Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Filomat **25:2** (2011), 91–108 DOI: 10.2298/FIL1102091K

ON CONFORMAL, HARMONIC MAPPINGS AND DIRICHLET'S INTEGRAL

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Abstract

This paper has an expository character, however we present as well some new results and new proofs. We prove a complex version of Dirichlet's principle in the plane and give some applications of it as well as estimates of Dirichlet's integral from below. Some of the results in the plane are generalized to higher dimensions. Roughly speaking, under the appropriate conditions we estimate the n- Dirichlet integral of a mapping u defined on a domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, by the measure of $u(\Omega)$ and show that equality holds if and only if it is injective conformal. Also some sharp inequalities related to the L^2 norms of the radial derivatives of vector harmonic mappings from the unit ball in $\mathbb{R}^n$, $n \geq 2$, are given. As an application, we estimate the 2-Dirichlet integrals of mappings in the Sobolev space W_1^2.

1 Introduction, background, notation and motivation

Let Ω be an open set of the Euclidean space $\mathbb{R}^n$. A twice differentiable mapping $u = (u_1, \dots, u_m) : \Omega \to \mathbb{R}^m$ is called vector harmonic if the real functions u_i, $i = 1, \dots, m$, are harmonic. By $B = B^n$ we denote the unit ball in $\mathbb{R}^n$ and by S^{n-1} the unit $n-1$ dimensional sphere. For $n = 2$ we write U instead of B^2.

For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, we denote by $|x| = (x_1^2 + \dots + x_n^2)^{1/2}$ the norm of x. Let $m = m_n$ denote the usual Lebesgue measure on $\mathbb{R}^n$. Sometimes we use notation $dx = dx_1 \dots dx_n$ and $|A|$ instead of dm_n and $m(A)$, where $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and A is a Lebesgue measurable set in $\mathbb{R}^n$, respectively. By $d\sigma$ we denote the positive Borel measure on S^{n-1} invariant w.r.t. the orthogonal group $O(n)$ normalized such that $\sigma(S^{n-1}) = 1$.

Let $f : S^{n-1} \to \mathbb{R}^m$ be a Lebesgue integrable mapping. Then the mapping

$$u(x) = P[f](x) = \int_{S^{n-1}} \frac{1 - |x|^2}{|x - \eta|^n} f(\eta) d\sigma(\eta), \tag{1.1}$$

2010 *Mathematics Subject Classifications.* Primary 31B05 ; Secondary 31B10, 30C65.
Key words and Phrases. Harmonic maps, Dirichlet's integral,Möbius transformations.
Received: November 1, 2010
Communicated by Dragan S. Djordjević
Research of the second author partially supported by MNTRS, Serbia, Grant No. 144 020

is a harmonic mapping on the unit ball B^n. The Sobolev space $W_1^p(\Omega)$, $1 \le p < \infty$, consists of all real-valued functions u in $L^p(\Omega)$ with weak first partial derivatives in $L^p(\Omega)$. By considering component functions we extend this definition to $\mathbb{R}^m$-valued mappings using the same notation.
If $f \in W_1^p(B^n)$, then the radial limit

$$f_b(\zeta) = f^*(\zeta) = \lim_{r \to 1} f(r\zeta)$$

exists a.e. $\zeta \in S$. So we consider that every $f \in W_1^p(B^n)$ is defined a.e. on S.
For a given domain $\Omega \subset \mathbb{R}^n$, we say that a mapping $u : \Omega \to \mathbb{R}^n$ is quasiregular (abbreviated qr) if

1. u is ACL^n, and
2. there exists a real number K, $1 \le K \le \infty$, such that

$$|u'(x)|^n \le K J_u(x) \quad a.e. \quad on \quad \Omega, \tag{1.2}$$

where $|u'(x)| = \max_{|h|=1} |u'(x)h|$.
In this setting we shortly write that f is a $K-$qr mapping. For properties of qr-mappings see [26], [9]. If f is a $K-$qr and homeomophic mapping then it is called $K-$ quasiconformal or shortly $K-$q.c. Let

$$\|u'(x)\| = \sqrt{\sum_{i,j=1}^{n} (\partial_j u_i(x))^2}$$

denote the Hilbert-Schmidt norm of $u'(x)$, where $\partial_j = \partial_{x_j}$ denotes $j-th$ partial derivative. It is well known that if u is a $K-$qr mapping on Ω, then

$$\|u'(x)\|^n \le n^{n/2} K J_u(x) \quad \text{a.e. on} \quad \Omega.$$

Let

$$e(u) = \left(\sum_{i=1}^{n} |\partial_i u|^2 \right)^{1/2}$$

and $e_p(u) = e(u)^p$; we also use the notation $\frac{\partial u}{\partial x_i}$ instead of $\partial_i u$. In particular,

$$e_n(u)(x) = \left(\sum_{i=1}^{n} |\partial_i u|^2 \right)^{n/2};$$

note that $e(u)(x) = \|u'(x)\|$ and $e_n(f) = \|u'(x)\|^n$. For a given domain $\Omega \subset \mathbb{R}^n$, let

$$D_p(u) := \int_\Omega e(u)^p(x)\, dm_n(x).$$

We will mainly consider the integrals D_n and D_2 here.

We only need a version of Dirichlet's principle for the unit ball B^n.

