Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Filomat **25:2** (2011), 79–89 DOI: 10.2298/FIL1102079I

IMPACT OF OCCASIONALLY WEAKLY COMPATIBLE PROPERTY ON COMMON FIXED POINT THEOREMS FOR EXPANSIVE MAPPINGS

M. Imdad, Javid Ali and V. Popa

Abstract

A general common fixed point theorem for two pairs of occasionally weak compatible expansive mappings satisfying a significantly enriched implicit function is proved in symmetric spaces which generalizes several previously known results. Some related results and illustrative examples are also discussed.

1 Introduction and preliminaries

A metrical common fixed point theorem is broadly comprised of conditions on commutativity, continuity, completeness and contraction besides suitable containment of range of one map into the range of the other. To prove new results, the researchers of this domain are required to improve one or more of these conditions. With a view to improve the commutativity condition in such results, Sessa [15] initiated the idea of weak commutativity which was received well by the researchers of this direction. In process, several conditions of weak commutativity were introduced and utilized to prove new common fixed point theorems whose lucid survey (up to 2001) is available in Murthy [11]. In the last few years the notion of weak compatibility due to Jungck [7] has been extensively utilized to prove new results as it is a minimal condition merely requiring the commutativity at the set of coincidence points of the pair.

Definition 1.1[8]. A point $x \in X$ is called a coincidence point of a pair of self mappings (A, S) defined on a nonempty set X iff $Ax = Sx$. In what follows, we shall call $w = Ax = Sx$ a point of coincidence of A and S.

2010 *Mathematics Subject Classifications.* 47H10, 54H25.
Key words and Phrases. Symmetric spaces, fixed points, occasionally weakly compatible mappings and implicit function.
Received: May 12, 2010
Communicated by Vladimir Rakocevic

Definition 1.2. A pair of self mappings (A, S) defined on a nonempty set X is said to be weakly compatible if the pair commutes at the set of coincidence points of A and S.

Here it may be pointed out that a pair of self mappings without coincidence point is also weakly compatible as the requirement of weak compatibility is met out vacuously. But such pairs are uninteresting in common fixed point considerations as opposed to a pair of weakly compatible mappings with at least one coincidence point which one may term as nontrivial weakly compatible pair. In an attempt to coin a proper generalization of nontrivial weakly compatible pair, Al-Thagafi and Shahzad [1] introduced the notion of occasionally weakly compatible pair (abbreviated as OWC in the sequel) as follows.

Definition 1.3.[1] A pair of self mappings (A, S) defined on a nonempty set X is said to be occasionally weakly compatible iff there is at least one coincidence point of the pair at which pair commutes.

Lemma 1.1[8]. Let X be a set and A and S are OWC self mappings on X. If A and S have a unique point of coincidence $w = Ax = Sx$, then w is the unique common fixed point of A and S.

Let X be a nonempty set. A symmetric is a nonnegative real function d on $X \times X$ such that

(a) $d(x, y) = 0$ if and only if $x = y$,

(b) $d(x, y) = d(y, x) \ \forall \ x, y \in X$.

As expected by (X, d), we denote a nonempty set X equipped with a symmetric d on X and call it a symmetric space. A space (X, d) in which limiting points are defined in the usual way is also sometime called an E-space. The idea of E-spaces is due to Frechet and Menger. Note that every metric space is symmetric but not conversely.

The purpose of this paper is to prove a general common fixed point theorem for OWC expansive mappings satisfying an implicit function in symmetric spaces which generalizes several results from the literature. In process, our results generalize several fixed point theorems in following respects.

(a) The class of spaces is widened from the class of metric spaces to the class of symmetric spaces.

(b) The class of implicit functions is also enriched significantly as it requires merely one condition to satisfy.

(c) The condition on completeness/compactness of the space is completely relaxed.

(d) The condition of weak compatibility is weakened to OWC.

(e) The condition of surjectivity of the mappings is completely relaxed. Consequently the condition of the required containment of the ranges of the involved mappings is not essential.

(f) The condition of continuity of the involved mappings is also relaxed.

