ON SOME NEW SEQUENCE SPACES OF NON-ABSOLUTE TYPE RELATED TO THE SPACES $\ell_p$ AND $\ell_\infty$ I

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Abstract

In the present paper, we introduce the sequence space $\ell_\lambda^p$ of non-absolute type and prove that the spaces $\ell_\lambda^p$ and $\ell_p$ are linearly isomorphic for $0 < p \leq \infty$. Further, we show that $\ell_\lambda^p$ is a $p$-normed space and a BK-space in the cases of $0 < p < 1$ and $1 \leq p < \infty$, respectively. Furthermore, we derive some inclusion relations concerning the space $\ell_\lambda^p$. Finally, we construct the basis for the space $\ell_\lambda^p$, where $1 \leq p < \infty$.

1 Introduction

By $w$, we denote the space of all real or complex valued sequences. Any vector subspace of $w$ is called a sequence space.

A sequence space $X$ with a linear topology is called a $K$-space provided each of the maps $p_n : X \to \mathbb{C}$ defined by $p_n(x) = x_n$ is continuous for all $n \in \mathbb{N}$, where $\mathbb{C}$ denotes the complex field and $\mathbb{N} = \{0, 1, 2, \ldots\}$. A $K$-space $X$ is called an FK-space provided $X$ is a complete linear metric space. An FK-space whose topology is normable is called a BK-space [9, pp.272-273].

We shall write $\ell_\infty$, $c$ and $c_0$ for the sequence spaces of all bounded, convergent and null sequences, respectively, which are BK-spaces with the same sup-norm given by

$$\|x\|_{\ell_\infty} = \sup_k |x_k|,$$

where, here and in the sequel, the supremum $\sup_k$ is taken over all $k \in \mathbb{N}$. Also by $\ell_p$ (0 < $p < \infty$), we denote the sequence space of all sequences associated with $p$-absolutely convergent series. It is known that $\ell_p$ is a complete $p$-normed space and a BK-space in the cases of $0 < p < 1$ and $1 \leq p < \infty$ with respect to the usual $p$-norm and $\ell_p$-norm defined by

$$\|x\|_{\ell_p} = \sum_k |x_k|^p; \quad (0 < p < 1)$$

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and
\[ \|x\|_{\ell_p} = \left( \sum_k |x_k|^p \right)^{1/p}; \quad (1 \leq p < \infty), \]
respectively (see [11, pp.217-218]). For simplicity in notation, here and in what follows, the summation without limits runs from 0 to \( \infty \).

Let \( X \) and \( Y \) be sequence spaces and \( A = (a_{nk}) \) be an infinite matrix of real or complex numbers \( a_{nk} \), where \( n, k \in \mathbb{N} \). Then, we say that \( A \) defines a matrix mapping from \( X \) into \( Y \) if for every sequence \( x = (x_k) \in X \) the sequence \( Ax = \{A_n(x)\} \), the A-transform of \( x \), exists and is in \( Y \), where
\[ A_n(x) = \sum_k a_{nk}x_k; \quad (n \in \mathbb{N}). \tag{1} \]

By \( (X : Y) \), we denote the class of all infinite matrices that map \( X \) into \( Y \). Thus \( A \in (X : Y) \) if and only if the series on the right side of (1) converges for each \( n \in \mathbb{N} \) and every \( x \in X \), and \( Ax \in Y \) for all \( x \in X \).

For a sequence space \( X \), the matrix domain of an infinite matrix \( A \) in \( X \) is defined by
\[ X_A = \{ x \in w : Ax \in X \} \tag{2} \]
which is a sequence space.

We shall write \( c^{(k)} \) for the sequence whose only non-zero term is a 1 in the \( k^{th} \) place for each \( k \in \mathbb{N} \).

The approach of constructing a new sequence space by means of the matrix domain of a particular limitation method has recently been employed by several authors, e.g., Wang [19], Ng and Lee [18], Malkowsky [12], Başar and Altay [7], Malkowsky and Savaş [13], Aydın and Başar [3, 4, 5, 6], Altay and Başar [1], Altay, Başar and Mursaleen [2, 14] and Mursaleen and Noman [15, 16], respectively. They introduced the sequence spaces \( (\ell_\infty)_C \) and \( c_{N_0} \) in [19], \( (\ell_\infty)_C = X_{\infty} \) and \( (\ell_\infty)_C = X_{\infty} \) in [18], \( (\ell_\infty)_R = r_{\infty} \), \( c_{R} = r_{\infty} \) and \( \{c_0\} = r_{\infty} \) in [12], \( (\ell_\infty)_\Delta = b_e \) in [7], \( \mu_G = Z(u,v,\mu) \) in [13], \( \{c_0\}_A = a_0^c \) and \( \{c\}_A = a_0^c \) in [3], \( \{c_0\} = a_0^c \) in [4], \( \{c_0\} = a_0^c \) and \( \{c\} = a_0^c \) in [5], \( \{c_0\} = a_0^c \) and \( \{c\} = a_0^c \) in [6], \( \{c_0\} = c_0^c \) and \( \{c\} = c_0^c \) in [1], \( \{c_0\} = c_0^c \) and \( \{c\} = c_0^c \) in [2, 14], \( \{c_0\} = c_0^c \) and \( \{c\} = c_0^c \) in [4] and \( \{c_0\} = c_0^c \) in [5] and \( \{c_0\} = c_0^c \) in [15] and \( \{c_0\} = c_0^c \) in [6], \( \{c_0\} = c_0^c \) in [15] and \( \{c_0\} = c_0^c \) in [6]. For simplicity in notation, here and in what follows, the summation without limits runs from 0 to \( \infty \).
2 \(\lambda\)-boundedness and \(p\)-absolute convergence of type \(\lambda\)

Throughout this paper, let \(\lambda = (\lambda_k)_{k=0}^{\infty}\) be a strictly increasing sequence of positive reals tending to \(\infty\), that is

\[0 < \lambda_0 < \lambda_1 < \cdots \text{ and } \lambda_k \to \infty \text{ as } k \to \infty.\] (3)

We say that a sequence \(x = (x_k) \in w\) is \(\lambda\)-bounded if \(\sup_n |\Lambda_n(x)| < \infty\), where

\[\Lambda_n(x) = \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1})x_k; \quad (n \in \mathbb{N}).\] (4)

Also, we say that the associated series \(\sum_k x_k\) is \(p\)-absolutely convergent of type \(\lambda\) if \(\sum_n |\Lambda_n(x)|^p < \infty\), where \(0 < p < \infty\).

Here and in the sequel, we shall use the convention that any term with a negative subscript is equal to zero, e.g., \(\lambda_{-1} = 0\) and \(x_{-1} = 0\).

