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NOTES ON L-FUZZY γ-OPEN SETS

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Abstract

The aim of this paper is to present a common approach allowing to obtain families of L-fuzzy sets in an L-topological space of Goguen type generalizing the class of all open L-fuzzy sets. In particular, we study the notion of an L-fuzzy γ-open set where γ is a monotone operator on the family of all L-fuzzy subsets of an L-topological space X and discuss some properties of L-fuzzy γ-open sets.

1 Introduction

Since Chang defined the concept of a fuzzy topology in [1], many authors investigated different properties of L-fuzzy sets ([2]) which are weaker than the property of openness of a fuzzy set in an L-topological space of Goguen type ([3]). For example, Singal and Prakash, Azad, Zhong, Thakur and Singh considered such kind of properties of fuzzy sets in the papers [4], [5], [6], [7], respectively. All these properties were introduced with the help of operators defined by different combinations of the interior and the closure operators in the corresponding L-topological space. As an important consequence of this was that all operators involved in the definition of these properties were monotone. Taking into account this fact and motivated by Á. Császár's work ([8]), in this paper we present a general construction which allows to obtain different classes of families of L-fuzzy sets in L-topological spaces generalizing the class of open L-fuzzy sets. To obtain this goal, we, basing on the monotony property, define L-fuzzy γ-open sets and study some fundamental properties of these sets. Besides, we investigate the behavior of L-fuzzy γ-open sets in L-topological spaces of Goguen type.

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2 Preliminaries

Let $(L, \leq)$ be a complete lattice and L^X denote the family of all L-fuzzy sets on a nonempty set X. By $\Gamma_L(X)$, we denote the set of all γ mappings where $\gamma : L^X \longrightarrow L^X$ satisfies the property $f \leq g$ implies $\gamma(f) \leq \gamma(g)$ for all $f, g \in L^X$, i.e., γ has the property of monotony. It is clear that the identity map, id, on L^X and L-interior operator of an L-topological space of Gougen type, int, are trivial elements of $\Gamma_L(X)$.

Throughout this paper, if $\gamma, \gamma' \in \Gamma_L(X)$ and $f \in L^X$, then for the sake of simplicity, we use the notation γf for $\gamma(f)$ and denote $\gamma \circ \gamma'$ by $\gamma\gamma'$ and, $\gamma \circ \gamma \circ ... \circ \gamma$ (n times) by γ^n.

We denote some basic subsets of $\Gamma_L(X)$ as follows:

$$
\begin{aligned}
&\Gamma_0 = \{\gamma \in \Gamma_L(X) : \gamma 1_\emptyset = 1_\emptyset\}\\
&\Gamma_1 = \{\gamma \in \Gamma_L(X) : \gamma 1_X = 1_X\}\\
&\Gamma_2 = \{\gamma \in \Gamma_L(X) : \forall f \in L^X, \gamma^2 f = \gamma f\}\\
&\Gamma_+ = \{\gamma \in \Gamma_L(X) : \forall f \in L^X, f \leq \gamma f\}\\
&\Gamma_- = \{\gamma \in \Gamma_L(X) : \forall f \in L^X, \gamma f \leq f\}\\
&\Gamma_{-2} = \{\gamma \in \Gamma_L(X) : \forall f \in L^X, \gamma^2 f \leq \gamma f\}
\end{aligned}
$$

In this paper, on the condition that $\Omega \subseteq \mathbb{Z} \bigcup \{+, -\}$, Γ_Ω represents the subset of $\Gamma_L(X)$ consisting of all γ functions which belong to Γ_n for all $n \in \Omega$. In addition, as an abbreviation, we use Γ_{nm} instead of $\Gamma_{\{n,m\}}$ where $n, m \in \Omega$. This abbreviation will be used for all other subsets of Ω having more than two elements as well. To illustrate, by Γ_{012+} we mean $\Gamma_{\{0,1,2,+\}}$.

It is clear that $\Gamma_+ \subseteq \Gamma_1$ and $\Gamma_- \subseteq \Gamma_0$. Also, $id \in \Gamma_{012+} \bigcap \Gamma_{012-}$ and L-interior operator $int \in \Gamma_{012-}$.

3 L-Fuzzy γ-Open Sets

Definition 1. Let X be a nonempty set and $\gamma \in \Gamma_L(X)$. $f \in L^X$ is called an *L-fuzzy γ-open set* iff $f \leq \gamma f$.

Corollary 1. *Let $\gamma \in \Gamma_L(X)$.*

(a) $1_\emptyset$ *is always an L-fuzzy γ-open set.*

(b) 1_X *is an L-fuzzy γ-open set iff $\gamma \in \Gamma_1$.*

(c) *If $\gamma \in \Gamma_2$, then γf is an L-fuzzy γ-open set for all $f \in L^X$.*

(d) *If $\gamma \in \Gamma_+$, then f is an L-fuzzy γ-open set for all $f \in L^X$.*

(e) *Let $\gamma \in \Gamma_-$. $f \in L^X$ is an L-fuzzy γ-open set iff $f = \gamma f$.*

Proposition 1. *Let $\gamma \in \Gamma_L(X)$. The arbitrary union of L-fuzzy γ-open sets is an L-fuzzy γ-open set.*