2 Implicit Function

Recently, motivated by Popa [13], Imdad and Khan [5] defined the implicit function to prove common fixed point theorems for expansive type mappings. In order to describe the implicit function, let Φ the set of all continuous functions $F : \Re_+^6 \to \Re$ satisfying the following conditions:

(F_1) : F is nondecreasing in variables t_5 and t_6,
(F_2) : there exists $h \in (1, \infty)$ such that for $u, v \geq 0$ with
(F_{2a}) : $F(u, v, u, v, u + v, 0) \geq 0$, or
(F_{2b}) : $F(u, v, v, u, 0, u + v) \geq 0$ implies $u \geq hv$,
(F_3) : $F(u, u, 0, 0, u, u) < 0$ for all $u > 0$.

The following examples along with required verifications are available in [5].

Example 2.1. Define $F(t_1, \ldots, t_6) : \Re_+^6 \to \Re$ as

$$F(t_1, t_2, \ldots, t_6) = t_1 - at_2 + \frac{b\sqrt{t_5 t_6}}{1 + ct_3 + dt_4}$$

with $b, c, d \geq 0$ and $a > 1 + b$.

Example 2.2. Define $F(t_1, \ldots, t_6) : \Re_+^6 \to \Re$ as

$$F(t_1, t_2, \ldots, t_6) = t_1 - at_2 - b\frac{t_1}{1 + \sqrt{t_5 t_6}} + c \ \min\{t_3, t_4\}$$

with $a, b, c \geq 0$ and $a > 1$.

Example 2.3. Define $F(t_1, \ldots, t_6) : \Re_+^6 \to \Re$ as

$$F(t_1, t_2, \ldots, t_6) = t_1 - at_2 - \frac{bt_3 + ct_4}{1 + \sqrt{t_5 t_6}}$$

where $0 \leq b < 1,\ 0 \leq c < 1$ and $a > 1$.

Example 2.4. Define $F(t_1, \ldots, t_6) : \Re_+^6 \to \Re$ as

$$F(t_1, t_2, \ldots, t_6) = t_1 - [at_2^p + bt_3^p + ct_4^p]^{1/p} - \frac{1}{2}\left(\frac{t_1}{1 + \sqrt{t_5 t_6}}\right)$$

where $a > 1$ and $0 \leq b, c < \frac{1}{2^p}, p \in N$.

Example 2.5. Define $F(t_1, \ldots, t_6) : \Re_+^6 \to \Re$ as

$$F(t_1, t_2, \ldots, t_6) = t_1 - \left[at_2^2 + \frac{bt_3^2 + ct_4^2}{1 + t_5 t_6} \right]^{1/2}$$

where $0 \leq b, c < 1$ and $a^{\frac{1}{2}} > 1$.

In the presence of OWC, it has become possible to enlarge the class Φ of implicit function by defining the following class. To describe it, let Ψ be the class of all functions $F : \Re_+^6 \to \Re$ satisfying the following condition.

$(F_3') : F(u, u, 0, 0, u, u) \leq 0$ for all $u > 0$.

Naturally the class Ψ is larger than Φ. To substantiate this view point, we furnish the following examples.

Example 2.6. Define $F(t_1, \ldots, t_6) : \Re_+^6 \to \Re$ as

$$F(t_1, t_2, \ldots, t_6) = t_1 - \max \left\{ t_2, t_3, t_4, \frac{t_5 + t_6}{2} \right\} - at_2 \min\{t_2, t_3, t_4\}$$

where $a \geq 0$.

Example 2.7. Define $F(t_1, \ldots, t_6) : \Re_+^6 \to \Re$ as

$$F(t_1, t_2, \ldots, t_6) = t_1 - \min\{t_2, t_3 + t_5, t_4 + t_6\}.$$

Example 2.8. Define $F(t_1, \ldots, t_6) : \Re_+^6 \to \Re$ as

$$F(t_1, t_2, \ldots, t_6) = t_1 - \max \left\{ t_2, \frac{t_3 + t_4}{2}, \frac{t_5 + t_6}{2} \right\}.$$

Example 2.9. Define $F(t_1, \ldots, t_6) : \Re_+^6 \to \Re$ as

$$F(t_1, t_2, \ldots, t_6) = t_1^2 - \max\{t_2^2, t_3^2, t_4^2\} - a \max\{t_3 t_5, t_4 t_6\} - bt_5 t_6$$

where $a, b \geq 0$.