Now, let \(x = (x_k)\) be a bounded sequence in the ordinary sense of boundedness, i.e., \(x \in \ell_\infty\). Then, there is a constant \(M > 0\) such that \(|x_k| \leq M\) for all \(k \in \mathbb{N}\). Thus, we have for every \(n \in \mathbb{N}\) that

\[|\Lambda_n(x)| \leq \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1})|x_k| \leq \frac{M}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) = M\]

which shows that \(x\) is \(\lambda\)-bounded. Therefore, we deduce that the ordinary boundedness implies the \(\lambda\)-boundedness. This leads us to the following basic result:

**Lemma 2.1.** Every bounded sequence is \(\lambda\)-bounded.

We shall later show that the converse implication need not be true. Further, we shall show that for every \(0 < p < \infty\) there is a sequence \(\lambda = (\lambda_k)\) satisfying (3) such that the convergence of the series \(\sum_n |\Lambda_n(x)|^p\) does not imply the convergence of the series \(\sum_n |\Lambda_n(x)|^p\), and conversely. Before that, we define the infinite matrix \(\Lambda = (\lambda_{nk})_{n,k=0}^{\infty}\) by

\[\lambda_{nk} = \begin{cases} \frac{\lambda_k - \lambda_{k-1}}{\lambda_n}; & (0 \leq k \leq n), \\ 0; & (k > n) \end{cases}\] (5)

for all \(n, k \in \mathbb{N}\). Then, for any sequence \(x = (x_k) \in w\), the \(\Lambda\)-transform of \(x\) is the sequence \(\Lambda(x) = \{\Lambda_n(x)\}\), where \(\Lambda_n(x)\) is given by (4) for all \(n \in \mathbb{N}\). Therefore, the sequence \(x\) is \(\lambda\)-bounded if and only if \(\Lambda(x) \in \ell_\infty\). Also, the notion of \(p\)-absolute convergence of type \(\lambda\) of the sequence \(x\) is equivalent to say that \(\Lambda(x) \in \ell_p\), where \(0 < p < \infty\). Further, it is obvious by (5) that the matrix \(\Lambda = (\lambda_{nk})\) is a triangle, i.e., \(\lambda_{nn} \neq 0\) and \(\lambda_{nk} = 0\) for all \(k > n\) \((n \in \mathbb{N})\).
Recently, the sequence spaces $c^\lambda_0$ and $c^\lambda$ have been defined in [15] as the matrix domains of the triangle $\Lambda$ in the spaces $c_0$ and $c$, respectively, that is

$$c^\lambda_0 = \left\{ x = (x_k) \in w : \lim_{n \to \infty} \left( \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) x_k \right) = 0 \right\}$$

and

$$c^\lambda = \left\{ x = (x_k) \in w : \lim_{n \to \infty} \left( \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) x_k \right) \text{ exists} \right\}.$$

Also, it has been shown that the inclusions $c_0 \subset c^\lambda_0$ and $c \subset c^\lambda$ hold and the inclusion $c^\lambda_0 \subset c^\lambda$ strictly holds.

Finally, we define the sequence $y(\lambda) = \{y_k(\lambda)\}$, which will be frequently used, as the $\Lambda$-transform of a sequence $x = (x_k)$, i.e., $y(\lambda) = \Lambda(x)$ and so we have

$$y_k(\lambda) = \sum_{j=0}^{k} \left( \frac{\lambda_j - \lambda_{j-1}}{\lambda_k} \right) x_j; \ (k \in \mathbb{N}). \quad (6)$$

3 The sequence spaces $\ell^\lambda_p$ and $\ell^\lambda_\infty$ of non-absolute type

In the present section, as a natural continuation of Mursaleen and Noman [15], we introduce the sequence spaces $\ell^\lambda_p$ and $\ell^\lambda_\infty$, as the sets of all sequences whose $\Lambda$-transforms are in the spaces $\ell_p$ and $\ell_\infty$, respectively, where $0 < p < \infty$, that is

$$\ell^\lambda_p = \left\{ x = (x_k) \in w : \sum_{n=0}^{\infty} \left| \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) x_k \right|^p < \infty \right\}; \ (0 < p < \infty)$$

and

$$\ell^\lambda_\infty = \left\{ x = (x_k) \in w : \sup_{n} \left| \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) x_k \right| < \infty \right\}.$$

With the notation of (2), we can redefine the spaces $\ell^\lambda_p$ and $\ell^\lambda_\infty$ as follows:

$$\ell^\lambda_p = (\ell_p)_\Lambda \quad (0 < p < \infty) \quad \text{and} \quad \ell^\lambda_\infty = (\ell_\infty)_\Lambda. \quad (7)$$

Then, it is obvious by (7) that $\ell^\lambda_\infty$ and $\ell^\lambda_p \ (0 < p < \infty)$ are sequence spaces consisting of all sequences which are $\lambda$-bounded and $p$-absolutely convergent of type $\lambda$, respectively. Further, we have the following result which is essential in the text.

**Theorem 3.1.** We have the following:

(a) If $0 < p < 1$, then $\ell^\lambda_p$ is a complete $p$-normed space with the $p$-norm $\|x\|_{\ell^\lambda_p} = \|\Lambda(x)\|_{\ell_p}$, that is

$$\|x\|_{\ell^\lambda_p} = \sum_{n} |\Lambda_n(x)|^p; \quad (0 < p < 1). \quad (8)$$
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(b) If $1 \leq p \leq \infty$, then $\ell_p^\Lambda$ is a BK-space with the norm $\|x\|_{\ell_p^\Lambda} = \|\Lambda(x)\|_{\ell_p}$, that is

$$\|x\|_{\ell_p^\Lambda} = \left(\sum_n |\Lambda_n(x)|^p\right)^{1/p}; \quad (1 \leq p < \infty) \quad (9)$$

and

$$\|x\|_{\ell_\infty^\Lambda} = \sup_n |\Lambda_n(x)|. \quad (10)$$

Proof. Since the matrix $\Lambda$ is a triangle, this result is immediate by (7) and Theorem 4.3.12 of Wilansky [20, p.63]. □

Remark 3.2. One can easily check that the absolute property does not hold on the space $\ell_p^\Lambda$, that is $\|x\|_{\ell_p^\Lambda} \neq \||x||\|_{\ell_p^\Lambda}$ for at least one sequence in the space $\ell_p^\Lambda$, and this tells us that $\ell_p^\Lambda$ is a sequence space of non-absolute type, where $|x| = (|x_k|)$ and $0 < p \leq \infty$.

Theorem 3.3. The sequence space $\ell_p^\Lambda$ of non-absolute type is isometrically isomorphic to the space $\ell_p$, that is $\ell_p^\Lambda \cong \ell_p$ for $0 < p \leq \infty$.

Proof. To prove this, we should show the existence of an isometric isomorphism between the spaces $\ell_p^\Lambda$ and $\ell_p$, where $0 < p \leq \infty$. For, let $0 < p \leq \infty$ and consider the transformation $T$ defined, with the notation of (6), from $\ell_p^\Lambda$ to $\ell_p$ by $x \mapsto y(\lambda) = Tx$. Then, we have $Tx = y(\lambda) = \Lambda(x) \in \ell_p$ for every $x \in \ell_p^\Lambda$. Also, the linearity of $T$ is trivial. Further, it is easy to see that $x = 0$ whenever $Tx = 0$ and hence $T$ is injective.

Furthermore, let $y = (y_k) \in \ell_p$ be given and define the sequence $x = \{x_k(\lambda)\}$ by

$$x_k(\lambda) = \sum_{j=k-1}^{k} (-1)^{k-j} \frac{\lambda_j}{\lambda_k - \lambda_{k-1}} y_j; \quad (k \in \mathbb{N}). \quad (11)$$

Then, by using (4) and (11), we have for every $n \in \mathbb{N}$ that

$$\Lambda_n(x) = \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) x_k(\lambda)$$

$$= \frac{1}{\lambda_n} \sum_{k=0}^{n} \sum_{j=k-1}^{k} (-1)^{k-j} \lambda_j y_j$$

$$= \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k y_k - \lambda_{k-1} y_{k-1})$$

$$= y_n.$$

This shows that $\Lambda(x) = y$ and since $y \in \ell_p$, we obtain that $\Lambda(x) \in \ell_p$. Thus, we deduce that $x \in \ell_p^\Lambda$ and $Tx = y$. Hence $T$ is surjective.
Moreover, for any \( x \in \ell^\lambda_p \), we have by (8), (9) and (10) of Theorem 3.1 that
\[
\|Tx\|_{\ell^p} = \|y(\lambda)\|_{\ell^p} = \|\Lambda(x)\|_{\ell^p} = \|x\|_{\ell^\lambda_p}
\]
which shows that \( T \) is \( p \)-norm and norm preserving in the cases of \( 0 < p < 1 \) and \( 1 \leq p \leq \infty \), respectively. Hence \( T \) is isometry. Consequently, the spaces \( \ell^\lambda_p \) and \( \ell^p \) are isometrically isomorphic for \( 0 < p \leq \infty \). This concludes the proof. \( \square \)

Now, one may expect the similar result for the space \( \ell^\lambda_p \) as was observed for the space \( \ell^p \), and ask the natural question: Is not the space \( \ell^\lambda_p \) a Hilbert space with \( p \neq 2 \)? The answer is positive and is given by the following theorem:

**Theorem 3.4.** Except the case \( p = 2 \), the space \( \ell^\lambda_p \) is not an inner product space, hence not a Hilbert space for \( 1 \leq p < \infty \).

