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ON $\widetilde{\Psi}_{\mathcal{G}}$-SETS IN GRILL TOPOLOGICAL SPACES

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Abstract

In this paper, we introduce and study $\widetilde{\Psi}_{\mathcal{G}}$-sets and utilize the $\Psi_{\mathcal{G}}$-operator to define interesting generalized open sets and study their properties.

1 Introduction

The idea of grills on a topological space was first introduced by Choquet [4]. The concept of grills has shown to be a powerful supporting and useful tool like nets and filters, for getting a deeper insight into further studying some topological notions such as proximity spaces, closure spaces and the theory of compactifications and extension problems of different kinds (see [3], [2], [18] for details). In [15], Roy and Mukherjee defined and studied a typical topology associated rather naturally to the existing topology and a grill on a given topological space.

The notion of ideal topological spaces was first studied by Kuratowski [8] and Vaidyanathaswamy [19]. Compatibility of the topology τ with an ideal $\mathcal{I}$ was first defined by Njåstad [13]. In 1990, Jankovic and Hamlett [7, 5] investigated further properties of ideal topological spaces and another operator called Ψ-operator defined as $\Psi(A) = X - (X - A)^*$. In 2007 Modak and Bandyopadhyay [11] used the Ψ-operator to define interesting generalized open sets.

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Definition 1. *[17] Let $(X, \tau, \mathcal{G})$ be a grill topological space. An operator $\Psi_{\mathcal{G}} : \mathcal{P}(X) \to \mathcal{P}(X)$ is defined as follows for every $A \in X$, $\Psi_{\mathcal{G}}(A) = \{x \in X :$ there exists a $U \in \tau(x)$ such that $U - A \notin \mathcal{G}\}$ and observes that $\Psi_{\mathcal{G}}(A) = X - \Phi(X - A)$ (we used the notion $\Psi_{\mathcal{G}}$ is instead of $\Gamma_{\mathcal{G}}$ which used in [17]).*

In this paper, we introduce and study $\widetilde{\Psi}_{\mathcal{G}}$-sets and utilize the $\Psi_{\mathcal{G}}$-operator to define interesting generalized open sets and study their properties.

2 Preliminaries

Let (X, τ) be a topological space with no separation properties assumed. For a subset A of a topological space (X, τ), $Cl(A)$ and $Int(A)$ denote the closure and the interior of A in (X, τ), respectively. The power set of X will be denoted by $\mathcal{P}(X)$. A subcollection $\mathcal{G}$ (not containing the empty set) of $\mathcal{P}(X)$ is called a grill [4] on X if $\mathcal{G}$ satisfies the following conditions:

1. $A \in \mathcal{G}$ and $A \subseteq B$ implies that $B \in \mathcal{G}$,

2. $A, B \subseteq X$ and $A \cup B \in \mathcal{G}$ implies that $A \in \mathcal{G}$ or $B \in \mathcal{G}$.

For any point x of a topological space (X, τ), $\tau(x)$ denotes the collection of all open neighborhoods of x.

Definition 2. *[15] Let (X, τ) be a topological space and $\mathcal{G}$ be a grill on X. A mapping $\Phi : \mathcal{P}(X) \to \mathcal{P}(X)$ is defined as follows: $\Phi(A) = \Phi_{\mathcal{G}}(A, \tau) = \{x \in X : A \cap U \in \mathcal{G}$ for all $U \in \tau(x)\}$ for each $A \in \mathcal{P}(X)$. The mapping Φ is called the operator associated with the grill $\mathcal{G}$ and the topology τ.*

Proposition 1. *[15] Let (X, τ) be a topological space and $\mathcal{G}$ be a grill on X. Then for all $A, B \subseteq X$:*

