_{n+1}) \le \left(\frac{2\lambda}{2-\lambda}\right)^n \cdot \frac{d(x_1,x_0) + d(y_1,y_0)}{2}, \quad \forall n \in \mathbb{N}.$$

It follows from $\lambda \in [0,\frac{2}{3})$ that $0 \le \frac{2\lambda}{2-\lambda} < 1$. Then, analogously to the corresponding proof of Theorem 2.2, one can show that there exist $x_*, y_* \in X$ such that

$$x_n \to x_*,\ y_n \to y_*, \quad n \to \infty.$$

It remains to prove

$$F(x_*,y_*) = x_*, \quad F(y_*,x_*) = y_*. \tag{2.1}$$

If F is continuous, (2.1) obviously holds. Now, suppose that (a) and (b) of (H3) hold. Since $x_n \to x_*,\ y_n \to y_*,\ n \to \infty$, for every $c \gg \theta$, there exists $N \in \mathbb{N}$ such that for all $n > N$, there hold

$$d(x_*,x_{n-1}) \ll c,\ d(x_*,x_n) \ll c,\ d(y_*,y_{n-1}) \ll c,\ d(x_{n-1},x_n) \ll c.$$

On the other hand, noting that $x_n \sqsubseteq x_*$ and $y_* \sqsubseteq y_n$ for all $n \in \mathbb{N}$, by (H1), we have

$$d(F(x_*,y_*),x_n) = d(F(x_*,y_*),F(x_{n-1},y_{n-1})) \le \lambda w_n,$$

where

$$w_n \in \left\{\frac{d(x_*,x_{n-1}) + d(y_*,y_{n-1})}{2}, \frac{d(x_*,F(x_*,y_*)) + d(x_{n-1},x_n) + d(y_*,y_{n-1})}{2}, \frac{d(x_*,x_n) + d(x_{n-1},F(x_*,y_*)) + d(y_*,y_{n-1})}{2}\right\}.$$

For each $n > N$, we consider three cases:

(i) Let $w_n = \frac{d(x_*,x_{n-1})+d(y_*,y_{n-1})}{2}$. Then $d(F(x_*,y_*),x_n) \le \lambda w_n \le c \le \frac{3}{2}c$.

(ii) Let $w_n = \frac{d(x_*,F(x_*,y_*))+d(x_{n-1},x_n)+d(y_*,y_{n-1})}{2}$. Then

$$d(F(x_*,y_*),x_n) \quad \le \quad \lambda w_n \le \frac{d(x_*,F(x_*,y_*)) + d(x_{n-1},x_n) + d(y_*,y_{n-1})}{3}$$

$$\begin{aligned} &\leq \frac{d(x_*, x_n) + d(x_n, F(x_*, y_*)) + d(x_{n-1}, x_n) + d(y_*, y_{n-1})}{3} \\ &\leq \frac{3c + d(x_n, F(x_*, y_*))}{3}, \end{aligned}$$

which yields $d(F(x_*, y_*), x_n) \leq \frac{3}{2}c$.
(iii) Let $w_n = \frac{d(x_*,x_n)+d(x_{n-1},F(x_*,y_*))+d(y_*,y_{n-1})}{2}$. Then

$$\begin{aligned} d(F(x_*, y_*), x_n) &\leq \lambda w_n \leq \frac{d(x_*, x_n) + d(x_{n-1}, F(x_*, y_*)) + d(y_*, y_{n-1})}{3} \\ &\leq \frac{3c + d(x_n, F(x_*, y_*))}{3}, \end{aligned}$$

which yields $d(F(x_*, y_*), x_n) \leq \frac{3}{2}c$.

It follows from the above proof that $x_n \to F(x_*, y_*)$ as $n \to \infty$. By Lemma 2.1, $x_* = F(x_*, y_*)$. Analogously, one can also show that $y_* = F(y_*, x_*)$. □

Theorem 2.7. *Suppose that all the assumptions of Theorem 2.6 are satisfied, and x_0, y_0 are comparable. Then $x_* = y_*$.*

Proof. Without loss of generality, one can assume that $x_0 \sqsubseteq y_0$. Then, by the mixed monotone property of F, $x_n \sqsubseteq y_n$ for all $n \in \mathbb{N}$. Thus, we have

$$\begin{aligned} d(x_*, y_*) &\leq d(x_*, x_n) + d(x_n, y_n) + d(y_n, y_*) \\ &= d(F(y_{n-1}, x_{n-1}), F(x_{n-1}, y_{n-1})) + d(x_*, x_n) + d(y_n, y_*) \\ &\leq \lambda v_n + d(x_*, x_n) + d(y_n, y_*), \end{aligned}$$

where v_n belongs to

$$\left\{ d(y_{n-1}, x_{n-1}), \frac{d(y_{n-1}, y_n) + d(x_{n-1}, x_n) + d(x_{n-1}, y_{n-1})}{2}, \frac{d(y_{n-1}, x_n) + d(x_{n-1}, y_n) + d(x_{n-1}, y_{n-1})}{2} \right\}.$$