Proof. Let $\{f_i : i \in J\} \subseteq L^X$. If $f_i \leq \gamma f_i$ for all $i \in J$ and $\bigvee f_i = f$, then for all $i \in J$,

$$\begin{aligned} f_i \leq f &\Rightarrow \gamma f_i \leq \gamma f \\ &\Rightarrow \bigvee \gamma f_i \leq \bigvee \gamma f \\ &\Rightarrow f = \bigvee f_i \leq \bigvee \gamma f_i \leq \gamma f. \end{aligned}$$

□

Definition 2. Let $f \in L^X$ and $\gamma \in \Gamma_L(X)$. The union of all L-fuzzy γ-open sets contained in f is called the *L-fuzzy γ-interior of f* and denoted by $i_\gamma f$, i.e.,

$$i_\gamma f = \bigvee \{g \in L^X : g \leq f, \ \ g \leq \gamma g\}.$$

Corollary 2. *Let $\gamma \in \Gamma_L(X)$ and $f \in L^X$.*

(a) *It is clear that $i_\gamma f$ is the greatest L-fuzzy γ-open set contained in f.*

(b) *Let (X, τ) be an L-topological space of Gougen type. Then, for the L-interior operator int, $i_{int} f = int(f)$, for all $f \in L^X$.*

Proposition 2. *Let $\gamma \in \Gamma_L(X)$. Then,*

(a) $i_\gamma \in \Gamma_{02-}$.

(b) $i_\gamma \in \Gamma_1 \Leftrightarrow \gamma \in \Gamma_1$.

(c) $\gamma \in \Gamma_{02-} \Leftrightarrow \gamma = i_\gamma$.

Proof. Let $f, g \in L^X$.

(a) $f \leq g$ implies that $i_\gamma f \leq f \leq g$. Since $i_\gamma g$ is the biggest L-fuzzy γ-open set contained in g, we have $i_\gamma f \leq i_\gamma g$ and then, we get $i_\gamma \in \Gamma_L(X)$. Furthermore, we have $i_\gamma \in \Gamma_{0-}$, since we have $i_\gamma 1_\emptyset = 1_\emptyset$ and $i_\gamma f \leq f$. We also have $i_\gamma i_\gamma f \leq i_\gamma f$, since $i_\gamma \in \Gamma_-$. Now it is enough to show the converse part. Since $i_\gamma f \leq i_\gamma f$ and $i_\gamma f$ is an L-fuzzy γ-open set, we get $i_\gamma f \leq i_\gamma i_\gamma f$ which completes the proof.

(b) Let $i_\gamma 1_X = 1_X$ which means that 1_X is an L-fuzzy γ-open set. Hence, $1_X \leq \gamma 1_X \leq 1_X$. On the other hand, if $\gamma 1_X = 1_X$, then 1_X is an L-fuzzy γ-open set, i.e., $i_\gamma 1_X = 1_X$.

(c) Let $\gamma \in \Gamma_{02-}$. It is enough to show that γf is the biggest L-fuzzy γ-open set contained in f. Since $\gamma \in \Gamma_{2-}$, we obtain that $\gamma f \leq f$ and $\gamma\gamma f = \gamma f$, i.e., γf is an L-fuzzy γ-open set contained in f. Now, suppose that there exists a fuzzy set $g \in L^X$ such that $g \leq f$ and $g \leq \gamma g$. Since $\gamma \in \Gamma_L(X)$, we have $g \leq \gamma g \leq \gamma f$ which shows that γf is the biggest L-fuzzy γ-open set contained in f, i.e., $i_\gamma f = \gamma f$. For the converse part, $\gamma = i_\gamma \in \Gamma_{02-}$ by (a). □

Proposition 3. *Let $f \in L^X$ and $\gamma \in \Gamma_L(X)$. Then, the following statements are equivalent:*

(a) *f is an L-fuzzy γ-open set.*

(b) $f = i_\gamma f$.

(c) *f is an L-fuzzy i_γ-open set.*

Proof. **(a)$\Rightarrow$ (b):** Since $f \leq f$ and f is an L-fuzzy γ-open set, $f \leq i_\gamma f$. Since the inverse inequality always holds, $f = i_\gamma f$.

(b)$\Rightarrow$ (a): Since $i_\gamma f$ is an L-fuzzy γ-open set, f is an L-fuzzy γ-open set.

(b)$\Rightarrow$ (c): If $f = i_\gamma f$, then $f \leq i_\gamma f$. Hence, f is an L-fuzzy i_γ-open set.

(c)$\Rightarrow$ (b): If f is an L-fuzzy i_γ-open set, then $f \leq i_\gamma f$. Since the inverse inequality always exits, $f = i_\gamma f$. □

Remark 1. Let X be a nonempty set, and (X, τ) be an L-topological space of Chang type ([1]). Let *int* represent the interior operator and *cl* denote the closure operator of the topological space (X, τ). Then,

- **(a)** for $\gamma = int$, L-fuzzy γ-open sets coincide with L-fuzzy open sets,
- **(b)** for $\gamma = intcl$, L-fuzzy γ-open sets coincide with L-fuzzy preopen sets ([4]),
- **(c)** for $\gamma = clint$, L-fuzzy γ-open sets coincide with L-fuzzy semiopen sets ([5]),
- **(d)** for $\gamma = intclint$, L-fuzzy γ-open sets coincide with L-fuzzy strongly semiopen (called as L-fuzzy α-sets in [9]) sets ([6]),
- **(e)** for $\gamma = clintcl$, L-fuzzy γ-open sets coincide with L-fuzzy semipreopen (called as L-fuzzy β-open sets in [10]) sets ([7]).
- **(f)** for $\gamma = intcl \vee clint$, L-fuzzy γ-open sets coincide with L-fuzzy γ-open sets ([11]),
- **(g)** for $\gamma = V_b$ where $V_b(f) = \bigwedge\{g \in I^X | g \leq f,\ intcl(g) \wedge clint(g) \leq g\}$ for all $f \in I^X$, L-fuzzy γ-open sets coincide with L-fuzzy V_b-sets ([12]).