1. *$A \subseteq B$ implies that $\Phi(A) \subseteq \Phi(B)$,*

2. *$\Phi(A \cup B) = \Phi(A) \cup \Phi(B)$,*

3. *$\Phi(\Phi(A)) \subseteq \Phi(A) = Cl(\Phi(A)) \subseteq Cl(A)$.*

Let $\mathcal{G}$ be a grill on a space X. Then a mapping $\Psi : \mathcal{P}(X) \to \mathcal{P}(X)$ is defined by $\Psi(A) = A \cup \Phi(A)$ for all $A \in \mathcal{P}(X)$ [15]. The map Ψ is a Kuratowski closure axiom. Corresponding to a grill $\mathcal{G}$ on a topological space (X, τ), there exists a unique topology $\tau_{\mathcal{G}}$ on X given by $\tau_{\mathcal{G}} = \{U \subseteq X : \Psi(X - U) = X - U\}$, where for any $A \subseteq X$, $\Psi(A) = A \cup \Phi(A) = \tau_{\mathcal{G}}\text{-}Cl(A)$. For any grill $\mathcal{G}$ on a topological space (X, τ), $\tau \subseteq \tau_{\mathcal{G}}$. By $\tau_{\mathcal{G}}\text{-}Int(A)$, we denote the interior of A with respect to $\tau_{\mathcal{G}}$. If (X, τ) is a topological space with a grill $\mathcal{G}$ on X, then we call it a grill topological space and denote it by $(X, \tau, \mathcal{G})$.

Theorem 1. *[15] Let $(X, \tau, \mathcal{G})$ be a grill topological space. Then $\beta(\mathcal{G}, \tau) = \{V - G : V \in \tau, G \notin \mathcal{G}\}$ is an open base for $\tau_{\mathcal{G}}$.*

Corollary 1. *[15] Let $(X, \tau, \mathcal{G})$ be a grill topological space and suppose $A, B \subseteq X$ with $B \notin \mathcal{G}$. Then $\Phi(A \cup B) = \Phi(A) = \Phi(A - B)$.*

Several basic facts concerning the behavior of the operator $\Psi_{\mathcal{G}}$ are included in the following theorem.

Theorem 2. *[17]*

Let $(X, \tau, \mathcal{G})$ be a grill topological space. Then the following properties hold:

1. *If $A, B \in \mathcal{P}(X)$, then $\Psi_{\mathcal{G}}(A \cap B) = \Psi_{\mathcal{G}}(A) \cap \Psi_{\mathcal{G}}(B)$.*
2. *If $U \in \tau_{\mathcal{G}}$, then $U \subseteq \Psi_{\mathcal{G}}(U)$.*
3. *If $A \notin \mathcal{G}$, then $\Psi_{\mathcal{G}}(A) = X - \Phi(X)$.*
4. *If $A, B \subseteq X$ and $(A - B) \cup (B - A) \notin \mathcal{G}$, then $\Psi_{\mathcal{G}}(A) = \Psi_{\mathcal{G}}(B)$.*
5. *If $A \subseteq X$, then $\Psi_{\mathcal{G}}(A)$ is open in (X, τ).*
6. *If $A \subseteq B$, then $\Psi_{\mathcal{G}}(A) \subseteq \Psi_{\mathcal{G}}(B)$.*
7. *If $A \subseteq X$, then $\Psi_{\mathcal{G}}(A) \subseteq \Psi_{\mathcal{G}}(\Psi_{\mathcal{G}}(A))$.*
8. *If $A \subseteq X$, then $A \cap \Psi_{\mathcal{G}}(A) = \tau_{\mathcal{G}}\text{-}Int(A)$.*
9. *If $A \subseteq X$ and $G \notin \mathcal{G}$, then $\Psi_{\mathcal{G}}(A - G) = \Psi_{\mathcal{G}}(A)$.*
10. *If $A \subseteq X$ and $G \notin \mathcal{G}$, then $\Psi_{\mathcal{G}}(A \cup G) = \Psi_{\mathcal{G}}(A)$.*

3 $\widetilde{\Psi}_{\mathcal{G}}$-Sets

Definition 3. *Let $(X, \tau, \mathcal{G})$ be a grill topological space. A subset A of X is called a $\widetilde{\Psi}_{\mathcal{G}}$-set if $A \subseteq Cl(\Psi_{\mathcal{G}}(A))$.*

The collection of all $\widetilde{\Psi}_{\mathcal{G}}$-sets in $(X, \tau, \mathcal{G})$ is denoted by $\widetilde{\Psi}_{\mathcal{G}}(X, \tau)$.