**Proof.** We have to prove that the space \( \ell^2_2 \) is the only Hilbert space among the \( \ell^\lambda_p \) spaces for \( 1 \leq p < \infty \). Since the space \( \ell^2_2 \) is a BK-space with the norm \( \|x\|_{\ell^2_2} = \|\Lambda(x)\|_{\ell^2} \) by Theorem 3.1 and its norm can be obtained from an inner product, i.e., the equality
\[
\|x\|_{\ell^2_2} = \langle x, x \rangle^{1/2} = \langle \Lambda(x), \Lambda(x) \rangle^{1/2}_2
\]
holds for every \( x \in \ell^2_2 \), the space \( \ell^2_2 \) is a Hilbert space, where \( \langle \cdot, \cdot \rangle_2 \) denotes the inner product on \( \ell^2 \).

Let us now consider the sequences
\[
u = \{v_k(\lambda)\} = \left(1, -\frac{\lambda_1}{\lambda_2 - \lambda_1}, 0, 0, \ldots\right)
\]
and
\[
u = \{v_k(\lambda)\} = \left(1, \frac{-\lambda_1 + \lambda_0}{\lambda_1 - \lambda_0}, \frac{\lambda_1}{\lambda_2 - \lambda_1}, 0, 0, \ldots\right).
\]
Then, we have
\[
\Lambda(u) = (1, 1, 0, 0, \ldots) \quad \text{and} \quad \Lambda(v) = (1, -1, 0, 0, \ldots).
\]
Thus, it can easily be seen that
\[
\|u + v\|^2_{\ell^2_2} + \|u - v\|^2_{\ell^2_2} = 8 = 4(2^{2/p}) = 2\left(\|u\|^2_{\ell^p_p} + \|v\|^2_{\ell^p_p}\right); \quad (p \neq 2),
\]
that is, the norm of the space \( \ell^\lambda_p \) with \( p \neq 2 \) does not satisfy the parallelogram equality which means that this norm cannot be obtained from an inner product. Hence, the space \( \ell^\lambda_p \) with \( p \neq 2 \) is a Banach space which is not a Hilbert space, where \( 1 \leq p < \infty \). This completes the proof. \( \square \)

**Remark 3.5.** It is obvious that \( \ell^\lambda_\infty \) is also a Banach space which is not a Hilbert space.
4 Some inclusion relations

In the present section, we establish some inclusion relations concerning the spaces \( \ell_p^\lambda \) and \( \ell_{p}^\infty \), where \( 0 < p < \infty \). We essentially prove that the inclusion \( \ell_{\infty}^\lambda \subset \ell_{\infty}^\lambda \) holds and characterize the case in which the inclusion \( \ell_p^\lambda \subset \ell_p^\lambda \) holds for \( 1 \leq p < \infty \).

We may begin with quoting the following two lemmas (see [15]) which are needed in the proofs of our main results.

Lemma 4.1. For any sequence \( x = (x_k) \in w, \) the equalities

\[
S_n(x) = x_n - \Lambda_n(x); \quad (n \in \mathbb{N}) \quad (12)
\]

and

\[
S_n(x) = \frac{\lambda_{n-1}}{\lambda_n - \lambda_{n-1}} [\Lambda_n(x) - \Lambda_{n-1}(x)]; \quad (n \in \mathbb{N}) \quad (13)
\]

hold, where \( S(x) = \{S_n(x)\} \) is the sequence defined by

\[
S_0(x) = 0 \quad \text{and} \quad S_n(x) = \frac{1}{\lambda_n} \sum_{k=1}^{n} \lambda_{k-1}(x_k - x_{k-1}); \quad (n \geq 1).
\]

Lemma 4.2. For any sequence \( \lambda = (\lambda_k)_{k=0}^{\infty} \) satisfying (3), we have

(a) \( \left( \frac{\lambda_k}{\lambda_k - \lambda_{k-1}} \right)_{k=0}^{\infty} \notin \ell_{\infty} \) if and only if \( \liminf_{k \to \infty} \frac{\lambda_{k+1}}{\lambda_k} = 1. \)

(b) \( \left( \frac{\lambda_k}{\lambda_k - \lambda_{k-1}} \right)_{k=0}^{\infty} \in \ell_{\infty} \) if and only if \( \liminf_{k \to \infty} \frac{\lambda_{k+1}}{\lambda_k} > 1. \)

It is obvious that Lemma 4.2 still holds if the sequence \( \{\lambda_k/(\lambda_k - \lambda_{k-1})\} \) is replaced by \( \{\lambda_k/(\lambda_{k+1} - \lambda_k)\}. \)

Now, we prove the following:

Theorem 4.3. If \( 0 < p < q < \infty \), then the inclusion \( \ell_p^\lambda \subset \ell_q^\lambda \) strictly holds.

Proof. Let \( 0 < p < q < \infty \). Then, it follows by the inclusion \( \ell_p < \ell_q \) that the inclusion \( \ell_p^\lambda \subset \ell_q^\lambda \) holds. Further, since the inclusion \( \ell_p \subset \ell_q \) is strict, there is a sequence \( y = (y_k) \in \ell_q \) but not in \( \ell_p \), i.e., \( x \in \ell_q \setminus \ell_p \). Let us now define the sequence \( y = (y_k) \) in terms of the sequence \( x \) as follows:

\[
y_k = \frac{\lambda_k x_k - \lambda_{k-1} x_{k-1}}{\lambda_k - \lambda_{k-1}}; \quad (k \in \mathbb{N}).
\]

Then, we have for every \( n \in \mathbb{N} \) that

\[
\Lambda_n(y) = \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k x_k - \lambda_{k-1} x_{k-1}) = x_n
\]

which shows that \( \Lambda(y) = x \) and hence \( \Lambda(y) \in \ell_q \setminus \ell_p \). Thus, the sequence \( y \) is in \( \ell_q^\lambda \) but not in \( \ell_p^\lambda \). Hence, the inclusion \( \ell_p^\lambda \subset \ell_q^\lambda \) is strict. This concludes the proof. \( \square \)
Theorem 4.4. The inclusions $\ell_p^\lambda \subset c_0^\lambda \subset c^\lambda \subset \ell_\infty^\lambda$ strictly hold, where $0 < p < \infty$.

Proof. Since the inclusion $c_0^\lambda \subset c^\lambda$ strictly holds [15, Theorem 4.1], it is enough to show that the inclusions $\ell_p^\lambda \subset c_0^\lambda$ and $c^\lambda \subset \ell_\infty^\lambda$ are strict, where $0 < p < \infty$.