Theorem A (Dirichlet's principle)**.** *Let $u \in W_1^2(B^n)$. Then there exists $u_0 \in W_1^2(B^n)$ such that u_0 is harmonic in B^n and $u_0^* = u^*$ on S^{n-1}. Furthermore, $D_2(u) \geq D_2(u_0)$. For the proof see for example [20], [5].*

For a similar statement for n-energy functional $D_n(u)$ see [9]. Note if u^* is continuous on S, then u_0 is continuous on $\overline{B}$.

First we consider the case $n = 2$. In subsection 2.1, using Dirichlet's principle, the inequalities (2.3), (2.5) and (2.6) are proved, which give the estimate of D_2 from below. Note that the results of this subsection can be generalized to planar and space domains and they should be considered as the motivation for the rest of the paper.

In subsection 2.2 the complex version of Dirichlet's principle is proved and as an immediate application of it the Kühnau-Lehto area theorem for qc mappings with conformal extension is obtained.

In section 3, we present results obtained in [10]. We present Theorem 3.3, which roughly speaking states that "absolutely conformal" mappings are conformal, and a several dimension version of (2.3) stated as Theorem 3.5.

If f is qr we estimate Dirichlet's integral $D_n(f)$ from above; this can be related to the estimates in section 2 which give short proof of Kühnau-Lehto area theorem. Note that, using Theorems C, D and E stated in this section, the interested reader can derive various versions and generalizations of Theorem 3.5.

In section 4, several dimension versions of (2.6) and (2.5) are proved: we estimate L^2-norm of the radial derivative of a harmonic function defined on B^n and using it Dirichlet's integral D_2 of a function $f \in W_1^2$ defined on B^n, from below by the square of L^2 norm of the boundary function on S.

After writing this paper we realized that if we limit the considerations to injective mappings then some results presented here can be related to the recent results of K.Astala, T. Iwaniec, G. Martin, J. Onninen, S. Hencl and P. Koskela; namely, the study of connection between extremal mappings of finite distortion and harmonic mappings was initiated in [1] and further investigated in [8].

In particular, Theorem 10.8 [1] and Theorem 2.1 [8] (stated here as Theorem B) are interesting concerning our work:

Theorem B. *Let Ω and Ω' be planar domains, f a homemorphism of Ω onto Ω' and g the inverse of f. Suppose that $g \in W_{loc}^{1,1}$ and $\int_{\Omega'} \mathbb{K}(w,g)du\,dv < \infty$ (where $\mathbb{K}(z,g)$ is the outer distortion function), then $f \in W_{loc}^{1,2}$ and*

$$D_2(f) = \int_\Omega \|f'(z)\|^2 dx\,dy = 2\int_{\Omega'} \mathbb{K}(w,h)du\,dv < \infty\,. \tag{1.3}$$

Using the theorem we can prove: If f satisfies the hypothesis of the theorem then $D_2(f) \geq 2|\Omega'|$; the equality holds here if and only if f is injective conformal.

In a similar way, we can use Theorem 10.8 [1] to prove a Sobolev version of Theorem 3.5.

2 Dirichlet's integral in the plane

2.1 2-Dirichlet in the plane

In the plane we use the notation $z = x + iy = re^{i\theta}$, where $r = |z|$ and θ are polar coordinates; and $dm = dx\,dy$. If $h = u + iv$ is a differentiable mapping, it is convenient to use the notation
$\partial h = h_z = \frac{1}{2}(\partial_x h - i\partial_y h)$, $\overline{\partial} h = h_{\overline{z}} = \frac{1}{2}(\partial_x h + i\partial_y h)$ and $|\nabla h|^2 = |\nabla u|^2 + |\nabla v|^2$.
Suppose

1. h is continuous function on $\overline{U}$ and $h \in W_1^2(U)$ and
2. $a_n := \frac{1}{2\pi}\int_0^{2\pi} h^*(e^{i\theta})e^{-in\theta}d\theta$, $n \in \mathbb{Z}$.

We write then

$$h(e^{i\theta}) \sim \sum_{n=-\infty}^{\infty} a_n e^{in\theta} = \sum_{n=1}^{\infty} a_{-n}\, e^{-in\theta} + \sum_{n=0}^{\infty} a_n e^{in\theta}. \tag{2.1}$$

Let

$$H = P[h](z) = \sum_{n=-\infty}^{\infty} a_n r^{|n|} e^{in\theta}, \tag{2.2}$$

$$f = H_1 = C[h](z) = \sum_{n=0}^{\infty} a_n\, z^n \text{ and } g = H_2 = \sum_{n=1}^{\infty} \overline{a_{-n}}\, z^n.$$

It is easy to verify that

$$H = H_1 + \overline{H_2},\ \partial H = H_1',\ \bar{\partial} H = \overline{H_2'} \text{ and } |\nabla H|^2 = 2\left(|H_1'|^2 + |H_2'|^2\right).$$

Define

$$A := \iint_U |\,\partial H\,|^2\,dxdy = \pi\sum_{n=1}^{\infty} n\,|\,a_n\,|^2, \quad B := \iint_U |\,\bar{\partial} H\,|^2\,dxdy = \pi\sum_{n=1}^{\infty} n\,|\,a_{-n}\,|^2\,.$$

It is clear that $D_2(H) = 2(A + B)$. Let

$$P := \int_U J_h(z)dxdy.$$

Using the index of the curve γ defined by $w = h(e^{i\theta}), 0 \le \theta \le 2\pi$, or Lemma 3.7.2 [9], one can verify that

$$P = \int_U J_H(z)dxdy.$$

Hence

$$P = \int_U (|f'(z)|^2 - |g'(z)|^2)dxdy = A - B$$

and therefore, by Dirichlet's principle, $D_2(h) \geq D_2(H) \geq 2P$.