Proposition 2. *Let $\{A_\alpha : \alpha \in \Delta\}$ be a collection of nonempty $\widetilde{\Psi}_{\mathcal{G}}$-sets in a grill topological space $(X, \tau, \mathcal{G})$, then $\cup_{\alpha\in\Delta} A_\alpha \in \widetilde{\Psi}_{\mathcal{G}}(X, \tau)$.*

Proof. For each $\alpha \in \Delta$,

$$A_\alpha \subseteq Cl(\Psi_{\mathcal{G}}(A_\alpha)) \subseteq Cl(\Psi_{\mathcal{G}}(\cup_{\alpha\in\Delta} A_\alpha))$$

This implies that

$$\cup_{\alpha\in\Delta} A_\alpha \subseteq Cl(\Psi_{\mathcal{G}}(\cup_{\alpha\in\Delta} A_\alpha))$$

Thus $\cup_{\alpha\in\Delta} A_\alpha \in \widetilde{\Psi}_{\mathcal{G}}(X, \mathcal{G})$. □

The following example shows that the intersection of two $\widetilde{\Psi}_{\mathcal{G}}$-sets in $(X, \tau, \mathcal{G})$ may not be a $\widetilde{\Psi}_{\mathcal{G}}$-set.

Example 1. *Let $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{a\}, \{b, c\}, \{a, b, c\}\}$ and the grill $\mathcal{G} = \{\{a\}, \{b\}, \{a, c\}, \{a, b\}, \{a, d\}, \{a, b, c\}, \{c, b, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c\}, \{b, d\}, \{b, c, d\}, X\}$. Then $A = \{a, d\}$ and $B = \{b, c, d\}$ are $\widetilde{\Psi}_{\mathcal{G}}$-sets but $A \cap B$ is not a $\widetilde{\Psi}_{\mathcal{G}}$-set. For $A = \{a, d\}$, $\Phi(X - A) = \{b, c, d\}$ and $\widetilde{\Psi}_{\mathcal{G}}(A) = \{a\}$. Hence $A \subseteq Cl(\widetilde{\Psi}_{\mathcal{G}}(A))$ implies that A is a $\widetilde{\Psi}_{\mathcal{G}}$-set. For $B = \{b, c, d\}$, $\Phi(X - B) = \{a, d\}$ and $\widetilde{\Psi}_{\mathcal{G}}(B) = \{b, c\}$. Hence $B \subseteq Cl(\widetilde{\Psi}_{\mathcal{G}}(B))$ implies that B is a $\widetilde{\Psi}_{\mathcal{G}}$-set. On the other hand, since $A \cap B = \{d\}$, $\Phi(X - (A \cap B)) = X$ and $\widetilde{\Psi}_{\mathcal{G}}(B) = \phi$. Hence $A \cap B \nsubseteq Cl(\widetilde{\Psi}_{\mathcal{G}}(A \cap B))$ implies that $A \cap B$ is not a $\widetilde{\Psi}_{\mathcal{G}}$-set.*

Recall that a subset A of X in a topological space (X, τ) is called an α-set if $A \subseteq Int(Cl(Int(A)))$. The collection of α-sets in (X, τ) is denoted by τ^α. Njåstad [12] has shown that τ^α forms a topology. Although the intersection of two $\widetilde{\Psi}_{\mathcal{G}}$-sets need not be a $\widetilde{\Psi}_{\mathcal{G}}$-set but we shall prove that intersection of an α-set with a $\widetilde{\Psi}_{\mathcal{G}}$-set is a $\widetilde{\Psi}_{\mathcal{G}}$-set.

Corollary 2. *Let $(X, \tau, \mathcal{G})$ be a grill topological space. Then $U \subseteq \Psi_{\mathcal{G}}(U)$ for every open set $U \in \tau$.*

Proof. Since $\tau \subset \tau_{\mathcal{G}}$, the proof follows easily from Theorem 2. $\square$

Theorem 3. *Let $(X, \tau, \mathcal{G})$ be a grill topological space and $A \in \widetilde{\Psi}_{\mathcal{G}}(X, \tau)$. If $U \in \tau^{\alpha}$, then $U \cap A \in \widetilde{\Psi}_{\mathcal{G}}(X, \tau)$.*