Firstly, it is trivial that the inclusion $\ell_p^\lambda \subset c_0^\lambda$ holds for $0 < p < \infty$, since $x \in \ell_p^\lambda$ implies $\Lambda(x) \in \ell_p$ and hence $\Lambda(x) \in c_0^\lambda$ which means that $x \in c_0^\lambda$. Further, to show that this inclusion is strict, let $0 < p < \infty$ and consider the sequence $x = (x_k)$ defined by

$$x_k = \frac{1}{(k+1)^{1/p}}; \quad (k \in \mathbb{N}).$$

Then $x \in c_0^\lambda$ and hence $x \in c_0^\lambda$, since the inclusion $c_0^\lambda \subset c_0^\lambda$ holds. On the other hand, we have for every $n \in \mathbb{N}$ that

$$|\Lambda_n(x)| = 1 \frac{n}{\lambda_n} \sum_{k=0}^{n} \frac{\lambda_k - \lambda_{k-1}}{(k+1)^{1/p}}$$

$$\geq 1 \frac{1}{\lambda_n (n+1)^{1/p}} \sum_{k=0}^{n} (\lambda_k - \lambda_k - 1)$$

$$= 1 \frac{1}{(n+1)^{1/p}}$$

which shows that $\Lambda(x) \notin \ell_p$ and hence $x \notin \ell_p^\lambda$. Thus, the sequence $x$ is in $c_0^\lambda$ but not in $\ell_p^\lambda$. Therefore, the inclusion $\ell_p^\lambda \subset c_0^\lambda$ is strict for $0 < p < \infty$.

Similarly, it is also clear that the inclusion $c^\lambda \subset \ell_\infty^\lambda$ holds. To show that this inclusion is strict, we define the sequence $y = (y_k)$ by

$$y_k = (-1)^k \frac{\lambda_k + \lambda_{k-1}}{\lambda_k - \lambda_{k-1}}; \quad (k \in \mathbb{N}).$$

Then, we have for every $n \in \mathbb{N}$ that

$$\Lambda_n(y) = 1 \frac{n}{\lambda_n} \sum_{k=0}^{n} (-1)^k (\lambda_k + \lambda_{k-1}) = (-1)^n$$

which shows that $\Lambda(y) \notin \ell_\infty$ \& $c$. Thus, the sequence $y$ is in $\ell_\infty^\lambda$ but not in $c^\lambda$ and hence $c^\lambda \subset \ell_\infty^\lambda$ is a strict inclusion. This completes the proof. □

Lemma 4.5. The inclusion $\ell_p^\lambda \subset \ell_p$ holds if and only if $S(x) \in \ell_p$ for every sequence $x \in \ell_p^\lambda$, where $0 < p \leq \infty$.

Proof. Suppose that the inclusion $\ell_p^\lambda \subset \ell_p$ holds, where $0 < p \leq \infty$, and take any $x = (x_k) \in \ell_p^\lambda$. Then $x \in \ell_p$ by the hypothesis. Thus, we obtain from (12) that

$$\|S(x)\|_{\ell_p} \leq \|x\|_{\ell_p} + \|\Lambda(x)\|_{\ell_p} = \|x\|_{\ell_p} + \|x\|_{\ell_\infty} < \infty$$

which yields that $S(x) \in \ell_p$. 

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Conversely, let \( x \in \ell_p^\lambda \) be given, where \( 0 < p \leq \infty \). Then, we have by the hypothesis that \( S(x) \in \ell_p \). Again, it follows by (12) that

\[
\|x\|_p \leq \|S(x)\|_p + \|\Lambda(x)\|_p = \|S(x)\|_p + \|x\|_p < \infty
\]

which shows that \( x \in \ell_p \). Hence, the inclusion \( \ell_p^\lambda \subset \ell_p \) holds and this concludes the proof. \( \square \)

**Theorem 4.6.** The inclusion \( \ell_\infty \subset \ell_\infty^\lambda \) holds. Further, the equality holds if and only if \( S(x) \in \ell_\infty \) for every sequence \( x \in \ell_\infty^\lambda \).

**Proof.** The first part of the theorem is immediately obtained from Lemma 2.1, and so we turn to the second part. For, suppose firstly that the equality \( \ell_\infty^\lambda = \ell_\infty \) holds. Then, the inclusion \( \ell_\infty^\lambda \subset \ell_\infty \) holds which leads us with Lemma 4.5 to the consequence that \( S(x) \in \ell_\infty \) for every \( x \in \ell_\infty^\lambda \).

Conversely, suppose that \( S(x) \in \ell_\infty \) for every \( x \in \ell_\infty^\lambda \). Then, we deduce by Lemma 4.5 that the inclusion \( \ell_\infty^\lambda \subset \ell_\infty \) holds. Combining this with the inclusion \( \ell_\infty \subset \ell_\infty^\lambda \), we get the equality \( \ell_\infty^\lambda = \ell_\infty \). This completes the proof. \( \square \)

Now, the following theorem gives the necessary and sufficient condition for the matrix \( \Lambda \) to be stronger than boundedness, i.e., for the inclusion \( \ell_\infty \subset \ell_\infty^\lambda \) to be strict.

**Theorem 4.7.** The inclusion \( \ell_\infty \subset \ell_\infty^\lambda \) strictly holds if and only if \( \lim \inf_{n \to \infty} \lambda_{n+1}/\lambda_n = 1 \).

**Proof.** Suppose that the inclusion \( \ell_\infty \subset \ell_\infty^\lambda \) is strict. Then, Theorem 4.6 implies the existence of a sequence \( x \in \ell_\infty^\lambda \) such that \( S(x) = \{S_n(x)\} \notin \ell_\infty \). Since \( x \in \ell_\infty^\lambda \), we have \( \Lambda(x) = \{\Lambda_n(x)\} \in \ell_\infty \) and hence \( \{\Lambda_n(x) - \Lambda_{n-1}(x)\} \in \ell_\infty \). Combining this with the fact that \( \{S_n(x)\} \notin \ell_\infty \), we obtain by (13) that \( \{\lambda_{n-1}/(\lambda_n - \lambda_{n-1})\} \notin \ell_\infty \) and hence \( \{\lambda_n/(\lambda_n - \lambda_{n-1})\} \notin \ell_\infty \). This leads us with Lemma 4.2 (a) to the consequence that \( \lim \inf_{n \to \infty} \lambda_{n+1}/\lambda_n = 1 \) which shows the necessity of the condition.

Conversely, suppose that \( \lim \inf_{n \to \infty} \lambda_{n+1}/\lambda_n = 1 \). Then, we have by Lemma 4.2 (a) that \( \{\lambda_n/(\lambda_n - \lambda_{n-1})\} \notin \ell_\infty \). Let us now consider the sequence \( x = (x_k) \) defined by \( x_k = (-1)^k \lambda_k/(\lambda_k - \lambda_{k-1}) \) for all \( k \in \mathbb{N} \). Then, it is obvious that \( x \notin \ell_\infty \).

On the other hand, we have for every \( n \in \mathbb{N} \) that

\[
|\Lambda_n(x)| = \frac{1}{\lambda_n} \left| \sum_{k=0}^{n} (-1)^k \lambda_k \right| \leq \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) = 1
\]

which shows that \( \Lambda(x) \in \ell_\infty \) and hence \( x \in \ell_\infty^\lambda \). Thus, the sequence \( x \) is in \( \ell_\infty^\lambda \) but not in \( \ell_\infty \). Therefore, by combining this with the inclusion \( \ell_\infty \subset \ell_\infty^\lambda \), we deduce that this inclusion is strict. This concludes the proof. \( \square \)

Now, as a consequence of Theorem 4.7, the following corollary presents the necessary and sufficient condition for the matrix \( \Lambda \) to be equivalent to boundedness.
Corollary 4.8. The equality $\ell^\lambda_\infty = \ell_\infty$ holds if and only if $\lim \inf_{n \to \infty} \lambda_{n+1}/\lambda_n > 1$.