Using
$$\int_0^{2\pi} |h(e^{i\theta})|^2 d\theta = 2\pi \sum_{n=-\infty}^{\infty} n \mid a_n \mid^2$$
and
$$D_2(H) = 2\pi \sum_{n=-\infty}^{\infty} |n| \mid a_n \mid^2,$$
we find, if $a_0 = 0$,
$$D_2(H) \geq \int_0^{2\pi} |h(e^{i\theta})|^2 d\theta.$$
On the other hand, by Dirichlet's principle,
$$D_2(h) \geq D_2(H).$$
Hence
$$D_2(h) \geq \int_0^{2\pi} |h(e^{i\theta})|^2 d\theta.$$
Since
$$e_2(h)(z) \geq 2(|\partial_x u\, \partial_y v| + |\partial_x v\, \partial_y u|) \geq 2|J_h(z)|,$$
we find
$$D_2(h) \geq 2\int_U |J_h(z)| dxdy \geq 2|h(U)|$$
and hence (2.4) (see below). If h is harmonic, then $h = H = f + \overline{g}$. An easy calculation shows
$$\partial_\theta h(z) = i(zf'(z) - \overline{zg'(z)}), \quad \partial_r h = e^{i\theta} f' + \overline{e^{i\theta} g'}$$
and therefore $r\,\partial_r h$ is the harmonic conjugate of $\partial_\theta h$. By using
$$|\partial_r h|^2 = |f'|^2 + |g'|^2 + 2\,\mathrm{Re}(e^{i2\theta} f' g'),$$
we find
$$D_2(h) = 2\int_U |\partial_r h|^2\, dxdy.$$
We can summarize the above consideration as the following result:

Theorem 2.1. *Suppose that h satisfies the condition* (1) *and the above notation. Then*
$$D_2(h) \geq 2P, \tag{2.3}$$
$$D_2(h) \geq 2|h(U)|. \tag{2.4}$$
If $a_0 = 0$, then
$$D_2(h) \geq \int_0^{2\pi} |h(e^{i\theta})|^2 d\theta. \tag{2.5}$$

If h harmonic and $a_0 = 0$ then

$$D_2(h) = 2\int_U |h_r|^2\,dxdy \geq \int_0^{2\pi} |h(e^{i\theta})|^2 d\theta. \tag{2.6}$$

The equality holds in (2.3) if and only if $g' \equiv 0$, i.e. h is conformal. If h is smooth then the equality holds in (2.4) if and only if h is constant or h is univalent conformal mapping. The equality holds in (2.5) if and only if $h(z) = az + b\overline{z}$, $a, b \in \mathbb{C}$.

The case of equality in (2.4) is considered in Theorem 3.5 below. We generalize those inequalities to $n \geq 2$; see Theorem 2.2, Theorem 3.5, Corollary 3.8, Theorem 3.9, Theorem 4.2, Theorem 4.5 and Corollary 4.6.

2.2 Complex version of Dirichlet's principle and Kühnau-Lehto theorem

Let h satisfy the relation (2.1), $H = P[h](z) = \sum_{n=-\infty}^{\infty} a_n r^{|n|} e^{in\theta}$ and

$$A := \pi \sum_{n=1}^{\infty} n \mid a_n \mid^2, \quad B = \pi \sum_{n=1}^{\infty} n \mid a_{-n} \mid^2 .$$

Theorem 2.2 (Complex version of Dirichlet's principle)**.** *With notation and hypothesis just stated,*

$$\iint_U \mid \partial H \mid^2 dxdy \leq \alpha := \iint_U \mid \partial h \mid^2 dxdy$$

and

$$\iint_U \mid \bar\partial H \mid^2 dxdy \leq \beta := \iint_U \mid \bar\partial h \mid^2 dxdy.$$

We can rewrite the theorem in the form:

$$A = \pi \sum_{n=1}^{\infty} n \mid a_n \mid^2 \leq \alpha$$

and

$$B = \pi \sum_{n=1}^{\infty} n \mid a_{-n} \mid^2 \leq \beta.$$

Proof. Since $h \in W_1^2(U)$, then $H \in W_1^2(U)$ and therefore

$$D[H] = \iint_U \mid \nabla H \mid^2 dxdy = 2\,(A + B), \quad \text{and} \tag{2.7}$$

$$D[h] = \iint_U \mid \nabla h \mid^2 dxdy = 2\,(\alpha + \beta). \tag{2.8}$$

By Dirichlet's principle $D[H] \leq D[h]$, so that $A + B \leq \alpha + \beta$.

Hence, since $I = P = \alpha - \beta = A - B$, it follows $\alpha \geq A$ and $\beta \geq B$. $\Box$

Remark 2.3. Note that we have just derived the complex version of Dirichlet's principle from the classical Dirichlet's principle. Note that we can also derive the classical Dirichlet's principle from the complex version of Dirichlet's principle.