Proof. We note that if G is open, for any $A \subseteq X$, $G \cap Cl(A) \subseteq Cl(G \cap A)$. Let $U \in \tau^{\alpha}$ and $A \in \widetilde{\Psi}_{\mathcal{G}}(X, \mathcal{G})$. Then by Theorem 2 and Corollary 2 we have

$$\begin{aligned} U \cap A &\subseteq Int(Cl(Int(U))) \cap Cl(\Psi_{\mathcal{G}}(A)) \\ &\subseteq Int(Cl(\Psi_{\mathcal{G}}(U))) \cap Cl(\Psi_{\mathcal{G}}(A)) \\ &\subseteq Cl[Int(Cl(\Psi_{\mathcal{G}}(U))) \cap \Psi_{\mathcal{G}}(A)] \\ &= Cl[Int(Cl[\Psi_{\mathcal{G}}(U) \cap \Psi_{\mathcal{G}}(A)])] \\ &= Cl[\Psi_{\mathcal{G}}(U) \cap \Psi_{\mathcal{G}}(A)] \\ &= Cl[\Psi_{\mathcal{G}}(U \cap A)]. \end{aligned}$$

Hence $U \cap A \in \widetilde{\Psi}_{\mathcal{G}}(X, \tau)$. $\square$

Corollary 3. *Let $(X, \tau, \mathcal{G})$ be a grill topological space and $A \in \widetilde{\Psi}_{\mathcal{G}}(X, \tau)$. If $U \in \tau$, then $U \cap A \in \widetilde{\Psi}_{\mathcal{G}}(X, \tau)$.*

A set D is called a relatively $\mathcal{G}$-dense in a set A if for every relatively nonempty open set $U \cap A$, $U \in \tau$, it is true that $(U \cap A) \cap D \in \mathcal{G}$. We now provide a necessary and sufficient condition for $A \notin \widetilde{\Psi}_{\mathcal{G}}(X, \tau)$.

Theorem 4. *A set A dose not belong to $\widetilde{\Psi}_{\mathcal{G}}(X, \tau)$ if and only if there exists $x \in A$ such that there is a neighborhood $V_x \in \tau$ of x for which $X - A$ is relatively $\mathcal{G}$-dense in V_x.*

Proof. Let $A \notin \widetilde{\Psi}_{\mathcal{G}}(X, \tau)$. We are to show that there exists an element $x \in A$ and a neighborhood $V_x \in \tau(x)$ satisfying that $(X - A)$ is relatively $\mathcal{G}$-dense in V_x. Since $A \nsubseteq Cl(\Psi_{\mathcal{G}}(A))$, there exists $x \in X$ such that $x \in A$ but $x \notin Cl(\Psi_{\mathcal{G}}(A))$. Hence there exists a neighborhood $V_x \in \tau(x)$ such that $V_x \cap \Psi_{\mathcal{G}}(A) = \phi$. This implies that $V_x \cap (X - \Phi(X - A)) = \phi$ and hence $V_x \subseteq \Phi(X - A)$. Let U be any nonempty open set in V_x. Since $V_x \subseteq \Phi(X - A)$, therefore $U \cap (X - A) \in \mathcal{G}$. This implies that $(X - A)$ is relatively $\mathcal{G}$-dense in V_x. Converse part follows by reversing the argument. $\square$

Recall that a subset $A \subseteq X$ is said to be preopen [10] (resp. Φ-open [6]) if $A \subseteq Int(Cl(A))$ (resp. $A \subseteq Int(\Phi(A))$). The collection of all peropen (resp. Φ-open) sets in a topological space (X, τ) is denoted by $PO(X, \tau)$ (resp. $\Phi O(X, \tau)$).

Definition 4. *Let $(X, \tau, \mathcal{G})$ be a grill topological space. A grill $\mathcal{G}$ is said to be anti-codense grill if $\tau - \{\phi\} \subseteq \mathcal{G}$.*

Theorem 5. *[16] Let $(X, \tau, \mathcal{G})$ be a grill topological space, where $\mathcal{G}$ is an anti-codense grill. Then $\Phi O(X, \tau) = PO(X, \tau_{\mathcal{G}})$.*

Recall that a subset A of X in a topological space (X, τ) is called a semi-open set [9] if $A \subseteq Cl(Int(A))$. The collection of semi-open sets in (X, τ) is denoted by $SO(X, \tau)$. We prove $SO(X, \tau_{\mathcal{G}}) = \widetilde{\Psi}_{\mathcal{G}}(X, \tau)$.

