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ON SPACES OF GROUP-VALUED FUNCTIONS

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Abstract

Several results concerning closure-type properties of function spaces of continuous real-valued functions on a Tychonoff space X are extended to the spaces of group-valued functions.

1 Introduction

We use standard notation and terminology following [2] and [7]. All spaces in this article are assumed to be Tychonoff, and all topological groups are assumed to be Hausdorff. For a group G, e denotes the identity element of G, and $\mathcal{N}_e$ denotes a local base of e in G.

Let X be a space and G a topological group. By $\mathsf{C_p}(X, G)$ we denote the multiplicative topological group of all continuous mappings from X into G endowed with the pointwise group operation and the topology of pointwise convergence τ_p. The symbol f_e denotes the function in $\mathsf{C_p}(X, G)$ defined by $f_e(x) = e$ for all $x \in X$. When G is the additive group $\mathbb{R}$ of real numbers (with the usual metric topology) we write $\mathsf{C}_p(X)$ instead of $\mathsf{C_p}(\mathsf{X}, \mathbb{R})$. Since $\mathsf{C_p}(X, G)$ is a topological group, hence a homogeneous space, we can consider $f_e \in \mathsf{C_p}(X, G)$ when study local properties of $\mathsf{C_p}(X, G)$. We study duality between spaces X and $\mathsf{C_p}(X, G)$ for a given group G. In the particular case $G = \mathbb{R}$ we get results known in the C_p-theory (as well as in C_k-theory [11]).

For $x \in X$ and an open set $U \subset X$, we denote by $W(x, U)$ the set

$$W(x, U) := \{f \in \mathsf{C_p}(X, G) : f(x) \in U\};$$

The pointwise topology on $\mathsf{C_p}(X, G)$ has a subbase consisting of the sets

$$W(x, U), x \in X, U \text{ open in } G.$$

Let X be a space and G a topological group. According to [20] X is said to be:

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(1) *G-regular* if for each closed set $F \subset X$ and each point $x \in X \setminus F$ there are $f \in \mathsf{C_p}(X, G)$ and $g \in G \setminus \{e\}$ such that $f(x) = g$ and $f(F) \subseteq \{e\}$;

(2) *G^*-regular* if there is $g \in G$ such that for each closed set $F \subset X$ and each point $x \in X \setminus F$ there is $f \in \mathsf{C_p}(X, G)$ such that $f(x) = g$ and $f(F) \subseteq \{e\}$;

(3) *G^{**}-regular* if for each closed set $F \subset X$ each point $x \in X \setminus F$ and each $g \in G$ there is $f \in \mathsf{C_p}(X, G)$ such that $f(x) = g$ and $f(F) \subseteq \{e\}$;

1.1 Selection principles

The following two selection principles are used in this paper. Let $\mathcal{A}$ and $\mathcal{B}$ denote collections of subsets of a set Z. Then:

$\mathsf{S}_{fin}(\mathcal{A}, \mathcal{B})$ denotes that for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$, $n \in \mathbb{N}$, and $\bigcup_{n\in\mathbb{N}} \mathcal{V}_n \in \mathcal{B}$.

$\mathsf{S}_1(\mathcal{A}, \mathcal{B})$ is defined similarly to $\mathsf{S}_{fin}(\mathcal{A}, \mathcal{B})$ but all $\mathcal{V}_n$'s are one-element sets.

An open cover $\mathcal{U}$ of a space X is said to be an *ω-cover* if $X \notin \mathcal{U}$ and each finite set $F \subset X$ is contained in a member $U \in \mathcal{U}$.

Ω denotes the family of ω-covers of a space.

A countable ω-cover $\mathcal{U}$ of a space X is *groupable* [14], [15] if there is a partition of $\mathcal{U}$ into countably many pairwise disjoint finite subsets $\mathcal{U}_n$, $n \in \mathbb{N}$, such that for each finite $F \subset X$ for all but finitely many n there is an element $U \in \mathcal{U}_n$ with $F \subset U$.

Ω^{gp} denotes the family of all groupable ω-covers of a space.

A space X has the *Hurewicz covering property* if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ with $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ for each $n \in \mathbb{N}$, and each $x \in X$ belongs to $\cup\mathcal{V}_n$ for all but finitely many n ([9], [10], [17]).