Namely, it follows from the complex version of Dirichlet's principle that $A+B \leq \alpha + \beta$. Since

$$D[H] = \iint_U \mid \nabla H \mid^2 dxdy = 2\,(A + B), \tag{2.9}$$

$$D[h] = \iint_U \mid \nabla h \mid^2 dxdy = 2\,(\alpha + \beta), \tag{2.10}$$

we have Dirichlet's principle $D[H] \leq D[h]$.

Corollary 2.4. *Suppose that h satisfies the condition* (1) *and the above notation. In addition, if h is $k - qr$, then $B \leq k^2 A$.*

Proof. Since h is $k - qr$, then $\beta \leq k^2\alpha$, and therefore $P \geq (1 - k^2)\alpha$. Hence, since $\alpha \geq A$, it follows that $A - B = P \geq (1 - k^2)A$, so that $B \leq k^2 A$. $\Box$

2.3 Area

Let us consider a conformal mapping h which belongs to class $\sum$, i.e. h is univalent in $E = \{z \,:\, |z| > 1\}$ and has a power series expansion of the form

$$h(z) = z + \sum_{n=1}^{\infty} a_{-n} z^{-n}$$

in E. If h has a quasiconformal extension to the plane with complex dilatation μ, satisfying the inequality $\|\mu\|_\infty = k < 1$, we say that h belongs to the subclass $\sum_k$ of $\sum$.

Lehto (see [12] and [13] and Kühnau (see [11]) established the area theorem for $\sum_k$.

Theorem KL (Kühnau-Lehto). *Let $h \in \sum_k$. Then*

$$B = \sum_{n=1}^{\infty} n\,|a_{-n}|^2 \leq k^2.$$

The estimate is sharp.

If we denote by P the area of the omitted set of $h(E)$, then Theorem KL states that

$$P \geq \pi(1 - k^2).$$

In this setting, $A = 1$, so that Theorem KL is an immediate consequence of Corollary 2.4.

In [15] (see Theorem M2) we announce a generalization of Theorem KL to univalent harmonic mappings using the Dirichlet's principle. Furthermore, our approach shows that we can get the corresponding result without assumption that mapping is univalent (see Theorem M3 [15], which we call Dirichlet's principle for qr mappings).

3 D_n estimate

Recall, in this section, we present some results obtained in [10] and some other essentially well-known results.

For convenient of the reader we first consider the case concerning C^1 mappings. We need the following proposition in the sequel (see Theorem D below for a more general result).

Proposition 3.1. *[22] Let Ω be an open subset of $\mathbb{R}^n$ and let $u : \Omega \to \mathbb{R}^n$ be a C^1 mapping. Then the function $y \mapsto N(y, u)$ is measurable on $\mathbb{R}^n$ and*

$$\int_{\mathbb{R}^n} N(y, f)\,\mathrm{d}y = \int_\Omega |J_u(x)|\,\mathrm{d}x, \tag{3.1}$$

where $J_u(x)$ is the Jacobian of u at x and $N(y, u)$ denotes the cardinal number of the set $u^{-1}(y)$ if the last set is finite and it is $+\infty$ in the other case.

Corollary 3.2. *Under the condition of the previous proposition there holds the inequality*

$$\int_\Omega |J_u(x)|\,\mathrm{d}x \geq |u(\Omega)|. \tag{3.2}$$

The equality holds in (3.2) if and only if u is univalent on Ω.

For 1-qr mapping we also say generalized conformal mapping. The generalized Liouville theorem states: for $n \geq 3$ every 1-qr mapping on a domain $\Omega \subset \mathbb{R}^n$, is a restriction of a Möbius transformation or a constant.

The proof of Theorem 3.5 is based on the following result, which is of independent interest.

Theorem 3.3. *[10] Let Ω be a domain in $\mathbb{R}^n$ and let $u : \Omega \to \mathbb{R}^n$ be a C^1 mapping such that*

$$\|u'(x)\|^n = n^{n/2}|J_u(x)|,\ x \in \Omega \text{ (We say that } u \text{ is absolutely conformal.)} \tag{3.3}$$

Then, for $n = 2$, u is analytic or anti-analytic function. For $n \geq 3$, u is a restriction of a Möbius transformation or a constant.

Note that if we suppose instead of (3.3)

$$\|u'(x)\|^n = n^{n/2}J_u(x),\ x \in \Omega, \tag{3.4}$$

then the Theorem 3.3 is a version of the Liouville theorem.

Example 3.4. [10] Let $x = (x_1, \dots, x_n)$ and

$$u(x) = \begin{cases} (x_1, x_2, \dots, x_n) & \text{if } x_n \geq 0 \\ (x_1, x_2, \dots, -x_n) & \text{if } x_n \leq 0 \end{cases}$$

u is not C^1 on the set $x_n = 0$ and the Theorem 3.3 does not hold.

Theorem 3.5. *[10] Let Ω be a domain in the Euclidean space $\mathbb{R}^n$, $n \geq 2$, and $u \in C^1(\Omega)$.*
Then

$$D_n(u) \geq n^{n/2}|u(\Omega)|\,. \tag{3.5}$$

If u is a C^1 mapping , then the equation in (3.5) holds if and only if u is an injective conformal mapping or a constant mapping.