Theorem 6. *Let $(X, \tau, \mathcal{G})$ be a grill topological space, where $\mathcal{G}$ is an anti-codense grill. Then $SO(X, \tau_{\mathcal{G}}) = \widetilde{\Psi}_{\mathcal{G}}(X, \tau)$.*

Proof. Let $A \in SO(X, \tau_{\mathcal{G}})$. Therefore by Theorem 2, $A \subseteq \tau_{\mathcal{G}}$-$Cl(\tau_{\mathcal{G}}$-$Int(A)) = \tau_{\mathcal{G}}$-$Cl(\Psi_{\mathcal{G}}(A) \cap A) \subseteq Cl(\Psi_{\mathcal{G}}(A) \cap A) \subseteq Cl(\Psi_{\mathcal{G}}(A))$ and hence $A \in \widetilde{\Psi}_{\mathcal{G}}(X, \tau)$. Therefore, $SO(X, \tau_{\mathcal{G}}) \subseteq \widetilde{\Psi}_{\mathcal{G}}(X, \tau)$.

Conversely, let $A \in \widetilde{\Psi}_{\mathcal{G}}(X, \tau)$ and $x \in A$. Consider a basic neighbourhood U_1 of x in $(X, \tau_{\mathcal{G}})$. Then U_1 is of the form $U - G$ where $U \in \tau$ and $G \notin \mathcal{G}$. This implies that $x \in U$. Since $A \subseteq Cl(\Psi_{\mathcal{G}}(A))$ and $U \in \tau(x)$, $U \cap \Psi_{\mathcal{G}}(A) \neq \phi$. Let $y \in U \cap \Psi_{\mathcal{G}}(A)$. Therefore there exists a neighbourhood W_y of y such that $W_y - A \notin \mathcal{G}$ (by definition of $\Psi_{\mathcal{G}}(A)$). Now we consider $U \cap W_y = V$, let $G_1 = V - A \notin \mathcal{G}$ (by heredity), then $V \neq \phi$, $V \in \tau$ and $V - G_1 \subseteq A$. Also $V \subseteq U$. Thus $M = V - (G_1 \cup G) \subseteq A$ (note that $M = V - (G_1 \cup G) \neq \phi$ since $\mathcal{G}$ is an anti-codense grill) and $M \subseteq A \cap (U - G)$. Hence A contains a nonempty $\tau_{\mathcal{G}}$-open set M contained in $U - G$. Since x is an arbitrary point of A, we get $A \subseteq \tau_{\mathcal{G}}$-$Cl(\tau_{\mathcal{G}}$-$Int(A))$ and hence $A \in SO(X, \tau_{\mathcal{G}})$. Therefore, $\widetilde{\Psi}_{\mathcal{G}}(X, \tau) \subseteq SO(X, \tau_{\mathcal{G}})$. Hence $SO(X, \tau_{\mathcal{G}}) = \widetilde{\Psi}_{\mathcal{G}}(X, \tau)$. $\square$

Definition 5. *Let $(X, \tau, \mathcal{G})$ be a grill topological space and $A \subseteq X$, A is called a $\Psi_{\mathcal{A}}$-set if $A \subseteq Int(Cl(\Psi_{\mathcal{G}}(A)))$.*

The collection of all $\Psi_{\mathcal{A}}$-sets in $(X, \tau, \mathcal{G})$ is denoted by $\tau^{\mathcal{A}}$. From Definitions 3 and 5 it follows that $\tau^{\mathcal{A}} \subseteq \widetilde{\Psi}_{\mathcal{G}}(X, \tau)$. We show that the collection $\tau^{\mathcal{A}}$ forms a topology.