Proof. The necessity follows immediately from Theorem 4.7. For, if the equality $\ell^\lambda_\infty = \ell_\infty$ holds, then $\lim \inf_{n \to \infty} \lambda_{n+1}/\lambda_n \neq 1$ and hence $\lim \inf_{n \to \infty} \lambda_{n+1}/\lambda_n > 1$.

Conversely, suppose that $\lim \inf_{n \to \infty} \lambda_{n+1}/\lambda_n > 1$. Then, Lemma 4.2 (b) gives us the bounded sequence $\{\lambda_n/(\lambda_n - \lambda_{n-1})\}$ and so $\{\lambda_{n-1}/(\lambda_n - \lambda_{n-1})\} \in \ell_\infty$.

Now, let $x \in \ell^\lambda_\infty$. Then $\Lambda(x) = \{\Lambda_n(x)\} \in \ell_\infty$ and hence $\{\Lambda_n(x) - \Lambda_{n-1}(x)\} \in \ell_\infty$. Thus, we obtain by (13) that $\{S_n(x)\} \in \ell_\infty$. This shows that $S(x) \in \ell_\infty$ for every $x \in \ell^\lambda_\infty$, which leads us with Theorem 4.6 to the equality $\ell^\lambda_\infty = \ell_\infty$. □

Although the inclusions $c_0 \subset c^\lambda_\infty \subset c_\infty$ always hold, the inclusion $\ell_p \subset \ell_p^\lambda$ need not be held, where $0 < p < \infty$. In fact, we are going to show, in the following lemma, that if $1/\lambda \notin \ell_p$, then the inclusion $\ell_p \subset \ell_p^\lambda$ fails, where $1/\lambda = (1/\lambda_k)$ and $0 < p < \infty$.

Lemma 4.9. The spaces $\ell_p$ and $\ell_p^\lambda$ overlap. Further, if $1/\lambda \notin \ell_p$ then neither of them includes the other one, where $0 < p < \infty$.

Proof. Obviously, the spaces $\ell_p$ and $\ell_p^\lambda$ overlap, since $(\lambda_1 - \lambda_0, -\lambda_0, 0, 0, \ldots) \in \ell_p \cap \ell_p^\lambda$ for $0 < p < \infty$.

Now, suppose that $1/\lambda \notin \ell_p$, where $0 < p < \infty$, and consider the sequence $x = e^{(0)} = (1, 0, 0, \ldots) \in \ell_p$. Then, we have for every $n \in \mathbb{N}$ that

$$\Lambda_n(x) = \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1})e^{(0)}_k = \frac{\lambda_0}{\lambda_n}$$

which shows that $\Lambda(x) \notin \ell_p$ and hence $x \notin \ell_p^\lambda$. Thus, the sequence $x$ is in $\ell_p$ but not in $\ell_p^\lambda$. Hence, the inclusion $\ell_p \subset \ell_p^\lambda$ does not hold when $1/\lambda \notin \ell_p$ ($0 < p < \infty$).

On the other hand, let $1 \leq p < \infty$ and define the sequence $y = (y_k)$ by

$$y_k = \begin{cases} \frac{1}{\lambda_k}; & (k \text{ is even}), \\ -\frac{1}{\lambda_{k-1}} \frac{(\lambda_{k-1} - \lambda_{k-2})}{\lambda_k - \lambda_{k-1}}; & (k \text{ is odd}) \end{cases}$$

for all $k \in \mathbb{N}$. Since $1/\lambda \notin \ell_p$, we have $y \notin \ell_p$. Besides, we have for every $n \in \mathbb{N}$ that

$$\Lambda_n(y) = \begin{cases} \frac{1}{\lambda_n} \left(\frac{\lambda_n - \lambda_{n-1}}{\lambda_n}\right); & (n \text{ is even}), \\ 0; & (n \text{ is odd}) \end{cases}$$
and hence
\[
\sum_n |\Lambda_n(y)|^p = \sum_n |\Lambda_{2n}(y)|^p \\
= \sum_n \frac{1}{\lambda_{2n}^p} \left( \frac{\lambda_{2n} - \lambda_{2n-1}}{\lambda_{2n}} \right)^p \\
\leq \frac{1}{\lambda_0^p} + \sum_{n=1}^{\infty} \frac{1}{\lambda_{2n-2}^p} \left( \frac{\lambda_{2n} - \lambda_{2n-2}}{\lambda_{2n}} \right)^p \\
\leq \frac{1}{\lambda_0^p} + \sum_{n=1}^{\infty} \frac{1}{\lambda_{2n-2}^p} \left( \frac{\lambda_{2n}^p - \lambda_{2n-2}^p}{\lambda_{2n}^p} \right) \\
= \frac{1}{\lambda_0^p} + \sum_{n=1}^{\infty} \left( \frac{1}{\lambda_{2n-2}^p} - \frac{1}{\lambda_{2n}^p} \right) \\
= \frac{2}{\lambda_0^p} < \infty.
\]

This shows that $\Lambda(y) \in \ell_p$ and so $y \in \ell_\lambda^p$. Thus, the sequence $y$ is in $\ell_\lambda^p$ but not in $\ell_p$, where $1 \leq p < \infty$.

Similarly, one can construct a sequence belonging to the set $\ell_\lambda^p \setminus \ell_p$ for $0 < p < 1$. Therefore, the inclusion $\ell_\lambda^p \subset \ell_p$ also fails when $1/\lambda \notin \ell_p$ ($0 < p < \infty$). Hence, if $1/\lambda \notin \ell_p$ then neither of the spaces $\ell_p$ and $\ell_\lambda^p$ includes the other one, where $0 < p < \infty$. This completes the proof. \hfill \square

**Lemma 4.10.** If the inclusion $\ell_p \subset \ell_\lambda^p$ holds, then $1/\lambda \in \ell_p$ for $0 < p < \infty$.

**Proof.** Suppose that the inclusion $\ell_p \subset \ell_\lambda^p$ holds, where $0 < p < \infty$, and consider the sequence $x = e^{(0)} = (1, 0, 0, \ldots) \in \ell_p$. Then $x \in \ell_\lambda^p$ and hence $\Lambda(x) \in \ell_p$. Thus, we obtain that
\[
\lambda_0^p \sum_n \left( \frac{1}{\lambda_n} \right)^p = \sum_n |\Lambda_n(x)|^p < \infty
\]
which shows that $1/\lambda \in \ell_p$ and this concludes the proof. \hfill \square

We shall later show that the condition $1/\lambda \in \ell_p$ is not only necessary but also sufficient for the inclusion $\ell_p \subset \ell_\lambda^p$ to be held, where $1 \leq p < \infty$. Before that, by taking into account the definition of the sequence $\lambda = (\lambda_k)$ given by (3), we find that
\[
0 < \frac{\lambda_k - \lambda_{k-1}}{\lambda_n} < 1; \quad (0 \leq k \leq n)
\]
for all $n, k \in \mathbb{N}$ with $n + k > 0$. Furthermore, if $1/\lambda \in \ell_1$ then we have the following lemma which is easy to prove.

**Lemma 4.11.** If $1/\lambda \in \ell_1$, then
\[
\sup_k \left( \frac{\lambda_k - \lambda_{k-1}}{\lambda_n} \sum_{n=k}^{\infty} \frac{1}{\lambda_n} \right) < \infty.
\]
Theorem 4.12. The inclusion $\ell_1 \subset \ell_\lambda^1$ holds if and only if $1/\lambda \in \ell_1$.

Proof. The necessity is immediate by Lemma 4.10.