Example 2.10. Define $F(t_1, \ldots, t_6) : \Re_+^6 \to \Re$ as

$$F(t_1, t_2, \ldots, t_6) = t_1 - t_2 - a \max\{t_3, t_4, t_5, t_6\} - b \max\{t_3 + t_4, t_5 + t_6\}$$

where $a, b \geq 0$.

Example 2.11. Define $F(t_1, \ldots, t_6) : \Re_+^6 \to \Re$ as

$$F(t_1, t_2, \ldots, t_6) = \psi(t_1) - \max\{t_2, t_3, t_4, \min\{t_5, t_6\}\}$$

where $\psi : \Re^+ \to \Re$ is a function such that $\psi(t) \geq t$ for all $t \geq 0$.

Example 2.12. Define $F(t_1, \ldots, t_6) : \Re_+^6 \to \Re$ as

$$F(t_1, t_2, \ldots, t_6) = t_1 - t_2 - a \min\{t_3, t_4\} - b \min\{t_3, t_5, t_6\}$$

where $a, b \geq 0$.

Example 2.13. Define $F(t_1, \ldots, t_6) : \Re_+^6 \to \Re$ as

$$F(t_1, t_2, \ldots, t_6) = t_1 - \phi\left(\max\left\{t_2, \frac{t_3 + t_4}{2}, \frac{t_5 + t_6}{2}\right\}\right)$$

where $\phi : \Re^+ \to \Re$ is a function such that $\phi(t) \geq t$ for all $t \geq 0$.

Example 2.14. Define $F(t_1, \ldots, t_6) : \Re_+^6 \to \Re$ as

$$F(t_1, t_2, \ldots, t_6) = t_1 - \phi(t_2, t_3, t_4, t_5, t_6)$$

where $\phi : \Re_+^5 \to \Re$ is a function such that $\phi(t, 0, 0, t, t) \geq t$ for all $t \geq 0$.

Example 2.15. Define $F(t_1, \ldots, t_6) : \Re_+^6 \to \Re$ as

$$F(t_1, t_2, \ldots, t_6) = t_1 - t_2 - a_1 t_3 - a_2 t_4 - a_3 t_5 - a_4 t_6$$

where $a_i \geq 0$ for $i = 1, 2, 3, 4$.

Example 2.16. Define $F(t_1, \ldots, t_6) : \Re_+^6 \to \Re$ as

$$F(t_1, t_2, \ldots, t_6) = t_1 - \frac{(at_2^2 + bt_3t_4 + ct_5t_6)}{t_2 + t_3 + t_4}$$

where $a + c \geq 1$.

Example 2.17. Define $F(t_1, \ldots, t_6) : \Re_+^6 \to \Re$ as

$$F(t_1, t_2, \ldots, t_6) = t_1 - t_2 - (t_3 + t_4) - \frac{t_5^2 + t_6^2}{t_5 + t_6}.$$

Example 2.18. Define $F(t_1, \ldots, t_6) : \Re_+^6 \to \Re$ as

$$F(t_1, t_2, \ldots, t_6) = t_1 - t_2 - \frac{t_3t_4 + t_5t_6}{t_5 + t_6}.$$

Example 2.19. Define $F(t_1, \ldots, t_6) : \Re_+^6 \to \Re$ as

$$F(t_1, t_2, \ldots, t_6) = t_1^2 - t_2^2 - a\frac{t_5t_6}{1 + t_3t_4}$$

where $a \geq 0$.