Let $\Omega \subset \mathbb{R}^n$ be a domain and $A : \Omega \to \mathbb{R}^n$, $n > 2$, be a Möbius transformation and $\Omega' = A(\Omega)$; define the class $\mathcal{F}$ of all C^1 mappings f such that $f(\Omega) \supset \Omega'$. Using Theorem 3.5, we conclude that the minimization problem has a solution:

$$\min_{f \in \mathcal{F}} D_n(f) = D_n(A)\,.$$

Roughly speaking, Möbius transformations (respectively injecitive conformal mappings) are absolute minimizers of the n- Dirichlet integral subject to the corresponding constraint in the space (respectively in the plane).

Remark 3.6. Note that Reshetnyak [23] has proved: every ACL^n-homeomorphisam satisfies the condition (N). The condition (N) means that a mapping maps sets of measure zeros to sets of measure zeros. Therefore, using Theorem E below, one can show that the inequality (3.5) holds.

Remark 3.7. Observe that the Möbius transformations in the space, except orthogonal transformations, are not harmonic. However, they are $n-$harmonic, i.e. those mappings satisfying the equation (see [25], [7] and [19]):

$$\operatorname{div}(|\nabla u_i|^{n-2}\nabla u_i) = 0\,.$$

The reader can verify that Möbius transformations also satisfy a similar equation:

$$\operatorname{div}(|\nabla u|^{n-2}\nabla u) := \sum_{i=1}^{n} \operatorname{div}(|\nabla u|^{n-2}\nabla u_i) \cdot e_i = 0.$$

We also have the estimate of opposite type related to the inequality (3.5) for quasiregular mappings. In order to prove the estimate of opposite type and to generalize the inequality (3.5) if u is in the corresponding Sobolev space, we first need the following results.

Theorem C (Theorem 1.8 [6], Change of variables). *Let Ω be an open subset of $\mathbb{R}^n$ and let $u : \Omega \to \mathbb{R}^n$ be continuous and satisfy the conditions :*

(K) u has K-differential a.e. in Ω.

(L.I) $J_u(x)$ *is locally integrable on* Ω.

(N) $m(f(F)) = 0$ *whenever* $F \subset D$ *is a compact set of measure zero.*

Then the function $y \mapsto N(y, u)$ *is measurable on* $\mathbb{R}^n$ *and*

$$\int_{\mathbb{R}^n} N(y, f)\,\mathrm{d}y = \int_\Omega |J_u(x)|\,\mathrm{d}x. \tag{3.6}$$

See also Theorem 5.2.1 [9].

Note that the condition $u \in W_1^n(\Omega)$ implies the conditions (K) and (L.I).

Theorem D. *Let* Ω *be an open subset of* $\mathbb{R}^n$ *and let* $u : \Omega \to \mathbb{R}^n$ *be a qr mapping. Then the function* $y \mapsto N(y, u)$ *is measurable on* $\mathbb{R}^n$ *and*

$$\int_{\mathbb{R}^n} N(y, f)\,\mathrm{d}y = \int_\Omega |J_u(x)|\,\mathrm{d}x \tag{3.7}$$

For the proof of the theorem see Proposition 4.14 [26].

Theorem E (Theorem 2.3 [6])**.** *Let* $f : \Omega \to \mathbb{R}^n$ *be continuous, satisfy the condition* (N) *and* $f \in W_1^n(\Omega)$. *Then for every compact set* $K \subset \Omega$

$$\int_K |J_f(x)|\,\mathrm{d}x = \int_K N(y, f)\,\mathrm{d}y \geq |u(K)|. \tag{3.8}$$

See also Theorem 1.6 in [6].

Corollary 3.8. *Under the conditions of the Theorem C, Theorem D or Theorem E there hold the inequalities*

$$\int_\Omega |J_u(x)|\,\mathrm{d}x \geq |u(\Omega)| \quad \text{and (3.5)}. \tag{3.9}$$

Theorem 3.9. *Let* $f : \Omega \to \mathbb{R}^n$ *be continuous, satisfy the condition* (N), $f \in W_1^n(\Omega)$ *and* $\Omega' = f(\Omega)$. *Then*

$$n^{n/2}|\Omega'| \leq D_n(u). \tag{3.10}$$

More generally

$$n^{n/2}|\Omega'| \leq n^{n/2} \int_{\Omega'} N(y, f) dy \leq D_n(u). \tag{3.11}$$

If, in addition, f is a K-qr mapping, then

$$D_n(u) \leq n^{n/2} K \int_{\Omega'} N(y, f) dy, \tag{3.12}$$

where $N(y, f)$ *is the multiplicity of* y *in* Ω.

Proof. Let $e_n(f) = \left(\sum_{i=1}^n |\partial_i f|^2\right)^{n/2}$. Suppose that partial derivatives of f exist at a point $x \in \Omega$. By Hadamard's inequality,

$$n^{n/2} J_f(x) \leq e_n(f)(x).$$

Now, integration and an application of Theorem E gives (3.11) and as a corollary (3.10). It remains to prove (3.12).