Conversely, suppose $1/\lambda \in \ell_1$. Then $M = \sup_k \left[(\lambda_k - \lambda_{k-1}) \sum_{n=k}^{\infty} 1/\lambda_n\right] < \infty$ by Lemma 4.11. Also, let $x = (x_k) \in \ell_1$ be given. Then, we have

$$
\|x\|_{\ell_\lambda^1} = \sum_n |\Lambda_n(x)| \\
\leq \sum_{n=0}^{\infty} \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1})|x_k| \\
= \sum_{k=0}^{\infty} |x_k|(\lambda_k - \lambda_{k-1}) \sum_{n=k}^{\infty} \frac{1}{\lambda_n} \\
\leq M \sum_{k=0}^{\infty} |x_k| \\
= M \|x\|_{\ell_1} < \infty.
$$

This shows that $x \in \ell_\lambda^1$. Hence, the inclusion $\ell_1 \subset \ell_\lambda^1$ holds. \hfill \Box

Corollary 4.13. If $1/\lambda \in \ell_1$, then the inclusion $\ell_p \subset \ell_\lambda^p$ holds for $1 \leq p < \infty$.

Proof. The inclusion trivially holds for $p = 1$, which is obtained by Theorem 4.12, above. Thus, let $1 < p < \infty$ and take any $x = (x_k) \in \ell_p$. Then, for every $n \in \mathbb{N}$, we obtain by applying the Hölder’s inequality that

$$
|\Lambda_n(x)|^p \leq \left[ \sum_{k=0}^{n} \left(\frac{\lambda_k - \lambda_{k-1}}{\lambda_n}\right)|x_k|^p \right]^{1/p} \left[ \sum_{k=0}^{n} \lambda_k - \lambda_{k-1} \right]^{(p-1)/p} \\
= \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1})|x_k|^p.
$$

Therefore, we derive that

$$
\sum_{n} |\Lambda_n(x)|^p \leq \sum_{n=0}^{\infty} \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1})|x_k|^p \\
= \sum_{k=0}^{\infty} |x_k|^p(\lambda_k - \lambda_{k-1}) \sum_{n=k}^{\infty} \frac{1}{\lambda_n} \\
\leq M \|x\|_{\ell_p} \leq M \|x\|_{\ell_p}^p < \infty,
$$

and hence

$$
\|x\|_{\ell_\lambda^p}^p \leq M \sum_{k=0}^{\infty} |x_k|^p = M \|x\|_{\ell_p}^p < \infty.
$$
where $M = \sup_k \left[ (\lambda_k - \lambda_{k-1}) \sum_{n=k}^{\infty} 1/\lambda_n \right] < \infty$ by Lemma 4.11. This shows that $x \in \ell^\lambda_p$. Hence, we deduce that the inclusion $\ell_p \subset \ell^\lambda_p$ also holds for $1 < p < \infty$. This completes the proof. \hfill $\square$

**Corollary 4.14.** The inclusion $\ell_p \subset \ell^\lambda_p$ holds if and only if $1/\lambda \in \ell_p$, where $1 \leq p < \infty$.

**Proof.** The necessity is immediate by Lemma 4.10.

Conversely, suppose that $1/\lambda \in \ell_p$, where $1 \leq p < \infty$. Then $1/\lambda_p = (1/\lambda_k) \in \ell_1$.

Thus, it follows by Lemma 4.11 that

$$\sup_k \left( (\lambda_k - \lambda_{k-1})^p \sum_{n=k}^{\infty} \frac{1}{\lambda_n^p} \right) \leq \sup_k \left( (\lambda_k^p - \lambda_{k-1}^p) \sum_{n=k}^{\infty} \frac{1}{\lambda_n^p} \right) < \infty.$$

Further, we have for every fixed $k \in \mathbb{N}$ that

$$\Lambda_n (e^{(k)}) = \begin{cases} \frac{\lambda_k - \lambda_{k-1}}{\lambda_n} ; & (n \geq k), \\ 0 ; & (n < k). \end{cases}$$

Thus, we obtain that

$$\|e^{(k)}\|_{\ell^\lambda_p}^p = (\lambda_k - \lambda_{k-1})^p \sum_{n=k}^{\infty} \frac{1}{\lambda_n^p} < \infty; \quad (k \in \mathbb{N})$$

which yields that $e^{(k)} \in \ell^\lambda_p$ for every $k \in \mathbb{N}$, i.e., every basis element of the space $\ell_p$ is in $\ell^\lambda_p$. This shows that the space $\ell^\lambda_p$ contains the Schauder basis of the space $\ell_p$, such that $\sup_k \|e^{(k)}\|_{\ell^\lambda_p} < \infty$. Hence, we deduce that the inclusion $\ell_p \subset \ell^\lambda_p$ holds and this concludes the proof. \hfill $\square$

Now, in the following example, we give an important special case of the space $\ell^\lambda_p$, where $1 \leq p < \infty$.

**Example 4.15.** Consider the special case of the sequence $\lambda = (\lambda_k)$ given by $

\lambda_k = k + 1$ for all $k \in \mathbb{N}$. Then $1/\lambda \notin \ell_1$ while $1/\lambda \in \ell_p$ for $1 < p < \infty$. Hence, the inclusion $\ell_1 \subset \ell^\lambda_p$ does not hold by Lemma 4.9.

On the other hand, by applying the well-known inequality (see [10, p.239])

$$\sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \frac{|x_k|}{n+1} \right)^p \leq \left( \frac{p}{p-1} \right)^p \sum_{n=0}^{\infty} |x_n|^p; \quad (1 < p < \infty),$$

we immediately deduce that the inequality

$$\|x\|_{\ell^\lambda_p} < \left( \frac{p}{p-1} \right)^p \|x\|_{\ell_p}$$

On the other hand, by applying the well-known inequality (see [10, p.239])

$$\sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \frac{|x_k|}{n+1} \right)^p \leq \left( \frac{p}{p-1} \right)^p \sum_{n=0}^{\infty} |x_n|^p; \quad (1 < p < \infty),$$

we immediately deduce that the inequality

$$\|x\|_{\ell^\lambda_p} < \left( \frac{p}{p-1} \right)^p \|x\|_{\ell_p}$$
holds for every $x \in \ell_p$, where $1 < p < \infty$. This shows that the inclusion $\ell_p \subset \ell_\lambda$ holds for $1 < p < \infty$. Further, this inclusion is strict. For example, the sequence $y = \{(−1)^k\}$ is not in $\ell_p$ but in $\ell_\lambda$, since

$$\sum_n |A_n(y)|^p = \sum_n \left| \frac{1}{n+1} \sum_{k=0}^n (-1)^k \right|^p = \sum_n \frac{1}{(2n+1)^p} < \infty; \quad (1 < p < \infty).$$

**Remark 4.16.** In the special case of the sequence $\lambda = (\lambda_k)$ given in Example 4.15, i.e., $\lambda_k = k+1$ for all $k \in \mathbb{N}$, we may note that the spaces $\ell_\lambda^0$ and $\ell_\infty^0$ are respectively reduced to the Cesàro sequence spaces $X_p$ and $X_\infty$ of non-negative type, which are defined as the spaces of all sequences whose $C_1$-transforms are in the spaces $\ell_p$ and $\ell_\infty$, respectively, where $1 \leq p < \infty$ (see [17, 18]).

Now, let $x = (x_k)$ be a null sequence of positive reals, that is

$$x_k > 0 \quad \text{for all} \quad k \in \mathbb{N} \quad \text{and} \quad x_k \to 0 \quad \text{as} \quad k \to \infty.$$

Then, as is easy to see, for every positive integer $m$ there is a subsequence $(x_{k_r})_{r=0}^\infty$ of the sequence $x$ such that

$$x_{k_r} = O \left( \frac{1}{(r+1)^{m+1}} \right)$$

and hence

$$(r+1)x_{k_r} = O \left( \frac{1}{(r+1)^m} \right).$$

Further, this subsequence can be chosen such that $k_{r+1} - k_r \geq 2$ for all $r \in \mathbb{N}$.