Recall also the following:

A space X has *countable fan tightness* if for each point $x \in X$ and each sequence $(A_n : n \in \mathbb{N})$ with $x \in \bigcap_{n\in\mathbb{N}} \overline{A_n}$ there are finite $B_n \subset A_n$, $n \in \mathbb{N}$, such that $x \in \overline{\bigcup_{n\in\mathbb{N}} B_n}$ ([1]).

X has *countable strong fan tightness* if for each point $x \in X$ and each sequence $(A_n : n \in \mathbb{N})$ with $x \in \bigcap_{n\in\mathbb{N}} \overline{A_n}$ there are $b_n \in A_n$, $n \in \mathbb{N}$, such that $x \in \overline{\{b_n : n \in \mathbb{N}\}}$ ([19]).

A space X is *selectively weakly Fréchet-Urysohn* if for each $x \in X$ and each sequence $(A_n : n \in \mathbb{N})$ of subsets X with $x \in \overline{A_n}$ for each n, there are finite sets $B_n \subset A_n$, $n \in \mathbb{N}$, such that each neighbourhood of x meets all but finitely many B_n (compare with [14], [15]).

For more details regarding selection principles we refer the reader to the survey papers [12], [13] (see also the classical papers [9], [10], [16], [18]).

2 Results

In [20, Prop. 7.2] it was proved the following generalization of the well-known theorem of Arhangel'skiĭ-Pytkeev [2]: if G is a metric topological group and X is a G^*-regular space, then $\mathsf{C_p}(X,G)$ has countable tightness if and only if all finite powers of X are Lindelöf.

In this section we prove (in a similar spirit) generalizations of three well-known results in C_p-theory.

The following lemma is actually proved in the proof of Lemma 7.3 in [20].

Lemma 2.1. *If G is a topological group and X is a G^*-regular space, then there is $g \in G \setminus \{e\}$ such that for each open set $U \subset X$ and each non-empty finite set $F \subset U$ there is $f_{F,U} \in \mathsf{C_p}(X,G)$ satisfying $f_{F,U}(F) \subseteq \{e\}$ and $f_{F,U}(X \setminus U) \subseteq \{g^{-1}\}$.*

Theorem 2.2. *Let G be a metric group and X a G^*-regular space. If X satisfies $\mathsf{S}_{fin}(\Omega,\Omega)$, then $\mathsf{C_p}(X,G)$ has countable fan tightness.*

Proof. Let $(A_n : n \in \mathbb{N})$ be a sequence of subsets of $\mathsf{C_p}(X,G)$ whose closures contain f_e. Fix a decreasing local base $\{O_m : m \in \mathbb{N}\}$ at e in G (because G is a metric group). For each $n, m \in \mathbb{N}$ let $\mathcal{U}_{n,m} = \{f^{\leftarrow}(O_m) : f \in A_n\}$. We prove that for each finite set $F \subset X$ and each $m, n \in \mathbb{N}$ there is a member of $\mathcal{U}_{n,m}$ containing F. Indeed, for every finite set $F \subset X$ every neighborhood $W(F; O_m)$ of f_e intersects A_n, hence there exists a function $f \in A_n$ such that $f(F) \subset O_m$. Then $F \subset f^{\leftarrow}(O_m) \in \mathcal{U}_{n,m}$. Without loss of generality one may suppose that $X \notin \mathcal{U}_{n,m}$ for each $n, m \in \mathbb{N}$, so every $\mathcal{U}_{m,n}$ is an ω-cover of X.

Apply (1) to the sequence $(\mathcal{U}_{n,n} : n \in \mathbb{N})$; there exists a sequence $(\mathcal{V}_{n,n} : n \in \mathbb{N})$ of finite subsets such that for each n, $\mathcal{V}_{n,n} \subset \mathcal{U}_{n,n}$ and $\bigcup_{n\in\mathbb{N}} \mathcal{V}_{n,n}$ is an ω-cover of X. Let $\mathcal{V}_{n,n} = \{U_{n,m_1}, \cdots, U_{n,m_n}\}$, $n \in \mathbb{N}$, where $U_{n,m_i} = \varphi^{\leftarrow}_{n,m_i}(O_n)$. For each n put $B_n = \{\varphi_{n,m_i} : i \leq n\}$