3 Main Results

Theorem 3.1. Let A, B, S and T be self mappings of a symmetric space (X, d) satisfying the inequality

$$F(d(Ax, By), d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(Ax, Ty), d(By, Sx)) > 0 \quad (3.1)$$

for all $x, y \in X$, where F satisfy condition (F_3'). Then A, B, S and T have at most one common fixed point.

Proof. The proof is similar to the proof of Theorem 4.1 from [3].

Theorem 3.2. Let A, B, S and T be self mappings of a symmetric space (X, d) satisfying the inequality (3.1). If the pairs (A, S) and (B, T) are each OWC, then A, B, S and T have a unique common fixed point.

Proof. Since the pairs (A, S) and (B, T) are each OWC, there exist $x, y \in X$ such that $Ax = Sx$ and $By = Ty$. We assert that $Ax = By$. Otherwise, by (3.1), we have

$$F(d(Ax, By), d(Ax, By), 0, 0, d(Ax, By), d(By, Ax)) > 0,$$

a contradiction to condition (F_3'). Therefore, $Ax = By$; i.e. $Ax = Sx = By = Ty$. Moreover, if there is another point $v \in X$ such that $Av = Sv$, then using (3.1) we obtain

$$F(d(Av, By), d(Av, By), 0, 0, d(Av, By), d(By, Av)) > 0,$$

a contradiction to condition (F_3'). Hence $Av = Sv = By = Ty$. Therefore $Av = By$ and $w = Ax = Sx$ is the unique point of coincidence of the pair (A, S). By Lemma 1.1, w is the only common fixed point of the pair (A, S). Similarly, there exists a unique point $z \in X$ such that $z = Bz = Tz$. Suppose that $w \neq z$. Using (3.1), we have

$$F(d(Aw, Bz), d(Sw, Tz), d(Aw, Sw), d(Bz, Tz), d(Aw, Tz), d(Bz, Sw)) > 0,$$

$$\text{or} \qquad F(d(w, z), d(w, z), 0, 0, d(w, z), d(z, w)) > 0,$$

a contradiction to condition (F_3'). Therefore $w = z$ and z is a common fixed point of A, B, S and T. In view of Theorem 3.1, z is the unique common fixed point of A, B, S and T.

Corollary 3.1. Let (X, d) be a symmetric space. Suppose that A and S are self mappings of X satisfy the inequality

$$F(d(Ax, Ay), d(Sx, Sy), d(Ax, Sx), d(Ay, Sy), d(Ax, Sy), d(Ay, Sx)) > 0 \quad (3.2)$$

for all $x, y \in X$, where F enjoys condition (F_3').

If the pair (A, S) is OWC, then A and S have a unique common fixed point.

Corollary 3.2. The conclusions of Theorem 3.2 remain true if the inequality (3.1) is replaced by the following conditions.

(a_1) $d(Ax,By) > ad(Sx,Ty) - b\dfrac{\sqrt{d(Ax,Ty)d(By,Sx)}}{1+cd(Ax,Sx)+dd(By,Ty)}$

where $b,c,d \geq 0$ and $a > 1+b$.

(a_2) $d(Ax,By) > ad(Sx,Ty) + b\dfrac{d(Ax,By)}{1+\sqrt{d(Ax,Ty)d(By,Sx)}} - c\min\{d(Ax,Sx),d(By,Ty)\}$

with $a,b,c \geq 0$ and $a > 1$.

(a_3) $d(Ax,By) > ad(Sx,Ty) + \dfrac{bd(Ax,Sx)+cd(By,Ty)}{1+\sqrt{d(Ax,Ty)d(By,Sx)}}$

where $0 \leq b < 1$, $0 \leq c < 1$ and $a > 1$.

(a_4) $d(Ax,By) > (ad^p(Sx,Ty) + bd^p(Ax,Sx) + cd^p(By,Ty))^{\frac{1}{p}} + \dfrac{1}{2}\left(\dfrac{d(Ax,By)}{1+\sqrt{d(Ax,Ty)d(By,Sx)}}\right)$

where $a > 1$ and $0 \leq b,c < \frac{1}{2^p}, p \in N$.