Next, we consider three cases:
(1) Let $v_n = d(y_{n-1}, x_{n-1})$. Then

$$\begin{aligned} d(x_*, y_*) &\leq \lambda d(y_{n-1}, x_{n-1}) + d(x_*, x_n) + d(y_n, y_*) \\ &\leq \lambda d(x_*, y_*) + [d(y_{n-1}, y_*) + d(x_*, x_{n-1}) + d(x_*, x_n) + d(y_n, y_*)]. \end{aligned}$$

(2) Let $v_n = \frac{d(y_{n-1},y_n)+d(x_{n-1},x_n)+d(x_{n-1},y_{n-1})}{2}$. Then

$$\begin{aligned} d(x_*, y_*) &\leq \lambda[d(y_{n-1}, y_n) + d(x_{n-1}, x_n) + d(x_{n-1}, y_{n-1})] + d(x_*, x_n) + d(y_n, y_*) \\ &\leq \lambda d(x_*, y_*) + [d(y_{n-1}, y_n) + d(x_{n-1}, x_n) + d(y_{n-1}, y_*) + d(x_*, x_{n-1}) + d(x_*, x_n) + d(y_n, y_*)]. \end{aligned}$$

(3) Let $v_n = \frac{d(y_{n-1},x_n)+d(x_{n-1},y_n)+d(x_{n-1},y_{n-1})}{2}$. Then

$$d(x_*, y_*) \leq \frac{\lambda}{2}[d(y_{n-1}, x_n) + d(x_{n-1}, y_n) + d(x_{n-1}, y_{n-1})] + d(x_*, x_n) + d(y_n, y_*)$$

$$\leq \quad \frac{3\lambda}{2} d(x_*, y_*) + 2[d(y_{n-1}, y_*) + d(x_*, x_n) + d(x_{n-1}, x_*) + d(y_*, y_n)].$$

Then, we conclude that for all $n \in \mathbb{N}$,

$$d(x_*, y_*) \leq \frac{3\lambda}{2} d(x_*, y_*) + d(y_{n-1}, y_n) + d(x_{n-1}, x_n) + 2[d(y_{n-1}, y_*) + d(x_*, x_n) + d(x_{n-1}, x_*) + d(y_*, y_n)].$$

Noticing the fact that $x_n \to x_*$, $y_n \to y_*$ and $\lambda \in [0, \frac{2}{3})$, it is not difficult to show that $\forall c \gg \theta$, $d(x_*, y_*) \ll c$. This means that $x_* = y_*$. □

References

[1] A. C. M. Ran, M. C. B. Reurings, *A fixed point theorem in partially ordered sets and some application to matrix equations*, Proc. Amer. Math. Soc. 132 (2004), 1435–1443.

[2] T. Grana Bhaskar, V. Lakshmikantham, *Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Anal. 65 (2006) 1379–1393.

[3] V. Lakshmikantham, L. Ćirić, *Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces*, Nonlinear Anal. 70 (2009), 4341–4349.

[4] B. Samet, *Coupled fixed point theorems for a generalized Meir–Keeler contraction in partially ordered metric spaces*, Nonlinear Anal. 72 (2010), 4508–4517.

[5] D. Ilić and V. Rakočević, *Common fixed points for maps on cone metric space*, J. Math. Anal. Appl. 341 (2008), 876–882.

[6] D. Ilić and V. Rakočević, *Quasi-contraction on a cone metric space*, Appl. Math. Lett. 22 (2009), 728–731.

[7] Z. Kadelburg, S. Radenović, V. Rakočević, *Remarks on quasi-contraction on a cone metric space*, Appl. Math. Lett. 22 (2009), 1674–1679.

[8] G. Jungck, S. Radenović, S. Radojević, V. Rakočević, *Common fixed point theorems for weakly compatible pairs on cone metric spaces*, Fixed Point Theory and Applications Volume 2009 (2009), Article ID 643840, 13 pages.

[9] Z. Kadelburg, M. Pavlović, S. Radenović, *Common fixed point theorems for ordered contractions and quasicontractions in ordered cone metric spaces*, Comput. Math. Appl., 59 (2010), 3148–3159.

[10] M. Abbas, G. Jungck, *Common fixed point results for noncommuting mappings without continuity in cone metric spaces*, J. Math. Anal. Appl. 341 (2008), 416–420.

[11] M. Abbas, B. E. Rhoades, *Fixed and periodic point results in cone metric spaces*, Appl. Math. Lett. 21 (2008), 511–515.

[12] I. Altun, B. Damjanović, D. Djorić, *Fixed point and common fixed point theorems on ordered cone metric spaces*, Appl. Math. Lett. 23 (2010), 310–316.

[13] Sh. Rezapour and R. Hamlbarani, *Some notes on the paper: Cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Anal. Appl. 345 (2008), 719–724.

[14] L. G. Huang, X. Zhang, *Cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Anal. Appl. 332 (2007), 1468–1476.

Hui-Sheng Ding:
College of Mathematics and Information Science, Jiangxi Normal University, Nanchang, Jiangxi 330022, People's Republic of China.
E-mail: `dinghs@mail.ustc.edu.cn`

Lu Li:
College of Mathematics and Information Science, Jiangxi Normal University, Nanchang, Jiangxi 330022, People's Republic of China.
E-mail: `181793361@qq.com`