Let us prove $f_e \in \overline{\bigcup_{n\in\mathbb{N}} B_n}$. Let $W(E; O)$, (E a finite subset of X, O a neighbourhood of $e \in G$), be a neighborhood of f_e in $\mathsf{C_p}(X,G)$ and let m be a natural number such that $O_m \subset O$. Since E is a finite subset of X and X satisfies $\mathsf{S}_{fin}(\Omega,\Omega)$, there are $i \in \mathbb{N}$, $i \geq m$, and $j \leq m_i$ such that $\varphi^{\leftarrow}_{i,j}(O_i) \supset E$. So,

$$\varphi_{i,j}(E) \subset O_i \subset O_m \subset O,$$

i.e. $\varphi_{i,j} \in W(E,O)$ which shows $f_e \in \overline{\bigcup_{n\in\mathbb{N}} B_n}$. □

Theorem 2.3. *Let G be a group and X a G^*-regular space. If $\mathsf{C_p}(X,G)$ has countable fan tightness, then X satisfies $\mathsf{S}_{fin}(\Omega,\Omega)$.*

Proof. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of ω-covers of X. For a fixed $n \in \mathbb{N}$ and a finite set F of X let $\mathcal{U}_{n,F} := \{U \in \mathcal{U}_n : F \subset U\}$. By Lemma 2.1 one can find $g \in G$ such that for each $U \in \mathcal{U}_{n,F}$ there exists $f_{K,U} \in \mathsf{C_p}(X,G)$ satisfying $f_{F,U}(F) \subset \{e\}$ and $f_{F,U}(X \setminus U) \subset \{g^{-1}\}$. Let for each n,

$$A_n = \{f_{F,U} : F \text{ finite in } X, U \in \mathcal{U}_{n,F}\}.$$

Let us prove that $f_e \in \overline{A_n}$ for each $n \in \mathbb{N}$. Indeed, fix $m \in \mathbb{N}$; let K be a non-empty finite set in X and O a neighborhood of e in G. There is $U \in \mathcal{V}_m$ with $K \subset U$. Then $f_{K,U} \in A_n \cap \bigcap_{x \in K} W(x, O)$.

Since $\mathsf{C_p}(X, G)$ has countable fan tightness there is a sequence $(B_n : n \in \mathbb{N})$ of finite sets such that for each $n \in \mathbb{N}$, $B_n \subset A_n$ and $f_e \in \overline{\bigcup_{n\in\mathbb{N}} B_n}$. Let, for a fixed n, $B_n = \{f_{F_{n,i},U_{n,i}} : i \le k_n\}$ and $\mathcal{V}_n = \{U_{n,i} : i \le k_n\}$. We claim that the sequence $(\mathcal{V}_n : n \in \mathbb{N})$ witnesses that X has property $\mathsf{S}_{fin}(\Omega, \Omega)$.

Let S be a finite subset of X. From $f_e \in \overline{\bigcup_{n\in\mathbb{N}} B_n}$ it follows that there is $m \in \mathbb{N}$, $i \le k_m$ and a function $f_{F_{m,i},U_{m,i}} \in B_m$ belonging to $\bigcap_{x\in S} W(x, G \setminus \{g^{-1}\})$. Then $S \subset U_{m,i}$. Otherwise, for some $x \in S$ one has $x \notin U_{m,i}$ so that $f_{F_{m,i},U_{m,i}}(x) = g^{-1}$, which contradicts the fact $f_{F_{m,i},U_{m,i}} \in W(x, G \setminus \{g^{-1}\})$. $\square$

Corollary 2.4. *Let G be a metric group and X a G^*-regular space. Then the following are equivalent:*

(1) *X satisfies $\mathsf{S}_{fin}(\Omega, \Omega)$;*

(2) *$\mathsf{C_p}(X, G)$ has countable fan tightness.*

In a similar way we prove the following theorem.

Theorem 2.5. *Let G be a metric group and X a G^*-regular space. Then the following are equivalent:*

(1) *X satisfies $\mathsf{S}_1(\Omega, \Omega)$;*

(2) *$\mathsf{C_p}(X, G)$ has countable strong fan tightness.*

Observe that here, as in Theorem 2.2, we use the fact that G is a metric group only for the proof (1) $\Rightarrow$ (2).

