

A FIXED POINT RESULT FOR MAPPINGS WITH CONTRACTIVE ITERATE AT A POINT IN G -METRIC SPACES

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Abstract

Using the setting of generalized metric space, so called G -metric space, a fixed point theorems for mappings with a contractive iterate at a point are proved. This result generalize well known comparable result.

1 Introduction

In [10] V.M. Sehgal generalized a well-know Banach theorem. He proved fixed point theorem for mappings with a contractive iterate at a point.

Theorem 1.1. *Let (X, d) be a complete metric space $A : X \rightarrow X$ be a continuous mapping with property that for every $x \in X$ there exists $n(x) \in \mathbb{N}$ so that for every $y \in X$*

$$d(A^{n(x)}x, A^{n(x)}y) \leq q \cdot d(x, y), \text{ where } q \in [0, 1).$$

Then A has a unique fixed point in X and $\lim_k A^k(x_0) = z$, for each $x_0 \in X$.

L.F. Guseman [4] proved that in this theorem the condition that A be continuous can be dropped. Our aim in this study is to show that Guseman results are valid in more general class of spaces.

On the other hand, there have been a number of generalizations of metric spaces.

In 1963. S.Gähler introduced the notion of 2-metric spaces but different authors proved that there is no relation between these two function and there is no easy relationship between results obtained in the two settings. Because of that, B.C

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Dhage [3] introduced a new concept of the measure of nearness between three or more object. But topological structure of so called D -metric spaces was incorrect. Finally, Z. Mustafa and B. Sims [6] introduced correct definition of generalized metric space in that sense.

Definition 1.1. [6] Let X be a nonempty set, and let $G : X \times X \times X \rightarrow \mathbb{R}^+$ be a function satisfying the following properties:

$$(G1) \quad G(x, y, z) = 0 \text{ if } x = y = z;$$

$$(G2) \quad 0 < G(x, x, y); \text{ for all } x, y \in X, \text{ with } x \neq y;$$

$$(G3) \quad G(x, x, y) \leq G(x, y, z), \text{ for all } x, y, z \in X, \text{ with } z \neq y;$$

$$(G4) \quad G(x, y, z) = G(x, z, y) = G(y, z, x) = \dots, \text{ (symmetry in all three variables);}$$

$$(G5) \quad G(x, y, z) \leq G(x, a, a) + G(a, y, z), \text{ for all } x, y, z, a \in X.$$

Then the function G is called a generalized metric, or, more specifically, a G -metric on X , and the pair (X, G) is called a G -metric space.

Definition 1.2. [6] Let (X, G) be a G -metric space, and let $\{x_n\}$ be sequence of points of X , a point $x \in X$ is said to be the limit of the sequence $\{x_n\}$, if $\lim_{n, m \rightarrow \infty} G(x, x_n, x_m) = 0$, and one says that the sequence $\{x_n\}$ is G -convergent to x

Thus, if $x_n \rightarrow x$ in a G -metric space (X, G) , then for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x, x_n, x_m) < \epsilon$, for all $n, m \geq N$.

Definition 1.3. [6] Let (X, G) be a G -metric space, a sequence $\{x_n\}$ is called G -Cauchy if for every $\epsilon > 0$, there is $N \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \epsilon$, for all $n, m, l \geq N$; that is, if $G(x_n, x_m, x_l) \rightarrow 0$ as $n, m, l \rightarrow \infty$.

A G -metric space (X, G) is said to be G -complete (or complete G -metric) if every G -Cauchy sequence in (X, G) is G -convergent in (X, G) .

Proposition 1.1. [6] Let (X, G) be a G -metric space, then the function $G(x, y, z)$ is jointly continuous in all three of its variables.

Definition 1.4. [6] A G -metric space (X, G) is called symmetric G -metric space if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$.

Proposition 1.2. [5] Every G -metric space (X, G) will define a metric space (X, d_G) by

$$d_G(x, y) = G(x, y, y) + G(y, x, x), \text{ for all } x, y \in X.$$

Note that if (X, G) is a symmetric G -metric space, then

$$d_G(x, y) = 2G(x, y, y), \text{ for all } x, y \in X.$$

Proposition 1.3. [5] A G -metric space (X, G) is G -complete if and only if (X, d_G) is a complete metric space.

2 The main theorem

Lemma 2.1. *Let (X, G) be a G -metric space and $f : X \rightarrow X$ a mapping. Let $B \subseteq X$ with $f(B) \subseteq B$ be a subset of X . If there exist a $u \in B$ and some positive integer $n(u)$ such that $f^{n(u)}(u) = u$ and*

$$G(f^{n(u)}(z), f^{n(u)}(u), f^{n(u)}(u)) \leq q \cdot G(z, u, u), \quad (1)$$

for some $q \in [0, 1)$ and all $z \in B$ then u is the unique fixed point of f in B and $\lim_k f^k(y_0) = u$ for each $y_0 \in B$.