Theorem 7. *Let $(X, \tau, \mathcal{G})$ be a grill topological space, where $\mathcal{G}$ is an anti-codense grill. Then the collection $\tau^{\mathcal{A}} = \{A \subseteq X : A \subseteq Int(Cl(\Psi_{\mathcal{G}}(A)))\}$ forms a topology on X.*

Proof. (1) It is observed that $\phi \subseteq Int(Cl(\Psi_{\mathcal{G}}(\phi)))$ and $X \subseteq Int(Cl(\Psi_{\mathcal{G}}(X)))$, and thus ϕ and $X \in \tau^{\mathcal{A}}$.
(2) Let $\{A_\alpha : \alpha \in \Delta\} \subseteq \tau^{\mathcal{A}}$, then $\Psi_{\mathcal{G}}(A_\alpha) \subseteq \Psi_{\mathcal{G}}(\cup A_\alpha)$ for every $\alpha \in \Delta$. Thus $A_\alpha \subseteq Int(Cl(\Psi_{\mathcal{G}}(A_\alpha))) \subseteq Int(Cl(\Psi_{\mathcal{G}}(\cup A_\alpha)))$ for every $\alpha \in \Delta$, which implies that $\cup A_\alpha \subseteq Int(Cl(\Psi_{\mathcal{G}}(\cup A_\alpha)))$. Therefore, $\cup A_\alpha \in \tau^{\mathcal{A}}$.
(3) Let $A, B \in \tau^{\mathcal{A}}$. Since $\Psi_{\mathcal{G}}(A)$ is open in (X, τ), by using Theorem 2 and Lemma 3.5 in [14] we obtain $A \cap B \subseteq Int(Cl(\Psi_{\mathcal{G}}(A))) \cap Int(Cl(\Psi_{\mathcal{G}}(A))) = Int(Cl(\Psi_{\mathcal{G}}(A) \cap \Psi_{\mathcal{G}}(B))) = Int(Cl(\Psi_{\mathcal{G}}(A \cap B)))$. Therefore, $A \cap B \subseteq Int(Cl(\Psi_{\mathcal{G}}(A \cap B)))$ and $A \cap B \in \tau^{\mathcal{A}}$. □

Proposition 3. *[1] Let $(X, \tau, \mathcal{G})$ be a grill topological space. Then $\Psi_{\mathcal{G}}(A) \neq \phi$ if and only if A contains a nonempty $\tau_{\mathcal{G}}$-interior.*

Corollary 4. *Let $(X, \tau, \mathcal{G})$ be a grill topological space. Then $\{x\} \in \widetilde{\Psi}_{\mathcal{G}}(X, \tau)$ if and only $\{x\} \in \tau^{\mathcal{A}}$*

Proof. Let $\{x\} \in \widetilde{\Psi}_{\mathcal{G}}(X, \tau)$, therefore $\{x\}$ is open in $(X, \tau_{\mathcal{G}})$ by Proposition 3. Since $\{x\} \subseteq \Psi_{\mathcal{G}}(\{x\})$ and $\Psi_{\mathcal{G}}(\{x\})$ is open in (X, τ), therefore $\{x\} \subseteq Int(Cl(\Psi_{\mathcal{G}}(\{x\})$. Hence $\{x\} \in \tau^{\mathcal{A}}$.

Conversely suppose that $\{x\} \in \tau^{\mathcal{A}}$. Then $\{x\} \subseteq Int(Cl(\Psi_{\mathcal{G}}(\{x\})))$ and $\{x\} \subseteq Cl(\Psi_{\mathcal{G}}(\{x\}))$. Hence $\{x\} \in \widetilde{\Psi}_{\mathcal{G}}(X, \tau)$. □

Theorem 8. *Let $(X, \tau, \mathcal{G})$ be a grill topological space. Then $\tau^{\mathcal{A}}$ consists of exactly those sets A for which $A \cap B \in \widetilde{\Psi}_{\mathcal{G}}(X, \tau)$ for all $B \in \widetilde{\Psi}_{\mathcal{G}}(X, \tau)$.*