A space X is said to be *ω-Lindelöf* if each ω-cover of X contains a countable ω-subcover. Equivalently, all finite powers of X are Lindelöf.

For the proof of the following theorem we need

Theorem 2.6. ([15]) *For an ω-Lindelöf space the following assertions are equivalent:*

(1) *Each finite power of X has the Hurewicz property;*

(2) *X has property $\mathsf{S}_{fin}(\Omega, \Omega^{gp})$.*

Theorem 2.7. *Let X be an ω-Lindelöf space and G a metric group. If X is G^*-regular, then the following are equivalent:*

(1) *$\mathsf{C_p}(X, G)$ is a selectively weakly Fréchet-Urysohn space;*

(2) *Each finite power of X has the Hurewicz property.*

Proof. (1) $\Rightarrow$ (2): (Here we use only the fact that X is G^*-regular.) Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of ω-covers of X.

For each finite set $F \subset X$ pick a $U_{n,F} \in \mathcal{U}_n$ with $F \subset U_{n,F}$. By Lemma 2.1 pick $g \in G$ and $f_{F,U_{n,F}} \in \mathsf{C_p}(X,G)$ such that $f_{F,U_{n,F}}(F) = \{e\}$, and $f_{F,U_{n,F}}(X \setminus U) \subset \{g^{-1}\}$. Put $A_n = \{f_{F,U_{n,F}} : F \subset X$ finite and $U \in \mathcal{U}_n\}$. Then $f_e \in \overline{A_n}$ for each $n \in \mathbb{N}$.

Using (1) find a sequence $(B_n : n \in \mathbb{N})$ such that B_n is a finite subset of A_n for each $n \in \mathbb{N}$ and each neighbourhood of f_e intersects B_n for all but finitely many n. Let $\mathcal{V}_n$, $n \in \mathbb{N}$, be the set of sets $U_{n,F}$ such that $f_{F,U_{n,F}} \in B_n$. Then each $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$. Let us prove that each finite subset of X is contained in an element $V \in \mathcal{V}_n$ for all but finitely many n.

Let E be a finite subset of X. Consider the neighbourhood $W(E, O_k)$ of f_e. There is $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$, $W(E, O_k) \cap B_n \neq \emptyset$; let $f_{F,U_{n,F}} \in W(E, O_k)$. This means that for all $n \geq n_0$ we have: for each $x \in E$, $f_{F,U_{n,F}}(x) \in O_k$, i.e. $E \subset U_{n,F} \in \mathcal{V}_n$.

(2) $\Rightarrow$ (1): Fix a decreasing countable local base $\{O_n : n \in \mathbb{N}\}$ at $e \in G$. Let $(A_n : n \in \mathbb{N})$ be a sequence of subsets of $\mathsf{C_p}(X,G)$ such that $f_e \in \bigcap_{n\in\mathbb{N}} \overline{A_n}$. For each finite subset F of X the neighborhood $W(F, O_1)$ of f_e has a nonempty intersection with A_1. Choose $f_{F,1} \in A_1$. Since $f_{F,1}$ is continuous, choose for each $x \in F$ an open set V_x such that $f_{F,1}(V_x) \subset O_1$, and set $V_{F,1} = \bigcup_{x\in F} V_x$, and $\mathcal{V}_1 = \{V_{F,1} : F \subset X \text{ finite}\}$. Then $\mathcal{V}_1$ is an ω–cover of X. Similarly (by using $O_n \in \mathcal{N}_e$, $n \geq 2$) one can construct an ω-cover $\mathcal{V}_n$ for each $n \geq 2$. Apply (2) (and Theorem 2.6) to the sequence $(\mathcal{V}_n : n \in \mathbb{N})$ and choose finite sets $\mathcal{W}_n \subset \mathcal{V}_n$, $n \in \mathbb{N}$, so that each finite subset of X is contained in a member of $\mathcal{W}_n$ for all but finitely many n. Note that, without loss of generality, one can assume that $\mathcal{W}_n$'s are pairwise disjoint, i.e. that $\mathcal{W} = \bigcup_{n\in\mathbb{N}} \mathcal{W}_n$ is a groupable ω-cover of X. Let $\mathcal{W}_n = \{V_{F_1^{(n)}}, \cdots, V_{F_{k_n}^{(n)}}\}$, $n \in \mathbb{N}$. For each n, let $B_n \subset A_n$ be the set of all functions in A_n such that for each $V_{F_i^{(n)}} \in \mathcal{W}_n$, $i \leq k_n$, it holds $f_{V_{F_i^{(n)}}}(V_{F_i^{(n)}}) \subset O_i$. Then the sequence $(B_n : n \in \mathbb{N})$ testifies that (1) is satisfied. □