Proof. By (1), u is the unique fixed point for the mapping $f^{n(u)}$ in B . Then $f(u) = f(f^{n(u)}(u)) = f^{n(u)}(f(u))$ implies that $f(u) = u$.

Let $y_0 \in B$. For m sufficiently large write $m = k \cdot n(u) + r$, $k \geq 1$ and $1 \leq r < n(u)$. Then

$$\begin{aligned} G(f^m(y_0), u, u) &= G(f^{k \cdot n(u) + r}(y_0), f^{n(u)}(u), f^{n(u)}(u)) \\ &\leq q \cdot G(f^{(k-1)n(u) + r}(y_0), u, u) \leq \dots \\ &\leq q^k \cdot G(f^r(y_0), u, u) \leq q^k \cdot M \end{aligned}$$

for $M = \max \{G(f^p(y_0), u, u) \mid 1 \leq p \leq n(u) - 1\}$, implies that $\lim_k f^k(y_0) = u$.

For $f : X \rightarrow X$ the set $\mathcal{O}(f; x_0) = \{f^n(x_0) : n \in \mathbb{N}\}$ is called the orbit for $x_0 \in X$.

Theorem 2.1. *Let (X, G) be a complete G -metric space and let $f : X \rightarrow X$ be a mapping. Suppose that exists $B \subseteq X$ such that:*

- a) $f(B) \subseteq B$;
- b) For some $x_0 \in B$ limit of any sequence in $\mathcal{O}(f; x_0)$, if exists, belongs to B ;
- c) For some $q \in [0, 1)$ and each $x \in \overline{\mathcal{O}(f; x_0)}$ there is an integer $n(x) \geq 1$ such that

$$G(f^{n(x)}(z), f^{n(x)}(x), f^{n(x)}(x)) \leq q \cdot G(z, x, x) \quad (2)$$

for all $z \in B$.

Then there exists a unique $u \in B$ such that $f(u) = u$ and $f^k(x_0) \rightarrow u$, $k \rightarrow \infty$. Moreover, $f^k(y_0) \rightarrow u$, $k \rightarrow \infty$, for all $y_0 \in B$. Furthermore, if

$$G(f^{n(u)}(z), f^{n(u)}(u), f^{n(u)}(u)) \leq q \cdot G(z, u, u)$$

for each $z \in X$, then u is the unique fixed point in X .

Proof. At first let us show that

$$\sup_m G(f^m(x_0), x_0, x_0) = M < +\infty.$$

For any $m \in \mathbb{N}$ there exist $k, r \in \mathbb{N}$, $1 \leq r \leq n(x_0) - 1$ such that $m = k \cdot n(x_0) + r$. Then

$$\begin{aligned}
G(f^m(x_0), x_0, x_0) &\leq G(f^{kn(x_0)+r}(x_0), f^{n(x_0)}(x_0), f^{n(x_0)}(x_0)) + G(f^{n(x_0)}(x_0), x_0, x_0) \\
&\leq qG(f^{(k-1)n(x_0)+r}(x_0), x_0, x_0) + G(f^{n(x_0)}(x_0), x_0, x_0) \\
&\leq qG(f^{(k-1)n(x_0)+r}(x_0), f^{n(x_0)}(x_0), f^{n(x_0)}(x_0)) + (1+q)G(f^{n(x_0)}(x_0), x_0, x_0) \\
&\leq q^2G(f^{(k-2)n(x_0)+r}(x_0), x_0, x_0) + (1+q)G(f^{n(x_0)}(x_0), x_0, x_0) \leq \dots \\
&\leq q^kG(f^r(x_0), x_0, x_0) + (1+q+\dots+q^{k-1})G(f^{n(x_0)}(x_0), x_0, x_0) \\
&\leq \frac{1}{1-q}C = M < +\infty,
\end{aligned}$$

for $C = \max\{G(f^p(x_0), x_0, x_0) : 1 \leq p \leq n(x_0)\}$.

Now, let us define the sequence $x_k = f^{n(x_{k-1})}(x_{k-1})$, $k \in \mathbb{N}$. Then for each $p \in \mathbb{N}$

$$\begin{aligned}
G(x_k, x_k, x_{k+p}) &= G(f^{n(x_{k-1})}(x_{k-1}), f^{n(x_{k-1})}(x_{k-1}), f^{n(x_{k+p-1})}(x_{k+p-1})) \\
&= G(f^{n(x_{k-1})}(x_{k-1}), f^{n(x_{k-1})}(x_{k-1}), f^{n(x_{k+p-1})}f^{n(x_{k+p-2})}\dots f^{n(x_{k-1})}(x_{k-1})) \\
&\leq qG(x_{k-1}, x_{k-1}, f^{n(x_{k+p-1})}\dots f^{n(x_k)}(x_{k-1})) \leq \dots \\
&\leq q^kG(x_0, x_0, f^{n(x_{k+p-1})}\dots f^{n(x_k)}(x_0)) \leq q^kM,
\end{aligned}$$

so $\{x_k\}$ is Cauchy sequence and there exists $u = \lim_k x_k$, for some $u \in B$.