Moreover, $i_{intcl}f$ is the L-fuzzy preinterior ([4]), $i_{clint}f$ is the L-fuzzy semi-interior ([13]), $i_{clintcl}f$ is the L-fuzzy semipre-interior ([7]), $i_{intclint}f$ is the L-fuzzy strong semi-interior ([6]) and $i_{intcl\vee clint}f$ is the L-fuzzy γ-interior ([11]) of $f \in I^X$.

Theorem 1. *Let $\gamma_1, \gamma_2 \in \Gamma_L(X)$.*

- **(a)** $\gamma_1\gamma_2 \in \Gamma_L(X)$.
- **(b)** *If $\gamma_1, \gamma_2 \in \Gamma_n$ for all $n \in \{0, 1, +, -\}$, then $\gamma_1\gamma_2 \in \Gamma_n$.*

Proof. One can easily prove this theorem by means of the definitions of Γ_n for all $n \in \{0, 1, +, -\}$. □

Remark 2. Let $\gamma_1, \gamma_2 \in \Gamma_2$. $\gamma_1\gamma_2$ may not be in Γ_2 as shown in the following example.

Example 1. Let $X = [0, 1]$ and consider an L-topology on X of Chang type defined as $\tau = \{1_\emptyset, 1_X, h\}$ where $h(x) = 0.4$ for all $x \in X$. Now consider γ defined as follows:

$$\begin{array}{llll} \gamma: & L^X & \rightarrow & L^X \\ & f & \rightarrow & \gamma f = \begin{cases} g, & g \leq f; \\ 1_\emptyset, & \text{otherwise.} \end{cases} \end{array}$$

where $g(x) = 0.5$ for all $x \in X$. It is clear that $\gamma \in \Gamma_2$. On the other hand, $\gamma int \gamma int 1_X = \gamma int \gamma 1_X = \gamma int g = \gamma h = 1_\emptyset$ and $\gamma int 1_X = \gamma 1_X = g$ implying that $\gamma int \gamma int 1_X \neq \gamma int 1_X$. Hence, $\gamma int \notin \Gamma_2$.

Proposition 4. *Let $\iota, \kappa \in \Gamma_2$ and for all $f \in L^X$,*

$$\iota\kappa f \leq \kappa f \tag{1}$$

and

$$\iota\kappa f \leq \kappa\iota\kappa f \tag{2}$$

If γ is a composition of alternating factors ι and κ, then $\gamma \in \Gamma_2$ except for the case $\gamma = \kappa\iota$.

Proof. Let $f \in L^X$.
For $\gamma = \iota\kappa$, we have $\kappa\iota\kappa f \leq \kappa\kappa f = \kappa f$ by (1). Hence, we obtain $\iota\kappa\iota\kappa f \leq \iota\kappa f$. On the other hand, by (2), we get $\iota\kappa f \leq \kappa\iota\kappa f$ implying that $\iota\kappa = \iota\iota\kappa f \leq \iota\kappa\iota\kappa f$. Hence, we obtain $\iota\kappa\iota\kappa f = \iota\kappa$, i.e., $\gamma = \iota\kappa \in \Gamma_2$.
For $\gamma = \iota\kappa\iota$, we have $(\iota\kappa\iota)(\iota\kappa\iota)f = \iota\kappa\iota\kappa\iota f = (\iota\kappa\iota\kappa)\iota f = \iota\kappa\iota f$. Hence,$\gamma = \iota\kappa\iota \in \Gamma_2$.
For $\gamma = \kappa\iota\kappa$, we have $(\kappa\iota\kappa)(\kappa\iota\kappa)f = \kappa\iota\kappa\iota\kappa f = \kappa(\iota\kappa\iota\kappa)f = \kappa\iota\kappa f$ implying that $\gamma = \kappa\iota\kappa \in \Gamma_2$.
For $\gamma = \kappa\iota\kappa\iota$, we have $(\kappa\iota\kappa\iota)(\kappa\iota\kappa\iota)f = \kappa(\iota\kappa\iota\kappa)\iota\kappa\iota f = \kappa(\iota\kappa\iota\kappa)\iota f = \kappa\iota\kappa\iota f$. Hence $\gamma = \kappa\iota\kappa\iota \in \Gamma_2$.
For $\gamma = \iota\kappa\iota\kappa$, we have $(\iota\kappa\iota\kappa)(\iota\kappa\iota\kappa)f = \iota\kappa\iota\kappa f$. Hence,$\gamma = \kappa\iota\kappa\iota \in \Gamma_2$.
One can easily show that $\gamma \in \Gamma_2$ for any γ composed of more than four factors. $\square$

Proposition 5. *Let $\iota, \kappa \in \Gamma_2$, the inequality* (1) *and*

$$\iota f \leq \kappa\iota f \tag{3}$$

are satisfied for all $f \in L^X$. Then, $\gamma = \kappa\iota \in \Gamma_2$.