Since $f'(x)e_j = \nabla f_j(x)$, $|\nabla f_j(x)| \leq |f'(x)|$ and therefore

$$\sum_{i=1}^n |\partial_i f|^2 \leq n|f'(x)|^2,$$

that is

$$e_n(f)(x) \leq n^{n/2}|f'(x)|^n.$$

Thus we have

$$n^{n/2} J_f(x) \leq e_n(f)(x).$$

Hence, since by definition of K-qr,

$$|f'(x)|^n \leq K J_f(x),$$

we find

$$e_n(f) \leq K n^{n/2} J_f(x).$$

Now, integration and an application of Theorem E completes the proof. □

An immediate consequence is:

Corollary 3.10. *Let $f : \Omega \to \mathbb{R}^n$ be K-qc and $\Omega' = f(\Omega)$.*
Then

$$n^{n/2}|\Omega'| \leq D_n(u) \leq n^{n/2} K |\Omega'|. \tag{3.13}$$

The interested reader can learn more about the subject related to the result presented in this section from the excellent Iwaniec-Martin book [9], which, of course, contains much more than is needed here. In particular, it seems that we can generalize some of the above results using ideas from section 14.2 [9]

4 D_2 estimate

The results of this section are related to the famous Poincaré inequality, however we believe that some of the inequalities, statements and proofs we preset here are new. For $u \in L^2(\Omega)$ and $f \in L^2(S^{n-1})$ we define L^2 norms by

$$\|u\|_\Omega = \|u\|_{2,\Omega} = \left(\int_\Omega |u(x)|^2 dm(x)\right)^{1/2},$$

and

$$\|f\|_S = \|f\|_{2,S^{n-1}} = \left(\int_{S^{n-1}} |f(\zeta)|^2 d\sigma(\zeta)\right)^{1/2}.$$

For the following result we refer to [2]:

Theorem F. *For $f \in L^2(S^{n-1})$ we have the following decomposition:*

$$f(\zeta) = \sum_{\nu=0}^{\infty} p_\nu(\zeta),$$

where p_ν are the spherical harmonics of degree ν which are orthogonal on the sphere: i.e. there holds the equality:

$$\langle p_m, p_k \rangle = \int_{S^{n-1}} p_m(\zeta) \cdot p_k(\zeta) d\sigma(\zeta) = 0, \text{ for all } m \neq k. \tag{4.1}$$

Note that harmonic polynomials of degree 1 on $\mathbb{R}^n$ are of the form $h(x) = \sum_{k=1}^n a_k x_k$, where $a_k \in \mathbb{R}$, $k = 1, \dots n$; and spherical harmonics of degree 1 on S^{n-1} are the restrictions to S of harmonic polynomials of degree 1, i.e of real linear mappings.

For a C^1 mapping u we define the radial derivative by

$$\frac{\partial u}{\partial r}(x) = \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} \frac{x_i}{r},$$

where $r = \|x\|$.

Let $\Omega_n = |B^n|$ and $\omega_n = |S^n|$. It is known $\omega_{n-1} = n\Omega_n$. Denote by ν the Lebesgue measure on $\mathbb{R}^n$ normalized such that $\nu(B) = 1$.

If $f \in L^1(B^n)$, then

$$\int_B f d\nu = n \int_0^1 r^{n-1} dr \int_S f(r\zeta) d\sigma(\zeta). \tag{4.2}$$

Since $dm = dx = \Omega_n d\nu$ and $\omega_{n-1} = n\Omega_n$, we can rewrite the formula (4.2) in the form:

$$\int_B f dx = \omega_{n-1} \int_0^1 r^{n-1} dr \int_S f(r\zeta) d\sigma(\zeta). \tag{4.3}$$

Using a standard procedure the reader can verify:
if $\left\| \frac{\partial f}{\partial r} \right\|_{B^n} < \infty$, then the radial limit f^* exists a.e. on S and $f^* \in L^2(S)$.

Lemma 4.1. *If f is a real harmonic function defined on the unit ball B^n, satisfying the condition $f(0) = 0$, and $\left\| \frac{\partial f}{\partial r} \right\|_{2,B^n} < \infty$, then*

$$\left\| \frac{\partial f}{\partial r} \right\|_{2,B^n} \geq \sqrt{\Omega_n} \cdot \|f^*\|_{2,S^{n-1}}. \tag{4.4}$$

The equality holds in (4.4) if and only if f is a real linear mapping.

Proof. First of all, by Theorem F, we have the following representation of the function f:

$$f(x) = \sum_{j=0}^{\infty} p_j(x), \quad x \in B,$$

where $p_j(x)$ are homogeneous harmonic polynomials on $\mathbb{R}^n$ of degree j. Hence we have that $f(r\zeta) = \sum_{j=0}^{\infty} r^j p_j(\zeta)$, where $r \in [0,1)$ and $\zeta \in S^{n-1}$. Therefore, for $x = r\zeta$, we find

$$\frac{\partial f}{\partial r}(x) = \sum_{j=1}^{\infty} j\, r^{j-1} p_j(\zeta).$$

Integrating the square of the previous expression over the unit ball and using relations (4.2) and (4.1), we obtain

$$\int_{B^n} \left(\frac{\partial f}{\partial r}(x)\right)^2 dx = \sum_{j=1}^{\infty} \omega_{n-1} j^2 \int_0^1 r^{n-1} dr \int_{S^{n-1}} r^{2j-2} p_j^2(\zeta) d\sigma(\zeta).$$