Further, for all $k \in \mathbb{N}$,

$$G(f^{n(u)}(u), f^{n(u)}(u), f^{n(u)}(x_k)) \leq qG(u, u, x_k)$$

so $\lim_k f^{n(u)}(x_k) = f^{n(u)}(u)$.

On the other side

$$\begin{aligned}
G(f^{n(u)}(x_k), x_k, x_k) &= G(f^{n(u)}f^{n(x_{k-1})}(x_{k-1}), f^{n(x_{k-1})}(x_{k-1}), f^{n(x_{k-1})}(x_{k-1})) \\
&\leq qG(f^{n(u)}(x_{k-1}), x_{k-1}, x_{k-1}) \leq \dots \leq q^kG(f^{n(u)}(x_0), x_0, x_0)
\end{aligned}$$

which implies that $\lim_k G(f^{n(u)}(x_k), x_k, x_k) = 0$.

Since G is continuous it means that

$$G(f^{n(u)}(u), u, u) = 0.$$

Hence $f^{n(u)}(u) = u$. By Lemma 2.1, u is the unique fixed point of f in B and $\lim_k f^k(y_0) = u$ for each $y_0 \in B$. With additional condition on X uniqueness in X is obvious. This complete the proof.

For $B = X$ we have the next corollary.

Corollary 2.1. *Let (X, G) be G -metric space and $f : X \rightarrow X$ be a mapping. Suppose that exist a $q \in [0, 1)$ and a point $x_0 \in X$ with $\overline{\mathcal{O}(f; x_0)}$ complete such that for each $x \in \overline{\mathcal{O}(f; x_0)}$ there exists an integer $n(x) \geq 1$ and*

$$G(f^{n(x)}(z), f^{n(x)}(x), f^{n(x)}(x)) \leq q \cdot G(z, x, x),$$

for all $z \in X$.

Then there exists a unique $u \in X$ such that $f(u) = u$ and $f^k(y_0) \rightarrow u$, $k \rightarrow \infty$, for any $y_0 \in X$.

For $(x) = 1$, independently on x , we have the next corollary.

Corollary 2.2. *Let (X, G) be a G -metric space and $f : X \rightarrow X$ be a mapping. Suppose that there exist $q \in [0, 1)$ and a point $x_0 \in X$, with $\overline{\mathcal{O}(f; x_0)}$ complete, such that for each $x \in \overline{\mathcal{O}(f; x_0)}$*

$$G(f(z), f(z), f(z)) \leq qG(z, x, x) \quad (3)$$

for all $z \in X$. Then there exists a unique $u \in X$ such that $f(u) = u$ and $f^k(y_0) \rightarrow u$, $k \rightarrow \infty$, for any $y_0 \in X$.

Remark 1. *If G is symmetric then $d_G(x, z) = 2G(z, x, x)$, and thus inequation (3) becomes*

$$d_G(f(x), f(z)) \leq qd_G(x, z).$$

Remark 2. *Let us note in Theorem 2.1 and so in Corollary 2.2 mapping f is not necessarily continuous.*

Remark 3. *For $B = \mathcal{O}(f; x_0)$ one can see that Theorem 2.1 in fact generalized Theorem 2.1, from [1].*

Now, using the same methods as in proof of Theorem 2.1 one can prove the next statement.

Theorem 2.2. *Let (X, G) be a complete G -metric space $f : X \rightarrow X$ a mapping and $u, x_0 \in X$ with $\lim_k f^k(x_0) = u$. If f satisfies (1) for some $q \in [0, 1)$ and all $z \in X$ then u is a unique fixed point for f in X . Moreover, $\lim_k f^k(y_0) = u$ for any $y_0 \in X$.*

As an easy consequence of Corollary 2.2 we have the next common fixed point result.

Theorem 2.3. *Let $\{f_k\}$ be a sequence of selfmappings on a complete G -metric space (X, G) . If for some $q \in [0, 1)$ and each $x \in X$ there exists integer $n(x) \geq 1$ such that for all $z \in X$*

$$G(f_i^{n(x)}(z), f_j^{n(x)}(x), f_k^{n(x)}(x)) \leq q \cdot G(z, x, x)$$

whenever at least two of i, j, k are equal, then there exists a unique common fixed point for sequence $\{f_k\}$.

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