Proof. Let $f \in L^X$. If $\iota f \leq \kappa\iota f$, then $\iota f \leq \iota\kappa\iota f$ implying that $\kappa\iota f \leq \kappa\iota\kappa\iota f$. If in (1) we replace f by ιf, then we get $\iota\kappa\iota f \leq \kappa\iota f$. Hence, we obtain $\kappa\iota\kappa\iota f \leq \kappa\iota f$. Thus, $\kappa\iota\kappa\iota f = \kappa\iota f$, i.e., $\kappa\iota \in \Gamma_2$. $\square$

Corollary 3. *Let (X, τ) be an L-topological space of Chang type. If ι and κ are chosen as the interior and closure operator respectively, since $int, cl \in \Gamma_2$ and* (1), (2) *and* (3) *are satisfied by int and cl operators, any composition of alternating factors cl and int is equal to one of the mappings int, cl, int(cl), cl(int), int(cl(int)) or cl(int(cl)) by Proposition* 4 *and Proposition* 5.

Corollary 4. *If $\iota \in \Gamma_{2-}$ and $\kappa \in \Gamma_{2+}$, then any composition of alternating factors ι and κ is an element of Γ_2.*

Proof. It follows from the fact that inequalities (1), (2), (3) are verified by ι and κ. $\square$

Proposition 6. *Let $\gamma \in \Gamma_L(X)$. Every L-fuzzy γ-open set is an L-fuzzy γ^n-open for all $n \in \mathbb{N}$. If $\gamma \in \Gamma_{-2}$, then L-fuzzy γ-open sets coincide with L-fuzzy γ^n-sets.*

Proof. Let $\gamma \in \Gamma_L(X)$ and $f \in L^X$ is an L-fuzzy γ-open set. Then, $f \leq \gamma f \leq \gamma\gamma f \leq ... \leq \gamma^{n-1} f \leq \gamma^n f$. Therefore, f is an L-fuzzy γ^n-open set.
For the case $\gamma \in \Gamma_{-2}$, assume that f is an L-fuzzy γ^n-open set. Then, $f \leq \gamma^n f \leq \gamma^{n-1} f \leq ... \leq \gamma^2 f \leq \gamma f$. Hence, f is an L-fuzzy γ-open set. $\square$

Proposition 7. *Let $\iota, \kappa \in \Gamma_{-2}$ and ι, κ satisfy the inequality (1). Then, any composition of factors ι and κ is an element of Γ_{-2}. Moreover, if $\gamma^{'} \in \Gamma_L(X)$ is an arbitrary composition of factors ι and κ, then*

- **(a)** *an L-fuzzy γ-open set is an L-fuzzy $\iota\kappa$-open set where $\gamma = \iota\gamma^{'}\kappa$.*
- **(b)** *an L-fuzzy γ-open set is an L-fuzzy $\iota\kappa\iota$-open set where $\gamma = \iota\gamma^{'}\iota$.*
- **(c)** *an L-fuzzy γ-open set is an L-fuzzy $\kappa\iota$-open set where $\gamma = \kappa\gamma^{'}\iota$.*
- **(d)** *an L-fuzzy γ-open set is an L-fuzzy $\kappa\iota\kappa$-open set where $\gamma = \kappa\gamma^{'}\kappa$.*

The converse implications hold if no factor ι is immediately followed by another factor ι.

Proof. Let $f \in L^X$. It can be easily seen that if $\iota, \kappa \in \Gamma_{-2}$, then $\iota^n f \leq \iota f$ and $\kappa^n f \leq \kappa f$, and by (1), we obtain $(\iota\kappa)^n f \leq \kappa^n f \leq \kappa f$ for all $n \in \mathbb{N}$. Since for all $n \in \mathbb{N}$,

$$\gamma\gamma f = \iota^n \iota^n f \leq \iota\iota^n f = \iota^2 \iota^{n-1} f \leq \iota\iota^{n-1} f = \iota^n f = \gamma f,$$

we have $\gamma \in \Gamma_{-2}$ for $\gamma = \iota^n$.

Now let $\gamma = \gamma_1\kappa\gamma_2$ where γ_1, γ_2 are any (may be empty) compositions of factors ι and κ, i.e., γ has at least one factor κ. Then,$\gamma\gamma f = \gamma_1\kappa\gamma_2\gamma_1\kappa\gamma_2 f$. If ι is replaced instead of each ι^n factors, and κ is replaced instead of each κ^m factors, and κ is replaced instead of each $(\iota\kappa)^p$ factors in the composition $\kappa\gamma_2\gamma_1\kappa$, we get $\gamma\gamma f = \gamma_1\kappa\gamma_2\gamma_1\kappa\gamma_2 f \leq \gamma_1\kappa\gamma_2 f = \gamma f$. Hence, $\gamma \in \Gamma_{-2}$.

(a) Let $\gamma = \iota\gamma^{'}\kappa$ and $f \in L^X$ is an L-fuzzy γ-open set. For a suitable $m \in \mathbb{N}$,

$$f \leq \gamma f = \iota\gamma^{'}\kappa f \leq (\iota\kappa)^m f = (\iota\kappa)(\iota\kappa)^{m-1} f \leq \iota\kappa\kappa^{m-1} f = \iota\kappa^m f \leq \iota\kappa f$$

by similar substitutions in the above manner. Hence, f is an L-fuzzy $\iota\kappa$-open set. Conversely, suppose that no factor ι is immediately followed by another factor ι and f is L-fuzzy $\iota\kappa$-open set. We have $f \leq \iota\kappa f$ implying that $f \leq (\iota\kappa)^n f$ where n is the number of the factors κ in γ, and also we get $f \leq \iota\kappa f \leq \kappa f$. Then, by the inequality (1) and $f \leq \kappa f$, we obtain $f \leq \gamma f$. Hence, f is an L-fuzzy γ-open set.