In general, if $x = (x_k)$ is a sequence of positive reals such that $\liminf_{k \to \infty} x_k = 0$, then there is a subsequence $x' = (x_{k_r'})_{r=0}^\infty$ of the sequence $x$ such that $\lim_{r \to \infty} x_{k_r'} = 0$. Thus $x'$ is a null sequence of positive reals. Hence, as we have seen above, for every positive integer $m$ there is a subsequence $(x_{k_r})_{r=0}^\infty$ of the sequence $x'$, and hence of the sequence $x$, such that $k_{r+1} - k_r \geq 2$ for all $r \in \mathbb{N}$ and

$$(r+1)x_{k_r} = O \left( \frac{1}{(r+1)^m} \right),$$

where $k_r = k_{0(r+1)}$ and $\theta : \mathbb{N} \to \mathbb{N}$ is a suitable increasing function.

Now, let $0 < p < \infty$. Then, we can choose a positive integer $m$ such that $mp > 1$. In this situation, the sequence $\{(r+1)x_{k_r}\}_{r=0}^\infty$ is in the space $\ell_p$.

Obviously, we observe that the subsequence $(x_{k_r})_{r=0}^\infty$ depends on the positive integer $m$ which is, in turn, depending on $p$. Thus, our subsequence depends on $p$.

Hence, from the above discussion, we conclude the following result:

**Lemma 4.17.** Let $x = (x_k)$ be a positive real sequence such that $\liminf_{k \to \infty} x_k = 0$. Then, for every positive number $0 < p < \infty$ there is a subsequence $x^{(p)} = (x_{k_r})_{r=0}^\infty$ of $x$, depending on $p$, such that $k_{r+1} - k_r \geq 2$ for all $r \in \mathbb{N}$ and $\sum_r |(r+1)x_{k_r}|^p < \infty$. 

Now, the following theorem gives the necessary and sufficient conditions for the matrix $\Lambda$ to be stronger than $p$-absolute convergence, i.e., for the inclusion $\ell_p \subset \ell^\lambda_p$ to be strict, where $1 \leq p < \infty$.

**Theorem 4.18.** The inclusion $\ell_p \subset \ell^\lambda_p$ strictly holds if and only if $1/\lambda \in \ell_p$ and $\lim\inf_{n \to \infty} \lambda_{n+1}/\lambda_n = 1$, where $1 \leq p < \infty$.

**Proof.** Suppose that the inclusion $\ell_p \subset \ell^\lambda_p$ is strict, where $1 \leq p < \infty$. Then, the necessity of the first condition is immediate by Lemma 4.10. Further, since the inclusion $\ell^\lambda_p \subset \ell_p$ cannot be held, Lemma 4.5 implies the existence of a sequence $x \in \ell^\lambda_p$ such that $S(x) = \{S_n(x)\} \notin \ell_p$. Since $x \in \ell^\lambda_p$, we have $\sum_n |A_n(x)|^p < \infty$. Thus, it follows by applying the Minkowski’s inequality that $\sum_n |A_n(x) - \Lambda_n(x)|^p < \infty$. This means that $\{A_n(x) - \Lambda_n(x)\} \in \ell_p$ and since $\{S_n(x)\} \notin \ell_p$, it follows by the relation (13) that $\{\lambda_{n-1}/(\lambda_n - \lambda_{n-1})\} \notin \ell_\infty$ and hence $\{\lambda_n/(\lambda_n - \lambda_{n-1})\} \notin \ell_\infty$. This leads us with Lemma 4.2 (a) to the necessity of the second condition.

Conversely, since $1/\lambda \in \ell_p$, we have by Corollary 4.14 that the inclusion $\ell_p \subset \ell^\lambda_p$ holds. Further, since $\lim\inf_{k \to \infty} \lambda_{k+1}/\lambda_k = 1$, we obtain by Lemma 4.2 (a) that

$$\lim_{k \to \infty} \left( \frac{\lambda_k - \lambda_{k-1}}{\lambda_k} \right) = 0.$$ 

Thus, it follows by Lemma 4.17 that there is a subsequence $\lambda^{(p)} = (\lambda_{k_r})_{r=0}^\infty$ of the sequence $\lambda = (\lambda_k)$, depending on $p$, such that $k_{r+1} - k_r \geq 2$ for all $r \in \mathbb{N}$ and

$$\sum_r \left| (r + 1) \frac{(k_{r+1} - \lambda_{k_r} - 1)}{\lambda_{k_r}} \right|^p < \infty. \quad (15)$$

Let us now define the sequence $y = (y_k)$ for every $k \in \mathbb{N}$ by

$$y_k = \begin{cases} 
  r + 1; & (k = k_r), \\
  -(r + 1) \frac{k_{r+1} - \lambda_k - 2}{\lambda_k - \lambda_{k-1}}; & (k = k_r + 1), \\
  0; & (\text{otherwise}).
\end{cases} \quad (16)$$

Then, it is clear that $y \notin \ell_p$. On the other hand, we have for every $n \in \mathbb{N}$ that

$$\Lambda_n(y) = \begin{cases} 
  (r + 1) \frac{\lambda_n - \lambda_{n-1}}{\lambda_n}; & (n = k_r), \\
  0; & (n \neq k_r).
\end{cases} \quad (r \in \mathbb{N})$$

This and (15) imply that $\Lambda(y) \in \ell_p$ and hence $y \in \ell^\lambda_p$. Thus, the sequence $y$ is in $\ell^\lambda_p$ but not in $\ell_p$. Therefore, we deduce by combining this with the inclusion $\ell_p \subset \ell^\lambda_p$ that this inclusion is strict, where $1 \leq p < \infty$. This completes the proof. □

Now, as an immediate consequence of Theorem 4.18, the following corollary presents the necessary and sufficient condition for the matrix $\Lambda$ to be equivalent to $p$-absolute convergence, where $1 \leq p < \infty$. 


Corollary 4.19. The equality $\ell_p^\lambda = \ell_p$ holds if and only if $\liminf_{n \to \infty} \lambda_{n+1}/\lambda_n > 1$, where $1 \leq p < \infty$.

Proof. The necessity follows from Theorem 4.18. For, if the equality holds, then the inclusion $\ell_p \subset \ell_p^\lambda$ holds and hence $1/\lambda \in \ell_p$, by Lemma 4.10. Further, since the inclusion $\ell_p \subset \ell_p^\lambda$ cannot be strict, we have by Theorem 4.18 that $\liminf_{n \to \infty} \lambda_{n+1}/\lambda_n \neq 1$ and hence $\liminf_{n \to \infty} \lambda_{n+1}/\lambda_n > 1$.

Conversely, suppose that $\liminf_{n \to \infty} \lambda_{n+1}/\lambda_n > 1$. Then, there exists a constant $a > 1$ such that $\lambda_{n+1}/\lambda_n \geq a$ and hence $\lambda_n \geq \lambda_0 a^n$ for all $n \in \mathbb{N}$. This shows that $1/\lambda \in \ell_1$ which leads us with Corollary 4.13 to the consequence that the inclusion $\ell_p \subset \ell_p^\lambda$ holds for $1 \leq p < \infty$.

On the other hand, we have by Lemma 4.2 (b) that $\{\lambda_n/(\lambda_n - \lambda_{n-1})\} \in \ell_\infty$ and hence $\{\lambda_{n-1}/(\lambda_n - \lambda_{n-1})\} \in \ell_\infty$.

Now, let $x \in \ell_p^\lambda$. Then $A(x) = \{\Lambda_n(x)\} \in \ell_p$ and hence $\{\Lambda_n(x) - \Lambda_{n-1}(x)\} \in \ell_p$. Thus, we obtain by the relation (13) that $\{S_n(x)\} \in \ell_p$, i.e., $S(x) \in \ell_p$ for every $x \in \ell_p^\lambda$. Therefore, we deduce by Lemma 4.5 that the inclusion $\ell_p \subset \ell_p^\lambda$ also holds.