3 Results concerning bornologies

In this section we extend the results from the previous section and from [5] (see also [6]) to the function space $\mathsf{C_p}(X,G)$ endowed with the topology of strong uniform convergence on bornologies. In [5], several results concerning tightness-type properties of the space $(\mathsf{C}_p(X), \tau^s_{\mathfrak{B}})$, X a metric space, are proved. At the end of this section we obtain another result of this kind (see Theorem 3.5) as a corollary of a more general result.

According to [8] a *bornology* on a topological space X is a family $\mathfrak{B}$ of nonempty subsets of X which is closed under taking finite unions, hereditary and is a cover of X.

A *base* for $\mathfrak{B}$ is a $\mathfrak{B}_0 \subset \mathfrak{B}$ which is cofinal in $\mathfrak{B}$ with respect to the inclusion.

A base is called *closed* (*compact*) if all its members are closed (compact) subsets of X.

Some examples of important bornologies on a space X are:

1. the family $\mathfrak{F}$ of all nonempty finite subsets of X,
2. the family of all nonempty subsets of X,
3. the family $\mathfrak{K}$ of nonempty relatively compact subsets of X,
4. the family of all nonempty totally bounded subsets of a metric space X.

In [3], Beer and Levy defined the notion of strong uniform continuity for functions between two metric spaces, as well as the topology of strong uniform convergence on a bornology. Following their ideas we define the topology of strong uniform convergence on spaces of functions from a uniform space $(X, \mathbb{D})$ into a topological group G.

For a uniform space $(X, \mathbb{D})$, a topological group G and a bornology $\mathfrak{B}$ **with closed base** on X, $\tau^s_{\mathfrak{B}}$ is the *topology of strong uniform convergence on* $\mathfrak{B}$ determined by the uniformity on G^X having as a base the sets

$$[B, O]^s := \{(f, g) : \exists D \in \mathbb{D} \text{ so that } g(x) \in f(x)O \text{ for each } x \in D[B]\},$$

$(B \in \mathfrak{B},\ O \in \mathcal{N}_e)$.

For $f \in (\mathsf{C}(X, G), \tau^s_{\mathfrak{B}})$ the standard local base of f is the collection of sets

$$[B, O]^s(f) = \{g : \exists D \in \mathbb{D} \text{ with } g(x) \in f(x)O \text{ for all } x \in D[B]\},$$

$(B \in \mathfrak{B}, O \in \mathcal{N}_e)$.

Call a space X G^**-normal* if there is $g \in G$ such that for each pair (F_1, F_2) of disjoint closed subsets of X there is $f \in \mathsf{C_p(X, G)}$ such that $f(F_1) \subseteq \{g\}$ and $f(F_2) \subseteq \{e\}$

An open cover $\mathcal{U}$ of a uniform space $(X, \mathbb{D})$ with a bornology $\mathfrak{B}$ is said to be a *strong* $\mathfrak{B}$*-cover* of X (or a $\mathfrak{B}^s$*-cover* of X) [4], [5], if $X \notin \mathcal{U}$ and for each $B \in \mathfrak{B}$ there exist $U \in \mathcal{U}$ and $D \in \mathbb{D}$ such that $D[B] \subset U$.

$\mathcal{O}_{\mathfrak{B}^s}$ is the collection of $\mathfrak{B}^s$-covers of a space.

Theorem 3.1. *Let G be a metric group, $(X, \mathbb{D})$ a G^*-normal uniform space and $\mathfrak{B}$ a bornology on X with closed base. The following are equivalent:*

(1) $(\mathsf{C}(X, G), \tau^s_{\mathfrak{B}})$ *has countable strong fan tightness;*

(2) X *satisfies* $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$.