(b) Let $\gamma = \iota\gamma^{'}\iota$ and f is an L-fuzzy γ-open set. By (1), we obtain

$$f \leq \iota\gamma^{'}\iota f \leq (\iota\kappa)^m \iota f \leq (\iota\kappa)(\iota\kappa)^{m-1}\iota f \leq (\iota\kappa)(\kappa)^{m-1}\iota f = \iota\kappa^m \iota f \leq \iota\kappa\iota f$$

for a suitable $m \in \mathbb{N}$. Hence, f is an L-fuzzy $\iota\kappa\iota$-open set. Conversely, if no factor ι is immediately followed by another factor ι and f is L-fuzzy $\iota\kappa\iota$-open set, then for $m \in \mathbb{N}$,

$$f \leq \iota\kappa\iota f \Rightarrow (\iota\kappa)^m \iota f \leq (\iota\kappa)^m \iota\iota\kappa\iota f \leq (\iota\kappa)^m \iota\kappa\iota f = (\iota\kappa)^{m+1}\iota f.$$

Therefore, by the inequality (1), $f \leq \iota\kappa\iota f \leq (\iota\kappa)^n \iota f \leq \gamma f$ where n is the number of the factors κ in γ. Thus, f is an L-fuzzy γ-open set.

(c) Let $\gamma = \kappa\gamma^{'}\iota$ and f is an L-fuzzy γ-open set. For $m \geq 2$ and $m \in \mathbb{N}$, we have

$$f \leq \gamma f \leq (\kappa\iota)^m f = \kappa(\iota\kappa)^{m-1}\iota f \leq \kappa\kappa^{m-1}\iota f = \kappa^m \iota f \leq \kappa\iota f.$$

Hence, f is an L-fuzzy $\kappa\iota$-open set. Conversely, if no factor ι is immediately followed by another factor ι and f is L-fuzzy $\kappa\iota$-open set, then by the inequality (1), we have

$$f \le \kappa\iota f \Rightarrow f \le (\kappa\iota)^n f \le \gamma f$$

, where n is the number of factors κ in γ. Hence, f is an L-fuzzy γ-open set.

(d) Let $\gamma = \kappa\gamma^{'}\kappa$ and f is an L-fuzzy γ-open set. For a suitable $m \in \mathbb{N}$, we have

$$f \le \kappa\gamma^{'}\kappa f \le (\kappa\iota)^m \kappa f = \kappa(\iota\kappa)^{m-1}\iota\kappa f \le \kappa\kappa^{m-1}\iota\kappa f \le \kappa^m\iota\kappa f \le \kappa\iota\kappa f.$$

Thus, f is an L-fuzzy $\kappa\iota\kappa$-open set. Conversely, if no factor ι is immediately followed by another factor ι and f is L-fuzzy $\kappa\iota\kappa$-open set, then we get

$$f \le \kappa\iota\kappa f \Rightarrow (\kappa\iota)^m\kappa f \le (\kappa\iota)^m\kappa\kappa\iota\kappa f \le (\kappa\iota)^m\kappa\iota\kappa f = (\kappa\iota)^{m+1}\kappa f,$$

for all $m \in \mathbb{N}$. Thus, by the inequality (1), we obtain $f \le \kappa\iota\kappa f \le (\kappa\iota)^{n-1}\kappa f \le \gamma f$ where n is the number of factors κ in γ. Hence, f is an L-fuzzy γ-open set. □

Corollary 5. *Let $\iota \in \Gamma_-$ and $\kappa \in \Gamma_{-2}$. Then, the statements of Proposition 7 are valid. Furthermore,*

(a) *For an L-fuzzy set $f \in L^X$, the following diagram holds:*

$$\begin{array}{ccccc} & & L-fuzzy\ \iota\kappa-open & & \\ & \nearrow & & \searrow & \\ L-fuzzy\ \iota\kappa\iota-open & & & & L-fuzzy\ \kappa\iota\kappa-open \rightarrow L-fuzzy\ \kappa-open \\ & \searrow & & \nearrow & \\ & & L-fuzzy\ \kappa\iota-open & & \end{array}$$

(b) *If $f \in L^X$ is an L-fuzzy $\iota\kappa$-open set and $\kappa\iota$-open set, then it is an L-fuzzy $\iota\kappa\iota$-open set.*

Proof. Let $f \in L^X$. It is clear from the fact that the inequality (1) and $\iota \in \Gamma_{-2}$ are satisfied under these conditions.

(a) Let f is an L-fuzzy $\iota\kappa\iota$-open set. Hence, $f \le \iota(\kappa\iota)f \le \kappa\iota f$ implying that f is an L-fuzzy $\kappa\iota$-open set. If $f \le (\iota\kappa)\iota f \le \iota\kappa f$, then f is an L-fuzzy $\iota\kappa$-open set. Providing that f is an L-fuzzy $\iota\kappa$-open set or $\kappa\iota$-open set, then $f \le \iota\kappa f \le \iota\kappa\iota\kappa f \le \kappa\iota\kappa f$ or $f \le \kappa\iota f \le (\kappa\iota\kappa)\iota f \le \kappa\iota\kappa f$, respectively. i.e, f is an L-fuzzy $\kappa\iota\kappa$-open set in both cases. For the case $f \le \kappa\iota\kappa f$, we get $f \le \kappa\iota\kappa f \le \kappa\kappa f \le \kappa f$ implying that f is an L-fuzzy κ-open set.