Hence we have

$$\int_{B^n} \left(\frac{\partial f}{\partial r}(x)\right)^2 dx = \sum_{j=1}^{\infty} \frac{\omega_{n-1} j^2}{n+2(j-1)} \int_{S^{n-1}} p_j^2(\zeta) d\sigma(\zeta).$$

Therefore, since $\frac{j^2}{n+2(j-1)} \geq \frac{1}{n}$ for $j \geq 1, n \geq 2$, we obtain

$$\int_{B^n} \left(\frac{\partial f}{\partial r}(x)\right)^2 dx \geq \frac{\omega_{n-1}}{n} \sum_{j=1}^{\infty} \int_{S^{n-1}} p_j^2(\zeta) d\sigma(\zeta)\,. \tag{4.5}$$

Observe that, the equality holds in the previous inequality for $f(x) = x_i$, where x_i is i-th coordinate of x. From (4.5), using $\frac{\omega_{n-1}}{n} = \Omega_n$, we now get (4.4).

Now we consider the equality case. If $j > 1, n \geq 2$, then $\frac{j^2}{n+2(j-1)} > \frac{1}{n}$. Hence, if the equality holds in (4.4), then $p_j = 0$ for $j > 1$ and therefore $f^* = p_1$ on S.

□

Theorem 4.2. *If $f : B^n \to \mathbb{R}^m$ is a vector harmonic function, satisfying the condition $f(0) = 0$ and $\left\|\frac{\partial f}{\partial r}\right\|_{2,B^n} < \infty$, then*

$$\left\|\frac{\partial f}{\partial r}\right\|_{2,B^n} \geq \sqrt{\Omega_n} \cdot \|f^*\|_{2,S^{n-1}}. \tag{4.6}$$

The equality holds if and only if f_i, $i = 1, \ldots, m$, are spherical harmonic of degree 1, i.e. if and only if f is a linear mapping.

Proof. Let $f = (f_1, \ldots, f_m)$. Then by (4.4), we obtain

$$\left\|\frac{\partial f_i}{\partial r}\right\|_{2,B^n}^2 \geq \Omega_n \cdot \|f_i\|_{2,S^{n-1}}^2. \tag{4.7}$$

Summing from $i = 1, \ldots, m$ the previous inequality, we obtain (4.6):

$$\left\|\frac{\partial f}{\partial r}\right\|_{2,B^n}^2 \geq \Omega_n \cdot \sum_{i=1}^{m} \|f_i\|_{2,S^{n-1}}^2 = \Omega_n \int_{S^{n-1}} |f^*(\zeta)|^2 d\sigma(\zeta).$$

We now consider the equality case. An inspection of the proof shows that if equality holds in (4.6), then equality holds in (4.7) for $i = 1, \dots, m$. Hence the equality holds in (4.7) if and only if f is a linear mapping. □

Corollary 4.3. *If f is a harmonic mapping of the unit ball into itself satisfying the conditions $\|f^*(\zeta)\| = 1$ for a.e ζ on S and $f(0) = 0$, then*

$$\left|\left|\frac{\partial f}{\partial r}\right|\right|^2_{2,B^n} \geq \Omega_n\,. \tag{4.8}$$

The inequality is sharp.

Proof. Since $\|f^*(\zeta)\| = 1$ for a.e ζ on S and therefore $\|f\|_S = 1$, we obtain (4.8).

If A is an orthogonal transformation of B^n, then $|\frac{\partial A}{\partial r}| = 1$. Hence the equality holds in (4.8) for $f(x) = Ax$, where A is an arbitrary orthogonal transformation of B^n. Thus the inequality is sharp.

□

Lemma 4.4. *If $u \in W_1^2(B^n)$, then $\left|\left|\frac{\partial u}{\partial r}\right|\right|_{2,B^n} \leq D_2(u) < \infty$.*

Proof. By the definition we have

$$\left|\frac{\partial u}{\partial r}\right|^2 = \sum_{j=1}^n \left|\frac{\partial u_j}{\partial r}\right|^2 = \sum_{j=1}^n \left(\sum_{i=1}^n \frac{\partial u_j}{\partial x_i}\frac{\partial x_i}{\partial r}\right)^2 .$$

It follows that

$$\left|\frac{\partial u}{\partial r}\right|^2 \leq \sum_{j=1}^n\sum_{i=1}^n \left(\frac{\partial u_j}{\partial x_i}\right)^2 \left|\frac{\partial x}{\partial r}\right|^2 .$$

Therefore

$$\left|\frac{\partial u}{\partial r}\right|^2 \leq \|u'\|^2\,. \tag{4.9}$$

□

Let $\mathcal{S}_0$ (respectively $\mathcal{T}_0$) be the family of mappings: $f : S^{n-1} \to S^{n-1}$ (respectively $f : S^{n-1} \to \mathbb{R}^n$) satisfying the conditions $f \in L^1(S^{n-1})$ and

$$\int_{S^{n-1}} f(x)d\sigma = 0\,. \tag{4.10}$$

Theorem 4.5 (Poincaré inequality)**.** *Let $u : B^n \to \mathbb{R}^n$, $u \in W_1^2(B^n)$, and let u be continuous on $\overline{B}^n$. If $u^* \in \mathcal{T}_0$, then*

$$D_2(u) \geq \Omega_n \int_{S^{n-1}} |u^*(\zeta)|^2 d\sigma(\zeta). \tag{4.11}$$

Proof. Using the Theorem A, Lemma 4.4 and Theorem 4.2 it follows the desired conclusion. □

Corollary 4.6. *If $u : B^n \to B^n$, $u \in W_1^2(B^n)$, and $u^* \in \mathcal{S}_0$, then*

$$D_2(u) \geq \Omega_n. \tag{4.12}$$

Using the inequality (4.7) and taking $F = f - a$, we can prove the following generalization of the previous corollary.