Proof. Let $A \in \tau^{\mathcal{A}}$ and $B \in \widetilde{\Psi}_{\mathcal{G}}(X, \tau)$. Now we show that $A \cap B \in \widetilde{\Psi}_{\mathcal{G}}(X, \tau)$. If $A \cap B = \phi$, we are done. Let $A \cap B \neq \phi$ and $x \in A \cap B$, then $x \in Int(Cl(\Psi_{\mathcal{G}}(A)))$, since $A \in \tau^{\mathcal{A}}$. Consider any neighbourhood U_x of x then $U_x \cap Int(Cl(\Psi_{\mathcal{G}}(A)))$ is a neighbourhood of x. Since $x \in B \subseteq Cl(\Psi_{\mathcal{G}}(B))$, then $U_x \cap Int(Cl(\Psi_{\mathcal{G}}(A))) \cap \Psi_{\mathcal{G}}(B) \neq \phi$. Let $V = U_x \cap Int(Cl(\Psi_{\mathcal{G}}(A))) \cap \Psi_{\mathcal{G}}(B)$, then $V \subseteq Cl(\Psi_{\mathcal{G}}(A))$. This implies that $U_x \cap \Psi_{\mathcal{G}}(A) \cap \Psi_{\mathcal{G}}(B) = V \cap \Psi_{\mathcal{G}}(A) \neq \phi$, since $\Psi_{\mathcal{G}}(B)$ is open. Therefore $x \in Cl[\Psi_{\mathcal{G}}(A) \cap \Psi_{\mathcal{G}}(B)] = Cl[\Psi_{\mathcal{G}}(A \cap B)]$. Hence $A \cap B \subseteq Cl[\Psi_{\mathcal{G}}(A \cap B)]$, therefore

$A \cap B \in \widetilde{\Psi}_{\mathcal{G}}(X, \tau)$.

Next we consider a subset A of X such that $A \cap B \in \widetilde{\Psi}_{\mathcal{G}}(X, \tau)$ for each $B \in \widetilde{\Psi}_{\mathcal{G}}(X, \tau)$. We show that $A \in \tau^{\mathcal{A}}$. Suppose $A \nsubseteq Int(Cl(\Psi_{\mathcal{G}}(A)))$ then there exists $x \in A$ but $x \notin Int(Cl(\Psi_{\mathcal{G}}(A)))$. Therefore, $x \in A \cap [X - Int(Cl(\Psi_{\mathcal{G}}(A)))] = A \cap Cl[X - Cl(\Psi_{\mathcal{G}}(A))] = A \cap Cl(H)$, where $H = X - Cl(\Psi_{\mathcal{G}}(A))$. It is obvious that H is a nonempty open set. Since $x \in Cl(H)$, then for all open set V_x containing x, $V_x \cap H \neq \phi$. Therefore $V_x \cap \Psi_{\mathcal{G}}(H) \neq \phi$, since $H \subseteq \Psi_{\mathcal{G}}(H)$. This implies that $x \in Cl(\Psi_{\mathcal{G}}(H)) \subseteq Cl(\Psi_{\mathcal{G}}[H \cup \{x\}])$ and $H \subseteq Cl(\Psi_{\mathcal{G}}(H)) \subseteq Cl(\Psi_{\mathcal{G}}[H \cup \{x\}])$ and hence $\{x\} \cup H \subseteq Cl(\Psi_{\mathcal{G}}[H \cup \{x\}])$. Therefore $\{x\} \cup H \in \widetilde{\Psi}_{\mathcal{G}}(X, \tau)$. Now by hypothesis $A \cap [\{x\} \cup H] \in \widetilde{\Psi}_{\mathcal{G}}(X, \tau)$, since $[\{x\} \cup H] \in \widetilde{\Psi}_{\mathcal{G}}(X, \tau)$. We show that $A \cap [\{x\} \cup H] = \{x\}$. Suppose that there exists $y \in X$ and $x \neq y$ such that $y \in A \cap [\{x\} \cup H]$, then $y \in A$ and $y \in H$. Now $A = A \cap X$ and $X \in \widetilde{\Psi}_{\mathcal{G}}(X, \tau)$, hence by hypothesis $A \in \widetilde{\Psi}_{\mathcal{G}}(X, \tau)$. Since $y \in A \subseteq Cl(\Psi_{\mathcal{G}}(A)$, this is contrary to the fact that $y \in H = X - Cl(\Psi_{\mathcal{G}}(A))$. Thus $A \cap [\{x\} \cup H] = \{x\}$. Since $\{x\} \in \widetilde{\Psi}_{\mathcal{G}}(X, \tau)$, then $\{x\} \in \tau^{\mathcal{A}}$ (by Corollary 4). Hence, $\{x\} \subseteq Int(Cl(\Psi_{\mathcal{G}}(\{x\}) = Int(Cl(\Psi_{\mathcal{G}}(A \cap [\{x\} \cup H]) \subseteq Int(Cl(\Psi_{\mathcal{G}}(A)))$. But $x \in Int(Cl(\Psi_{\mathcal{G}}(A)))$ which is contrary to the fact that $x \notin Int(Cl(\Psi_{\mathcal{G}}(A)))$. Therefore we have $A \subseteq Int(Cl(\Psi_{\mathcal{G}}(A)))$ and hence $A \in \tau^{\mathcal{A}}$. □