(b) Suppose that f is an L-fuzzy $\iota\kappa$-open set and $\kappa\iota$-open set. Hence, we obtain $f \le \iota\kappa f \le \iota\kappa(\kappa\iota)f \le \iota\kappa\iota f$. Thus, f is an L-fuzzy $\iota\kappa\iota$-open set. □

Remark 3. If ι and κ are chosen as the interior and closure operator, respectively, and Remark 1 is considered, then one can easily see

(a) in [9] that an L-fuzzy $\iota\kappa$-open set need not be an L-fuzzy $\iota\kappa\iota$-open set,

(b) in [9] that L-fuzzy $\kappa\iota$-open set need not be an L-fuzzy $\iota\kappa\iota$-open set,

(c) in [7] that L-fuzzy $\kappa\iota\kappa$-open set need not be an L-fuzzy $\kappa\iota$-open set,

(d) in [7] that L-fuzzy $\kappa\iota\kappa$-open set need not be an L-fuzzy $\iota\kappa$-open set,

(e) in [12] that an L-fuzzy set need not be an L-fuzzy $\kappa\iota\kappa$-open set.

Also, in [4] it is shown that L-fuzzy $\iota\kappa$-open sets and L-fuzzy $\kappa\iota$-open sets are independent notions.

Remark 4. $\gamma_1\gamma_2$ need not belong to Γ_{-2} where $\gamma_1, \gamma_2 \in \Gamma_2$ as shown in the following example.

Example 2. Consider the L-fuzzy topology in Example 1 and let γ be defined as follows: For all $f \in L^X$,

$$\gamma f = \begin{cases} g, & f \leq g; \\ 1_X, & \text{otherwise.} \end{cases}$$

where $g(x) = 0.5$ for all $x \in X$. It is clear that $cl, \gamma \in \Gamma_2$. Since $\gamma cl \gamma cl 1_\emptyset = \gamma cl \gamma 1_\emptyset = \gamma cl g = 1_X \not\leq \gamma cl 1_\emptyset = g$, we get $\gamma cl \notin \Gamma_{-2}$.

Remark 5. Let $\iota \in \Gamma_-$ and $\kappa \in \Gamma_2$. In the following example, it is shown that an L-fuzzy $\iota\kappa$-open set may not be an L-fuzzy $\iota\iota\kappa$-open set.

Example 3. Consider the γ function defined in Example 1, and the L-fuzzy topology $\tau = \{1_\emptyset, 1_X, h\}$ of Chang type where $h(x) = 0.6$ for all $x \in X$. Let $\kappa : L^X \to L^X$ be defined as follows: $\kappa f = l$ for all $f \in L^X$ where $l(x) = 0.7$ for all $x \in X$. It is clear that $\iota = \gamma int \in \Gamma_-$ and $\kappa \in \Gamma_2$. For the L-fuzzy set f where $f(x) = 0.3$ for all $x \in X$, f is an L-fuzzy $\iota\kappa$-open set. However, it is not an L-fuzzy $\iota\iota\kappa$-open set.

4 L-Fuzzy γ-Open Sets In L-Topological Spaces

Let (X, τ) be an L-topological space of Gougen type. We denote the family of all $\gamma \in \Gamma(X)$ satisfying the property

$$\forall g \in \tau,\ \forall f \in L^X; \quad g \wedge \gamma f \leq \gamma(g \wedge f)$$

by Γ_3.

It is clear that for an L-topological space (X, τ), the interior operator $int \in \Gamma_3$. On the other hand, a closure operator may not be an element of Γ_3.

Proposition 8. *If $\gamma_1, \gamma_2 \in \Gamma_3$, then $\gamma_1 \circ \gamma_2 \in \Gamma_3$.*

Proof. Let (X, τ) be an L-topological space, $\gamma_1, \gamma_2 \in \Gamma_3$, $g \in \tau$, and $f \in L^X$. Since

$$g \wedge \gamma_1 \circ \gamma_2(f) = g \wedge \gamma_1(\gamma_2(f)) \leq \gamma_1(g \wedge \gamma_2(f)) \leq \gamma_1(\gamma_2(g \wedge f)) = \gamma_1 \circ \gamma_2(g \wedge f),$$

we have $\gamma_1 \circ \gamma_2 \in \Gamma_3$. □

Proposition 9. *Let (X, τ) be an L-topological space and $\gamma \in \Gamma_3$. If $g \in \tau$ and $f \in L^X$ is an L-fuzzy γ-open set, then $g \wedge f$ is an L-fuzzy γ-open set.*

Proof. Since $g \le g$ and $f \le \gamma f$, we have $g \wedge f \le g \wedge \gamma f \le \gamma(g \wedge f)$. Hence, $g \wedge f$ is an L-fuzzy γ-open set. □

Remark 6. An L-fuzzy open set need not be a fuzzy γ-open set even if $\gamma \in \Gamma_3$ or $\gamma \in \Gamma_{023}$ as shown in the following example.