Corollary 4.7. *Let $u : B^n \to \Omega \subset \mathbb{R}^n$, $u \in W_1^2(B^n)$, $u^* : S^{n-1} \to \partial\Omega$ and $P[u^*](0) = a$, where $a \in \Omega$. Then,*

$$D_2(u) \geq \Omega_n \cdot d(a, \partial\Omega). \tag{4.13}$$

Let

$$I_p(u) := \left(\int_{B^n} e(f)^p(x)\, d\nu(x) \right)^{1/p}.$$

Since $I_p \geq I_2$, $p \geq 2$, using Corollary 4.6, we get

Corollary 4.8. *Let $p \geq 2$, $u : B^n \to B^n$, $u \in W_1^2(B^n)$, and $u^* \in \mathcal{S}_0$. Then*

$$D_p(u) \geq \Omega_n. \tag{4.14}$$

In the following example it is shown that for validity of Theorem 4.5 the condition (4.10) is essential even for diffeomorphisms. It is also shown that we cannot obtain any analogous version of inequality (3.5) replacing D_n by D_2. More precisely, there is a family of diffeomorphisms F_n of B^3 onto itself such that $D_2(F_n)$ tends to 0 when n tends to infinity.

4.1 Example

Let f be a conformal mapping of the unit disk onto itself. The function

$$F(z,t) = (f(z), \sqrt{|f'(z)|}t),$$

$z = (x, y)$, $f(z) = (u(z), v(z))$, is a diffeomorphism of the unit ball in $\mathbb{R}^3$ onto itself. The mapping

$$\phi_a(z) = \frac{z - a}{1 - z\bar{a}}, \quad |a| < 1,$$

is a conformal mapping of the unit disk onto itself. The family of all conformal mapping of the unit disk onto itself, AUT($\mathbb{U}$) is given by

$$f(z) = e^{i\alpha}\phi_a(z), \quad |a| < 1,$$

and each $f \in$ AUT($\mathbb{U}$) satisfies the equation

$$|f'(z)| = \frac{1 - |f(z)|^2}{1 - |z|^2}, \text{ i.e. } \sqrt{1 - |z|^2}\sqrt{|f'(z)|} = \sqrt{1 - |f(z)|^2}.$$

Hence, for a fixed $|z| < 1$, F maps the segment $I_z = \{(z,t) : -\sqrt{1-|z|^2} \le t \le \sqrt{1-|z|^2}\}$ onto the segment $I_{f(z)} = \{(f(z),\tau) : -\sqrt{1-|f(z)|^2} \le \tau \le \sqrt{1-|f(z)|^2}\}$. Since $(z,t) \in \mathbb{B}^3$ if and only if $|z|^2 + t^2 < 1$, it follows that F maps the unit ball onto itself. For $f(z) = e^{i\alpha}\phi_a(z)$, $|a| < 1$, let $r = |a| < 1$, and let $\varphi(r) = D_2(F)$.

Using the Mathematica software (Mathematica for Windows Version 5.0), we obtain

$$\begin{aligned}\varphi(r) :&= \int_{B^3} \|F'\|^2 \mathrm{d}x\mathrm{d}y\mathrm{d}t = \int_{B^3} \|F_x\|^2 + \|F_y\|^2 + \|F_t\|^2 \mathrm{d}x\mathrm{d}y\mathrm{d}t \\ &= \frac{4\pi(1-r^2)(r - \sqrt{1-r^2}\arcsin r)}{r^3} \\ &+ \frac{2\pi(-15r + 17r^3 - 2r^5 - 3\sqrt{1-r^2}(-5+4r^2)\arcsin r)}{12r^3} \\ &- \frac{\pi^2\sqrt{1-r^2}(-8+4r^2-5r^4+8\sqrt{1-r^2})}{24r^2} \\ &+ \frac{2\pi\sqrt{1-r^2}(r\sqrt{1-r^2} + (-1+2r^2)\arcsin r)}{r^3}.\end{aligned}$$

Hence

$$\lim_{r\to 1-0} \varphi(r) = 0 \text{ and } \lim_{r\to 0+0} \varphi(r) = 4\pi.$$

Observe also that for $\alpha = 0$

$$\lim_{r\to 0+0} F = \mathrm{Id}$$

and that

$$\varphi(0) = \int_{B^3} |\mathrm{Id}'|^2 \mathrm{d}x\mathrm{d}y\mathrm{d}t = \int_{B^3} 3dxdydz = 3\cdot 4\pi/3 = 4\pi.$$

Acknowledgment

The authors wish to thank professors M. Vuorinen, O. Martio and V. Zorich for showing interest in this paper and giving comments concerning Theorem 3.3, and G. Martin for suggestion to consider recent work of T. Iwaniec and J. Onninen.

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