Proof. (1) $\Rightarrow$ (2): Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open $\mathfrak{B}^s$-covers of X. Fix a countable decreasing local base $\mathcal{N}_e = \{O_1, O_2, \cdots\}$ at $e \in G$. For every $B \in \mathfrak{B}$ and every $n \in \mathbb{N}$ there exists $U_{n,B} \in \mathcal{U}_n$ such that $D^2[B] \subset U_{n,B}$.

For a fixed $n \in \mathbb{N}$ and $B \in \mathfrak{B}$ set

$$\mathcal{U}_{n,B} := \{U \in \mathcal{U}_n : D^2[B] \subset U\}.$$

It is easy to check that $\overline{D[B]} \subset D^2[B]$. Therefore, for each $U \in \mathcal{U}_{n,B}$, there is a continuous function $f_{B,U}$ from X into G such that $f_{B,U}(D[B]) = \{e\}$ and $f_{B,U}(X \setminus U) = \{g\}$. Let for each n,

$$A_n = \{f_{B,U} : B \in \mathfrak{B}, U \in \mathcal{U}_{n,B}\}.$$

It is easy to verify that f_e belongs to the $\tau^s_{\mathfrak{B}}$-closure of A_n for each $n \in \mathbb{N}$. By (1) for each $n \in \mathbb{N}$ there is a function $f_{B_n,U_n} \in A_n$ so that f_e belongs to the $\tau^s_{\mathfrak{B}}$-closure of $\{f_{B_n,U_n} : n \in \mathbb{N}\}$. We claim that the sequence $(U_n : n \in \mathbb{N})$ witnesses for $(\mathcal{U}_n : n \in \mathbb{N})$ that X has property $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$.

Let $B \in \mathfrak{B}$. Since $f_e \in \overline{\{f_{B_n,U_n} : n \in \mathbb{N}\}}$ it follows that $[B, O_1]^s(f_e)$ contains the function f_{B_m,U_m} for some $m \in \mathbb{N}$. Therefore, there is $D \in \mathbb{D}$ such that for each $x \in D[B]$ it holds $f_{B_m,U_m}(x) \subset O_m$, which means $D[B] \subset U_m$.

(2) $\Rightarrow$ (1): Let $(A_n : n \in \mathbb{N})$ be a sequence of subsets of $(\mathsf{C}(X), \tau^s_{\mathfrak{B}})$ whose closures contain f_e.

For each $B \in \mathfrak{B}$ and each $O_m \in \mathcal{N}_e$ the neighborhood $[B, O_m]^s(f_e)$ of f_e meets each A_n in a function $f_{B,n,m} \in A_n$ satisfying: there is $D \in \mathbb{D}$ with $f_{B,n,m}(x) \in O_m$ for each $x \in D[B]$. For each n set

$$\mathcal{U}_{n,m} = \{f^{\leftarrow}(O_m) : m \in \mathbb{N}, f \in A_n\}.$$

We claim that for each $n, m \in \mathbb{N}$ and each $B \in \mathfrak{B}$ there is an element in $\mathcal{U}_{n,m}$ containing B. Indeed, if $B \in \mathfrak{B}$, then there is $f_{B,n,m} \in [B, O_m]^s(f_e) \cap A_n$. Hence there is $D \in \mathbb{D}$ such that $f_{B,n,m}(x) \in O_m$ for each $x \in D[B]$. This means $D[B] \subset f^{\leftarrow}_{B,n,m}(O_m) \in \mathcal{U}_{n,m}$.

Without loss of generality one may suppose that for all $m, n \in \mathbb{N}$, $X \notin \mathcal{U}_{n,m}$. (The case $\{m \in \mathbb{N} : X \in \mathcal{U}_{n,m}$ for some $n \in \mathbb{N}\}$ is infinite is trivial for consideration, while if the later set is finite one can suppose that it is equal to $\mathbb{N}$.)