Example 4. Let $X = [0, 1]$ and the fuzzy topology on X defined as $\tau = \{0_X, 1_X, g\}$ where

$$g(x) = \begin{cases} x, & 0 \le x \le 0.5; \\ 1 - x, & 0.5 \le x \le 1. \end{cases}$$

Consider the function $\gamma : I^X \to I^X$ defined as $\gamma f = h$ for all $f \in I^X$ where

$$h(x) = \begin{cases} 0.2, & 0 \le x < 0.5; \\ 0.8, & 0.5 \le x \le 1. \end{cases}$$

It is clear that $\gamma \in \Gamma_{23}$. On the other hand, the fuzzy open set g is not an L-fuzzy γ-open set.
If $\gamma : I^X \to I^X$ is chosen as

$$\gamma f = \begin{cases} 0_X, & f = 0_X; \\ h, & \text{otherwise}. \end{cases},$$

then it is obvious that $\gamma \in \Gamma_{023}$. Also, the fuzzy open set g is not an L-fuzzy γ-open set.

Proposition 10. *Let (X, τ) be an L-topological space. If $\gamma \in \Gamma_3$ and g is an L-fuzzy open set such that $g \le \gamma 1_X$, then g is an L-fuzzy γ-open set.*

Proof. Let $g \in \tau$ and $g \le \gamma 1_X$. Since

$$g = g \wedge \gamma 1_X \le \gamma(g \wedge 1_X) = \gamma g,$$

g is an L-fuzzy γ-open set. □

Corollary 6. *Let $\gamma \in \Gamma_{13}$. Then, every L-fuzzy open set is an L-fuzzy γ-open set.*

Proof. Let $g \in \tau$. Then,

$$g = g \wedge 1_X = g \wedge \gamma(1_X) \le \gamma(g \wedge 1_X) = \gamma g.$$

Hence, g is an L-fuzzy γ-open set. □

Corollary 7. *Let (X, τ) be an L-topological space and $\gamma \in \Gamma_3$. If g is an L-fuzzy open set such that there exits an L-fuzzy γ-open set f containing g, then g is an L-fuzzy γ-open set.*

Proof. Let $f \in L^X$, $f \le \gamma f$ and $g \le f$. Since $g \le f \le \gamma f \le \gamma 1_X$, g is an L-fuzzy γ-open set from Proposition 10. □

Proposition 11. *If $\gamma \in \Gamma_3$, then $i_\gamma \in \Gamma_3$.*

Proof. Let $g \in \tau$ and $f \in L^X$. Since $g \wedge i_\gamma f$ is an L-fuzzy γ-open set by Proposition 9 and $g \wedge i_\gamma f \leq g \wedge f$, we have $g \wedge i_\gamma f \leq i_\gamma(g \wedge f)$. Hence, we have $i_\gamma \in \Gamma_3$. □

Proposition 12. *Let $\gamma \in \Gamma_3$. $\gamma \in \Gamma_1$ if and only if $intf \leq \gamma f$ for all $f \in L^X$.*

Proof. Let $f \in L^X$.
$\Leftarrow$: Since $\gamma \in \Gamma_{13}$, we have

$$intf \wedge \gamma 1_X = intf \wedge 1_X = intf \leq \gamma(intf \wedge 1_X) = \gamma(intf) \leq \gamma f.$$

$\Rightarrow$: Since $int1_X = 1_X \leq \gamma 1_X \leq 1_X$, we get $\gamma \in \Gamma_1$. □

Remark 7. Consider the function $\gamma^{'}$ consisting of alternating compositions of *int* and γ where $\gamma \in \Gamma_{23}$. It is clear that $\gamma^{'} \in \Gamma_3$ by Proposition 8.

Proposition 13. *Let $\gamma \in \Gamma_{23}$ and $\gamma^{'}$ consist of alternating compositions of int and γ. Then $\gamma^{'} \in \Gamma_2$ except for the condition $\gamma^{'} = \gamma int$.*

Proof. Since the equalities *(1)* and *(2)* are satisfied for the cases $\iota = int$ and $\kappa = \gamma$, we have $\gamma^{'} \in \Gamma_2$ by Proposition 4. □

Example 5. Let $X = [0, 1]$ and the fuzzy topology on X be $\tau = \{1_X, 1_\emptyset\}$. Consider the function γ defined in Example 1. It is clear that $\gamma \in \Gamma_{23}$. On the other hand, $\gamma int \notin \Gamma_2$, since $\gamma int\gamma int1_X = \gamma int\gamma 1_X = \gamma intg = \gamma 1_\emptyset = 1_\emptyset \neq \gamma int1_X = g$.

Remark 8. If $\gamma \in \Gamma_{23}$, then the cases *int*, γ, $int\gamma$, γint, $int\gamma int$, $\gamma int\gamma$, $\gamma int\gamma int$ are enough to be considered.

Proposition 14. *If $\gamma \in \Gamma_{23}$, then*

- **(a)** $int\gamma int \leq int\gamma \leq \gamma int\gamma$.
- **(b)** $int\gamma int \leq \gamma int\gamma int \leq \gamma int\gamma$.
- **(c)** $\gamma int\gamma \leq \gamma$.
- **(d)** $\gamma int \leq \gamma$.