Hence, by combining the inclusions $\ell_p \subset \ell_p^\lambda$ and $\ell_p^\lambda \subset \ell_p$, we get the equality $\ell_p^\lambda = \ell_p$, where $1 \leq p < \infty$. This concludes the proof.

Remark 4.20. It can easily be shown that Corollary 4.19 still holds for $0 < p < 1$.

Finally, we end this section with the following corollary:

Corollary 4.21. Although the spaces $\ell_p^\lambda$, $c_0$, $c$ and $\ell_\infty$ overlap, the space $\ell_p^\lambda$ does not include any of the other spaces. Furthermore, if $\liminf_{n \to \infty} \lambda_{n+1}/\lambda_n = 1$, then none of the spaces $c_0$, $c$ and $\ell_\infty$ includes the space $\ell_p^\lambda$, where $0 < p < \infty$.

Proof. Let $0 < p < \infty$. Then, it is obvious that the spaces $\ell_p^\lambda$, $c_0$, $c$ and $\ell_\infty$ overlap, since the sequence $(\lambda_1 - \lambda_0, -\lambda_0, 0, 0, \ldots)$ belongs to all these spaces.

Further, the space $\ell_p^\lambda$ does not include the space $c_0$, since the sequence $x$ defined by (14), in the proof of Theorem 4.4, is in $c_0$ but not in $\ell_p^\lambda$. Hence, the space $\ell_p^\lambda$ does not include any of the spaces $c_0$, $c$ and $\ell_\infty$.

Furthermore, if $\liminf_{n \to \infty} \lambda_{n+1}/\lambda_n = 1$ then the space $\ell_\infty$ does not include the space $\ell_p^\lambda$. To see this, let $0 < p < \infty$. Then, Lemma 4.17 implies that the sequence $y$ defined by (16), in the proof of Theorem 4.18, is in $\ell_p^\lambda$ but not in $\ell_\infty$. Therefore, none of the spaces $c_0$, $c$ and $\ell_\infty$ includes the space $\ell_p^\lambda$ when $\liminf_{n \to \infty} \lambda_{n+1}/\lambda_n = 1$, where $0 < p < \infty$. This completes the proof.

5 The basis for the space $\ell_p^\lambda$

In this final section, we give a sequence of the points of the space $\ell_p^\lambda$ which forms a basis for this space, where $1 \leq p < \infty$.

If a normed space $X$ contains a sequence $(b_n)$ with the property that for every $x \in X$ there is a unique sequence $(\alpha_n)$ of scalars such that

$$\lim_{n \to \infty} \|x - (\alpha_0 b_0 + \alpha_1 b_1 + \cdots + \alpha_n b_n)\| = 0,$$

then $\ell_p^\lambda$ is a basis for $X$. This follows from the fact that $\ell_p^\lambda$ is isomorphic to $c_0$ or $c$.
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then \((b_n)\) is called a Schauder basis (or briefly basis) for \(X\). The series \(\sum_k \alpha_k b_k\) which has the sum \(x\) is then called the expansion of \(x\) with respect to \((b_n)\), and written as \(x = \sum_k \alpha_k b_k\).

Now, because of the transformation \(T\) defined from \(\ell_\lambda^p\) to \(\ell_p\), in the proof of Theorem 3.3, is an isomorphism, the inverse image of the basis \((e_k)_{k=0}^\infty\) of the space \(\ell_p\) is the basis for the new space \(\ell_\lambda^p\), where \(1 \leq p < \infty\). Therefore, we have the following:

**Theorem 5.1.** Let \(1 \leq p < \infty\) and define the sequence \(e_\lambda^{(k)} \in \ell_\lambda^p\) for every fixed \(k \in \mathbb{N}\) by

\[
(e_\lambda^{(k)})_n = \begin{cases}(-1)^{n-k} \frac{\lambda_k}{\lambda_n - \lambda_{n-1}}; & (k \leq n \leq k + 1), \\ 0; & \text{(otherwise)} \end{cases} (n \in \mathbb{N}) \tag{17}
\]

Then, the sequence \((e_\lambda^{(k)})_{k=0}^\infty\) is a basis for the space \(\ell_\lambda^p\) and every \(x \in \ell_\lambda^p\) has a unique representation of the form

\[
x = \sum_k \Lambda_k(x) e_\lambda^{(k)}. \tag{18}
\]

**Proof.** Let \(1 \leq p < \infty\). Then, it is obvious by (17) that \(\Lambda(e_\lambda^{(k)}) = e_\lambda^{(k)} \in \ell_p \ (k \in \mathbb{N})\) and hence \(e_\lambda^{(k)} \in \ell_\lambda^p\) for all \(k \in \mathbb{N}\).

Further, let \(x \in \ell_\lambda^p\) be given. For every non-negative integer \(m\), we put

\[
x^{(m)} = \sum_{k=0}^m \Lambda_k(x) e_\lambda^{(k)}.
\]

Then, we have that

\[
\Lambda(x^{(m)}) = \sum_{k=0}^m \Lambda_k(x) \Lambda(e_\lambda^{(k)}) = \sum_{k=0}^m \Lambda_k(x) e^{(k)}
\]

and hence

\[
\Lambda_n(x - x^{(m)}) = \begin{cases}0; & (0 \leq n \leq m), \\ \Lambda_n(x); & (n > m). \end{cases} (n, m \in \mathbb{N})
\]

Now, for any given \(\epsilon > 0\) there is a non-negative integer \(m_0\) such that

\[
\sum_{n=m_0+1}^\infty |\Lambda_n(x)|^p \leq \left(\frac{\epsilon}{2}\right)^p.
\]
Therefore, we have for every \( m \geq m_0 \) that
\[
\|x - x^{(m)}\|_{\ell^p} = \left( \sum_{n=m+1}^{\infty} |\Lambda_n(x)|^p \right)^{1/p} 
\leq \left( \sum_{n=m_0+1}^{\infty} |\Lambda_n(x)|^p \right)^{1/p} 
\leq \frac{\varepsilon}{2} < \varepsilon
\]
which shows that \( \lim_{m \to \infty} \|x - x^{(m)}\|_{\ell^p} = 0 \) and hence \( x \) is represented as in (18).

Finally, let us show the uniqueness of the representation (18) of \( x \in \ell^\lambda_p \). For this, suppose that \( x = \sum_k \alpha_k(x) e^{(k)}_\lambda \). Since the linear transformation \( T \) defined from \( \ell^\lambda_p \) to \( \ell_p \), in the proof of Theorem 3.3, is continuous, we have
\[
\Lambda_n(x) = \sum_k \alpha_k(x) \Lambda_n(e^{(k)}_\lambda) = \sum_k \alpha_k(x) \delta_{nk} = \alpha_n(x); \quad (n \in \mathbb{N}).
\]
Hence, the representation (18) of \( x \in \ell^\lambda_p \) is unique. This completes the proof. \( \square \)

Now, it is known by Theorem 3.1 (b) that \( \ell^\lambda_p \) \((1 \leq p < \infty)\) is a Banach space with its natural norm. This leads us together with Theorem 5.1 to the following corollary:

**Corollary 5.2.** The sequence space \( \ell^\lambda_p \) of non-absolute type is separable for \( 1 \leq p < \infty \).

Finally, we conclude our work by expressing from now on that the aim of the next paper is to determine the \( \alpha-, \beta- \) and \( \gamma- \)duals of the space \( \ell^\lambda_p \) and is to characterize some related matrix classes, where \( 1 \leq p \leq \infty \).

**References**


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