Apply now (2) to the sequence $(\mathcal{U}_{n,n} : n \in \mathbb{N})$ and choose for each $n \in \mathbb{N}$ a set $U_{n,n} \in \mathcal{U}_{n,n}$ so that $\{U_{n,n} : n \in \mathbb{N}\}$ is a $\mathfrak{B}^s$-cover of X. Consider the corresponding functions $f_{B_n,n,n}$, $n \in \mathbb{N}$. We prove $f_e \in \overline{\{f_{B_n,n,n} : n \in \mathbb{N}\}}$. Let $[B, O]^s(f_e)$ be a neighbourhood of f_e. There are $i \in \mathbb{N}$ and $D \in \mathbb{D}$ such that $D[B] \subset U_{i,i}$. In fact, the set K of all such i is infinite because $\{U_{n,n} : n \in \mathbb{N}\}$ is a $\mathfrak{B}^s$-cover of X. Take $k \in K$ so that $O_k \subset O$. Then $D[B] \subset f^{\leftarrow}_{B_k,k,k}(O_k) \subset f^{\leftarrow}_{B_k,k,k}(O)$, i.e. $f_{B_k,k,k} \in [B, O]^s(f_e)$. $\square$

Similarly we can prove

Theorem 3.2. *Let G be a metric group, $(X, \mathbb{U})$ a G^*-normal uniform space, and $\mathfrak{B}$ a bornology with closed base on X. The following are equivalent:*

(1) $(\mathsf{C}(X, G), \tau^s_{\mathfrak{B}})$ *has countable fan tightness;*

(2) X *satisfies* $\mathsf{S}_{fin}(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$.

By combining the proofs of Theorems 2.7 and 3.1 we obtain:

Theorem 3.3. *Let G be a metric group, $(X, \mathbb{D})$ a G^*-normal uniform space, and $\mathfrak{B}$ a bornology with closed base on X. The following are equivalent:*

(1) $(\mathsf{C}(X,G), \tau^s_{\mathfrak{B}})$ *is a selectively weakly Fréchet-Urysohn space;*

(2) *For each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of $\mathfrak{B}^s$-covers of X there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each n, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and for each $B \in \mathfrak{B}$) there are $n_0 \in \mathbb{N}$ and a sequence $(D_n : n \geq n_0)$ of elements of $\mathbb{D}$ such that $D_n[B] \subset V$ for some $V \in \mathcal{V}_n$ for all $n \geq n_0$.*

Proof. (1) $\Rightarrow$ (2): Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of $\mathfrak{B}^s$-covers of X. For every $n \in \mathbb{N}$ and every $B \in \mathfrak{B}$ there exist $U_{B,n} \in \mathcal{U}_n$ and $D \in \mathbb{D}$ with $D^2[B] \subset U_{B,n}$. Let $\mathcal{U}_{n,B} := \{U \in \mathcal{U}_n : D^2[B] \subset U\}$. There exists $g \in G$ such that for each $U \in \mathcal{U}_{n,B}$ there is a continuous function $f_{B,U} : X \to G$ satisfying $f_{B,U}(D[B]) = \{e\}$ and $f_{B,U}(X \setminus U) = \{g\}$. Let for each n, set

$$A_n = \{f_{B,U} : B \in \mathfrak{B}, U \in \mathcal{U}_{n,B}\}.$$

It is understood that the function f_e belongs to the $\tau^s_{\mathfrak{B}}$-closure of A_n for each $n \in \mathbb{N}$. Since $(\mathsf{C}(X,G), \tau^s_{\mathfrak{B}})$ is selectively weakly Fréchet-Urysohn, there are finite sets $\Phi_n \subset A_n$, $n \in \mathbb{N}$ such that each $\tau^s_{\mathfrak{B}}$-neighbourhood of f_e intersects all but finitely many Φ_n's. Suppose $\Phi_n = \{f_{B^1_n, U^1_n}, \cdots, f_{B^{k_n}_n, U^{k_n}_n}\}$, $n \in \mathbb{N}$. For each n let $\mathcal{V}_n = \{U^1_n, \cdots, U^{k_n}_n\}$ be the corresponding finite subset of $\mathcal{U}_n$. We prove that the sequence $(\mathcal{V}_n : n \in \mathbb{N})$ witnesses that is X satisfies (2). For the neighbourhood $[B, O_1]^s(f_e)$ of f_e there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$ there is some $f_{B^{i(n)}_n, U^{i(n)}_n}$, $i(n) \leq k_n$, in $[B, O_1]^s(f_e)$. So, for each $n > n_0$ there is $D_n \in \mathbb{D}$ such that $D_n[B] \subset f^{\leftarrow}_{B^{i(n)}_n, U^{i(n)}_n}(O_1)$, which means that $D_n[B] \subset U^{i(n)}_n$.