Proof. **(c)** and **(d)** are clear from the fact that $\gamma \in \Gamma_2$ and $int \in \Gamma_-$. We need to show that $int\gamma f \leq \gamma int\gamma f$ for all $f \in L^X$. Since $int\gamma f \leq int\gamma 1_X \leq \gamma 1_X$, $int\gamma f$ is an L-fuzzy γ-open set by Proposition 10. Hence, we have $int\gamma f \leq \gamma int\gamma f$ which completes for the proofs of **(a)** and **(b)**. □

Proposition 15. *Let (X, τ) be an L-topological space. If $\gamma \in \Gamma_{123}$, then for all $f \in L^X$,*

$$intf \leq int\gamma intf \leq \gamma intf = \gamma int\gamma intf \leq \gamma f.$$

Proof. Let $f \in L^X$. If $\gamma \in \Gamma_{123}$, then by Proposition 10, we have $intf \leq \gamma intf$. Since the inequalities (1) and (3) are satisfied for the case $\kappa = \gamma$ and $\iota = int$, we get $\gamma int \in \Gamma_2$. Thus, we obtain $intintf = intf \leq int\gamma int\gamma f \leq \gamma intf = \gamma int\gamma intf \leq \gamma\gamma f = \gamma f$. □

Corollary 8. *If $\gamma \in \Gamma_{123}$, then the implications in the following diagram hold:*

$$\begin{tikzcd}
 & & int\gamma\text{-}open \arrow[dr] & \\
open \arrow[r] & int\gamma int\text{-}open \arrow[ur] \arrow[dr] & & \gamma int\gamma\text{-}open \\
 & & \gamma int\text{-}open \Leftrightarrow \gamma int\gamma int\text{-}open \arrow[ur] &
\end{tikzcd}$$

Remark 9. The converses of the implications of the above diagram do not hold as shown in the following example.

Example 6. **(a)** Consider the fuzzy topology in Example 1 and the γ function in Example 2. It is easy to see that the fuzzy set f defined as $f(x) = 0.3$ for all $x \in [0, 1]$, is a fuzzy *intγint*-open set, however, it is not a fuzzy open set. Also, the fuzzy set κ defined as $\kappa(x) = 0.7$ for all $x \in [0, 1]$, is a fuzzy *intγ*-open set and a fuzzy *γintγ*-open set. On the other hand, it is neither a fuzzy *γint*-open set nor a fuzzy *intγint*-open set.

(b) Let the fuzzy topology on $X = [0, 1]$ be $\tau = \{1_X, 1_\emptyset\}$ and consider again the γ function in Example 2. The fuzzy set f defined as $f(x) = 0.2$ for all $x \in [0, 1]$, is both a fuzzy *γint*-open set and fuzzy *γintγ*-open set. However, it is neither a fuzzy *intγ*-open set nor a fuzzy *intγint*-open set.

Theorem 2. *Let $\gamma \in \Gamma_{-2}$. Then, for $n \in \mathbb{N}$, $f \in L^X$ is*

(a) $(int\gamma)^n - open \Leftrightarrow int\gamma - open.$

(b) $(int\gamma)^n int - open \Leftrightarrow int\gamma int - open.$

(c) $(\gamma int)^n - open \Leftrightarrow \gamma int - open.$

(d) $(\gamma int)^n \gamma - open \Leftrightarrow \gamma int\gamma - open.$

(e) $int\gamma int - open \Leftrightarrow int\gamma$ *and* $\gamma int - open.$

(f) $\left\{ \begin{array}{l} int\gamma - open \\ \gamma int - open \end{array} \right\} \Rightarrow \gamma int\gamma - open \Rightarrow \gamma - open.$

Proof. **(a)**, **(b)**, **(c)** and **(d)** are clear from Proposition 6. **(e)** and **(f)** are obvious from Corollary 5. $\square$

Remark 10. If γ is considered as the closure operator of a fuzzy topological space of Chang type and Remark 1 is considered, then it can be seen in [7], [12] and [4] that the reverses of the implications of Theorem 2 do not hold and L-fuzzy *intγ*-open sets and *γint*-open sets are independent notions.

Theorem 3. *Let (X, τ) be an L-topological space and $\gamma \in \Gamma_{13}$. Then, the family $\mathcal{O}$ of all γ-open sets satisfies the following properties:*

(a) *If $f \in \tau$, then $f \in \mathcal{O}$.*

(b) *If $f_i \in \mathcal{O}$ for all $i \in J$, then $\bigvee_{i \in J} f_i \in \mathcal{O}$.*

(c) *If $g \in \tau$ and $f \in \mathcal{O}$, then $f \wedge g \in \mathcal{O}$.*

Conversely, if $\mathcal{O}$ is a subset of an L-topological space satisfying the properties from **(a)** *to* **(c)***, then there exits a function $\gamma \in \Gamma_{0123-}$ such that $\mathcal{O}$ consists of all fuzzy γ-open sets.*

Proof. **(a), (b),** and **(c)** are obvious from Corollary 6, Proposition 1 and Proposition 9, respectively. Conversely, let $\mathcal{O}$ be a subset of an L-topological space satisfying the properties from **(a)** to **(c)**. Define

$$\begin{array}{llll} \gamma: & L^X & \to & L^X \\ & f & \to & \gamma f = \bigvee\{g \in \mathcal{O} | g \leq f\}. \end{array}$$

It is clear to see that $\gamma \in \Gamma_{012-}$. Let $g \in \mathcal{O}$ and $f \in L^X$. Since $g \wedge \gamma f \in \mathcal{O}$ by ***(c)*** and $g \wedge \gamma f \leq g \wedge f$, we have $g \wedge \gamma f \leq \gamma(g \wedge f)$. Hence, $\gamma \in \Gamma_3$. Now, it is enough to show that $\mathcal{O}$ consists of all fuzzy γ-open sets. Let $f \in \mathcal{O}$. Then, we have $\gamma f = f$ implying that f is an L-fuzzy γ-open set. Conversely, if $f \leq \gamma f$, then $f = \gamma f \in \mathcal{O}$. $\square$

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