(a_5) $d(Ax,By) > \left(ad^2(Sx,Ty) + \dfrac{bd^2(Ax,Sx)+cd^2(By,Ty)}{1+d(Ax,Ty)d(By,Sx)}\right)^{\frac{1}{2}}$

where $0 \leq b,c < 1$ and $a^{\frac{1}{2}} > 1$.

(a_6) $d(Ax,By) > \max\left\{d(Sx,Ty), d(Ax,Sx), d(By,Ty), \dfrac{d(Ax,Ty)+d(By,Sx)}{2}\right\}$

$$+ad(Sx,Ty)\min\{d(Sx,Ty),d(Ax,Sx),d(By,Ty)\}$$

where $a \geq 0$.

(a_7) $d(Ax,By) > \min\{d(Sx,Ty), d(Ax,Sx)+d(Ax,Ty), d(By,Ty)+d(By,Sx)\}$.

(a_8) $d(Ax,By) > \max\left\{d(Sx,Ty), \dfrac{d(Ax,Sx)+d(By,Ty)}{2}, \dfrac{d(Ax,Ty)+d(By,Sx)}{2}\right\}$.

(a_9) $d^2(Ax,By) > \max\{d^2(Sx,Ty), d^2(Ax,Sx), d^2(By,Ty)\} + a\max\{d(Ax,Sx)d(Ax,Ty),$

$$d(By,Ty)d(By,Sx)\} + bd(Ax,Ty)d(By,Sx)$$

where $a,b \geq 0$.

(a_{10}) $d(Ax,By) > d(Sx,Ty) + a\max\{d(Ax,Sx), d(By,Ty), d(Ax,Ty), d(By,Sx)\}$

$$+b\max\{d(Ax,Sx)+d(By,Ty), d(Ax,Ty)+d(By,Sx)\}$$

where $a, b \geq 0$.

(a_{11}) $\psi(d(Ax, By)) > \max\{d(Sx, Ty), d(Ax, Sx), d(By, Ty), \min\{d(Ax, Ty), d(By, Sx)\}\}$

where $\psi : \Re^+ \to \Re$ is a function such that $\psi(t) \geq t$ for all $t \geq 0$.

(a_{12}) $d(Ax, By) > d(Sx, Ty) + a\min\{d(Ax, Sx), d(By, Ty)\} + b\min\{d(Ax, Sx), d(Ax, Ty), d(By, Sx)\}$ where $a, b \geq 0$.

(a_{13}) $d(Ax, By) > \phi\left(\max\left\{d(Sx, Ty), \frac{d(Ax, Sx) + d(By, Ty)}{2}, \frac{d(Ax, Ty) + d(By, Sx)}{2}\right\}\right)$

where $\phi : \Re^+ \to \Re$ is a function such that $\phi(t) \geq t$ for all $t \geq 0$.

(a_{14}) $d(Ax, By) > \phi(d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(Ax, Ty), d(By, Sx))$

where $\phi : \Re^5_+ \to \Re$ is a function such that $\phi(t, 0, 0, t, t) \geq t$ for all $t \geq 0$.

(a_{15}) $d(Ax, By) > d(Sx, Ty) + a_1 d(Ax, Sx) + a_2 d(By, Ty) + a_3 d(Ax, Ty) + a_4 d(By, Sx)$

where $a_i \geq 0$ for $i = 1, 2, 3, 4$.

(a_{16}) $d(Ax, By) > \dfrac{ad^2(Sx, Ty) + bd(Ax, Sx)d(By, Ty) + cd(Ax, Ty)d(By, Sx)}{d(Sx, Ty) + d(Ax, Sx) + d(By, Ty)}$

where $a + c \geq 1$.

(a_{17}) $d(Ax, By) > d(Sx, Ty) + d(Ax, Sx) + d(By, Ty) + \dfrac{d^2(Ax, Ty) + d^2(By, Sx)}{d(Ax, Ty) + d(By, Sx)}$.