(2) $\Rightarrow$ (1): Fix a decreasing local base $\{O_n : n \in \mathbb{N}\}$ at $e \in G$.

Let $(A_n : n \in \mathbb{N})$ be a sequence of subsets of $(\mathsf{C}(X,G), \tau^s_{\mathfrak{B}})$ such that $f_e \in \bigcap_{n \in \mathbb{N}} \overline{A_n}$. For every $B \in \mathfrak{B}$ and every $m \in \mathbb{N}$ the neighborhood $[B, O_m]^s(f_e)$ of f_e intersects each A_n, and thus for each $n \in \mathbb{N}$ there is a function $f_{B,n,m} \in A_n$ such that there is $D \in \mathbb{D}$ with $f_{B,n,m}(x) \in O_m$ for each $x \in B^\delta$. For each $n \in \mathbb{N}$ let

$$\mathcal{U}_{n,m} = \{f^{\leftarrow}(O_m) : m \in \mathbb{N}, f \in A_n\}.$$

As in the proofs of the previous theorems we conclude that for each $n, m \in \mathbb{N}$, $\mathcal{U}_{n,m}$ is a $\mathfrak{B}^s$-cover of X. Apply now assumption (2) to the sequence $(\mathcal{U}_{n,n} : n \in \mathbb{N})$ and for each n choose a finite set $\mathcal{V}_n = \{U_{n,1}, \cdots, U_{n,k_n}\} \subset \mathcal{U}_{n,n}$ such that for each $B \in \mathfrak{B}$) there are $n_0 \in \mathbb{N}$ and a sequence $(D_n : n \geq n_0)$ of elements of $\mathbb{D}$ such that $D_n[B] \subset V$ for some $V \in \mathcal{V}_n$ for all $n \geq n_0$. To each $U_{n,i} \in \mathcal{V}_n$, $i \leq k_n$, associate the corresponding function $f_{B_n,n,i} \in A_n$; let $\Phi_n = \{f_{B_n,n,i} : i \leq k_n\}$. We prove that the sequence $(\Phi_n : n \in \mathbb{N})$ witnesses that (1) is satisfied.

Let $[B, O]^s(f_e)$ be a neighbourhood of f_e. There are $m \in \mathbb{N}$ and $D_n \in \mathbb{D}$, $n > m$, such that $O_m \subset O$ and for each $n > m$, $D^2[B] \subset U_{n,j}$ for some $j \leq k_n$. Therefore, for all $n > m$, $f_{B_n,n,j}(D_n[B]) \subset O_n \subset O$, i.e. $f_{B_n,n,j} \in [B, O]^s(f_e)$. $\square$

Observe that Theorem 3.3 is a new result even in the special case $G = \mathbb{R}$. In particular, we have the following two results below.

Theorem 3.4. *Let X be a metric space. The following are equivalent:*

(1) $(\mathsf{C}(X), \tau^s_{\mathfrak{F}})$ *is a selectively weakly Fréchet-Urysohn space;*

(2) *For each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of $\mathfrak{F}^s$-covers of X there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each n, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and for each finite set $F \subset X$ there are $n_0 \in \mathbb{N}$ and a sequence $(\delta_n : n \geq n_0)$ of positive real numbers such that $F^\delta \subset V$ for some $V \in \mathcal{V}_n$ for all $n \geq n_0$.*

It is known that $\tau^s_{\mathfrak{K}}$ coincides with the compact-open topology τ_k on $\mathsf{C}(X)$ (see [3, Corollary 6.6], [4, Remark 3.4], [5]).

An open cover of a space X is a k-cover of X if each compact set $K \subset X$ is contained in a member of the cover.

Theorem 3.5. *For a metric space X the following are equivalent:*

(1) $(\mathsf{C}(X), \tau_k)$ *is a selectively weakly Fréchet-Urysohn space;*

(2) *For each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of k-covers of X there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each n, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and each compact set $K \subset X$ is contained in an element $V \in \mathcal{V}_n$ for all but finitely many n.*

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