Now we refer to the following well known theorem.

Theorem 9. *[12] Let (X, τ) be a topological space. Then τ^{α} consists of exactly those sets A for which $A \cap B \in SO(X, \tau)$ for all $B \in SO(X, \tau)$.*

Now from Theorems 6, 8 and 9 we have the following theorem.

Theorem 10. *Let $(X, \tau, \mathcal{G})$ be a grill topological space, where $\mathcal{G}$ is an anti-codense grill. Then $\tau^{\mathcal{A}} = (\tau_{\mathcal{G}})^{\alpha}$.*

References

[1] A. Al-Omari, T. Noiri, *On $\Psi_{\mathcal{G}}$-operator in grill topological spaces*, submitted.

[2] K.C. Chattopadhyay, W.J. Thron, *Extensions of closure spaces*, Can. J. Math. 29 (1977), 1277-1286.

[3] K.C. Chattopadhyay, O. Njåstad, W.J. Thron, *Merotopic spaces and extensions of closure spaces*, Can. J. Math. 35 (1983), 613-629.

[4] G. Choqet, *Sur les notions de filter et grill*, Comptes Rendus Acad. Sci. Paris 224 (1947), 171-173.

[5] T.R. Hamlett, D. Janković, *Ideals in topological spaces and the set operator* Ψ, Boll. Un. Mat. Ital. 7 4-B (1990), 863-874.

[6] E. Hatir, S. Jafari, *On some new calsses of sets and a new decomposition of continuity via grills*, J. Adv. Math. Studies. 3 (2010), 33-40.

[7] D. Janković, T.R. Hamlett, *New topologies from old via ideals*, Amer. Math. Monthly 97 (1990), 295-310.

[8] K. Kuratowski, *Topology* I, Warszawa, 1933.

[9] N. Levine, *Semi-open sets and semi-continuity in topological spaces*, Amer Math. Monthly 70 (1963), 36-41.

[10] A.S. Mashhour, M.E. Abd El-Monsef, S.N. El-Deep, *On precontinuous and weak precontinuous mappings*, Proc. Math. Phys. Soc. Egypt 53 (1982), 47-53.

[11] S. Modak, C. Bandyopadyay, *A note on* Ψ*-operator*, Bull. Malays. Math. Sci. Soc. (2) 30 (1) (2007), 43-48.

[12] O. Njåstad, *On some classes of nearly open sets*, Pacific J. Math. 15 (1965), 961-970.

[13] O. Njåstad, *Remark on topologies defined by local properties*, Avh. Norske Vid. Akad. Oslo I (N. S) 8 (1966), 1-16.

[14] T. Noiri, *On* α*-continuous functions*, Časopis Pěat. Mat. 109 (1984), 118-126.

[15] B. Roy, M.N. Mukherjee, *On a typical topology induced by a grill*, Soochow J. Math. 33 (2007), 771-786.

[16] B. Roy, M.N. Mukherjee, S.K. Ghosh, On a subclass of preopen sets via grills, Univ. Dun Bacau Studii Si Stiintifice 18 (2008), 255-266.

[17] B. Roy, M.N. Mukherjee, S.K. Ghosh, *On a new operator based on a grill and its associated topology*, Arab Jour. Math Sc. 14 (2008), 21-32.

[18] W.J. Thron, *Proximity structure and grills*, Math. Ann. 206 (1973), 35-62.

[19] R. Vaidyanathaswamy, *Set Topology*, Chelsea Publishing Company, 1960.

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