(a_{18}) $d(Ax, By) > d(Sx, Ty) + \dfrac{d(Ax, Sx)d(By, Ty) + d(Ax, Ty)d(By, Sx)}{d(Ax, Ty) + d(By, Sx)}$.

(a_{19}) $d^2(Ax, By) > d^2(Sx, Ty) + a\dfrac{d(Ax, Ty)d(By, Sx)}{1 + d(Ax, Sx)d(By, Ty)}$

where $a \geq 0$.

Proof. The proof follows from Theorem 3.2 and Examples 2.1-2.19.

Remark 3.1. Corollaries corresponding to conditions (a_1) to (a_{19}) are new results as these results are proved under less commutativity requirement along with several other improvements as highlighted earlier. Some of these corollaries generalize several previously known relevant results contained in [2-5,9,10,12,14,16,17] whereas others can be viewed as expansion analogues of certain contraction conditions discussed in [6,8,13].

4 Illustrative Examples

In what follows, we furnish examples demonstrating the utility of Theorem 3.2 over the earlier results especially those contained in [2,4,5,10,12,16] and some others. The first of these examples exhibits how extensively OWC facilitates the fixed point theorems for expansive mappings despite the fact that the existence of coincidence point of the pairs is assumed.

Example 4.1. Consider $X = (0, 128)$ equipped with the symmetric $d(x, y) = (x - y)^2$. Define $A, B, S, T : X \to X$ as

$$A(x) = \begin{cases} 2x^4 - 1 & 1 \leq x \leq 2 \\ 7 & \text{otherwise,} \end{cases} \qquad B(x) = \begin{cases} 2x^6 - 1 & 1 \leq x \leq 2 \\ 3 & \text{otherwise,} \end{cases}$$

$$S(x) = \begin{cases} x^2 & 1 \leq x \leq 2 \\ 2 & \text{otherwise,} \end{cases} \qquad T(x) = \begin{cases} x^3 & 1 \leq x \leq 2 \\ \sqrt{2} & \text{otherwise.} \end{cases}$$

Notice that the mappings A, B, S and T are not continuous but the pairs (A, S) and (B, T) are OWC as $AS1 = SA1$ and $BT1 = TB1$. Define the implicit function $F : \Re_+^6 \to \Re$ as $F(t_1, t_2, \ldots, t_6) = t_1 - \min\{t_2, t_3 + t_5, t_4 + t_6\}$.

Towards the verification of implicit function, let $x, y \in [1, 2]$. Then we have

$$d(Ax, By) = (2x^4 - 2y^6)^2 = 4(x^2 + y^3)^2(x^2 - y^3)^2$$
$$> (x^2 - y^3)^2 = d(Sx, Ty) \geq m(x, y)$$

where $m(x, y) = \min\{d(Sx, Ty), d(Ax, Sx) + d(Ax, Ty), d(By, Ty) + d(By, Sx)\}$. When $x, y \notin [1, 2]$, then

$$d(Ax, By) = 4^2 > (2 - \sqrt{2})^2 = d(Sx, Ty) \geq m(x, y).$$

Finally, if $x \in [1, 2]$ and $y \notin [1, 2]$, then we have

$$d(Ax, By) = (2x^4 - 4)^2 = 4(x^2 + \sqrt{2})^2(x^2 - \sqrt{2})^2 > (x^2 - \sqrt{2})^2 = d(Sx, Ty) \geq m(x, y).$$

Thus all the conditions of Theorem 3.2 are satisfied and 1 is the unique common fixed point of the mappings A, B, S and T.

Here it may be pointed out that in the presence of OWC, we never require conditions of continuity, completeness, compactness and containment of ranges of the involved mappings.

Our next example is constructed to demonstrate the fact that the requirement of OWC is necessary in Theorem 3.2.

Example 4.2. Consider $X = \{1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots, \frac{1}{2^n}, \ldots\}$ equipped with a symmetric $d(x, y) = (x - y)^2$. Define $A, S : X \to X$ as $A(\frac{1}{2^n}) = \frac{1}{2^{n+1}}$ and $S(\frac{1}{2^n}) = \frac{1}{2^{n+2}}$, when $n = 0, 1, 2, \ldots$. Define implicit function F as in Example 4.1. Then

$$d\left(A(\frac{1}{2^n}), A(\frac{1}{2^m})\right) = \left(\frac{1}{2^{n+1}} - \frac{1}{2^{m+1}}\right)^2 = 4\left(\frac{1}{2^{n+2}} - \frac{1}{2^{m+2}}\right)^2$$

$$> \left(\frac{1}{2^{n+2}} - \frac{1}{2^{m+2}}\right)^2 = d(Sx, Sy) \geq m(x, y)$$

where $m(x, y)$ (for $B = A$ and $T = S$) is the same as in Example 4.1. Thus all the conditions of Theorem 3.2 are satisfied except OWC. Notice that A and S have no common fixed point.

Acknowledgements. The research of the second author is supported by the Department of Atomic Energy, Government of India under the NBHM-Post Doctoral Fellowship Program No. 40/2/2008-R& D II/6173.

References

[1] M. A. Al-Thagafi and N. Shahzad, *Generalized I-nonexpansive selfmaps and invariant approximations*, Acta Math. Sin.(Engl. Ser.) 24(2008), 867–876.

[2] Z. P. Daffer and H. Kaneko, *On expansive mappings*, Math. Japon. 37(1992), 733–735.

[3] A. Djoudi, *General fixed point theorems for weakly compatible maps*, Demonstratio Math. 38(2005), 197–205.

[4] A. A. Gillespie and B. B. Williams, *Fixed point theorems for expanding maps*, Applicable Anal. 14(3)(1983), 161–165.

[5] M. Imdad and T. I. Khan, *Fixed point theorems for some expansive mappings via implicit relations*, Nonlinear Anal. Forum 9(2)(2004), 209–218.

[6] G. Jungck, *Compatible mappings and common fixed points*, Internat. J. Math. Math. Sci. 9(1986), 771–779.

[7] G. Jungck, *Common fixed points for noncontinuous nonself maps on nonmetric spaces*, Far East J. Math. Sci. 4(1996), 199–215.

[8] G. Jungck and B. E. Rhoades, *Fixed point theorems for occasionally weakly compatible mappings*, Fixed Point Theory 7(2)(2006), 287–296.

[9] S. M. Kang and B. E. Rhoades, *Fixed points for four mappings*, Math. Japon. 37(1992), 1053–1059.

[10] S. Kumar, *Common fixed point theorems for expansions mappings in various spaces*, Acta Math. Hungar. 118(2008), 9–28.

[11] P. P. Murthy, *Important tools and possible applications of metric fixed point theory*, Nonlinear Anal. 47(2001), 3479–3490.

[12] H. K. Pathak and R. Tiwari, *Fixed point theorems for expansion mappings satisfying implicit relations*, Filomat 20(2006), 43–57.

[13] V. Popa, *Some fixed point theorems for compatible mappings satisfying an implicit relation*, Demonstratio Math. 32(1999), 157–163.

[14] B. E. Rhoades, *An expansion mapping theorem*, Jnanabha 23(1993), 151–152.

[15] S. Sessa, *On a weak commutativity condition of mappings in fixed point considerations*, Publ. Inst. Math. 32(1982), 149–153.

[16] T. Taniguchi, *Common fixed point theorems on expansive type mappings on complete metric spaces*, Math. Japon. 34(1)(1989), 139–142.

[17] S. Z. Wang, B. Y. Li, Z. M. Gao and K. Iséki, *Some fixed point theorems on expansion mappings*, Math. Japon. 29(1984), 631–636.

Addresses:

M. Imdad
Department of Mathematics, Aligarh Muslim University, Aligarh-202 002, India.
E-mail: mhimdad@yahoo.co.in

Javid Ali
Department of Mathematics & Statistics, Indian Institute of Technology Kanpur, Kanpur 208 016, India.
E-mail: javid@math.com

V. Popa
Department of Mathematics, University of Bacau, 5500 Bacau, Romania.
E-mail: vpopa@ub.ro