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ON ZERO SETS AND PARAMETRIC REPRESENTATIONS OF SOME NEW ANALYTIC AND MEROMORPHIC FUNCTION SPACES IN THE UNIT DISK

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Abstract

We introduce and study some certain new scales of analytic and meromorphic functions in the unit disk and solve some problems connected with parametric representations in these scales of spaces. Namely, we provide a complete description of zero sets and then, based on that description, present new parametric representations for functions in those spaces.

1 Introduction and Notation

We use standard notation: $\mathbb{D}$ denotes the open unit disk in the complex plane $\mathbb{C}$, $\mathbb{T}$ is the boundary of $\mathbb{D}$. Also, m_2 denotes the normalized Lebesgue area measure: $m_2(\mathbb{D}) = 1$. The space of all holomorphic, respectively meromorphic, functions in $\mathbb{D}$ is denoted by $H(\mathbb{D})$, respectively $M(\mathbb{D})$. For $f \in H(\mathbb{D})$ and $0 \leq t < 1$ $n_f(t) = n(t)$ denotes the number of zeroes, counting multiplicity, of f in the disk $\{z : |z| \leq t\}$. We set $n_f(1-2^{-k}) = n_k$, for $k = 1, 2, 3, \ldots$. The classical Nevanlinna characteristic of $f \in H(\mathbb{D})$ is defined by

$$T(r,f) = \frac{1}{2\pi} \int_{\mathbb{T}} \ln^+ |f(r\xi)| d\xi, \qquad 0 \leq r < 1,$$

where $a^+ = \max(0, a)$, $a \in \mathbb{R}$. We consider a scale of classes

$$N_\alpha^\infty = \{f \in H(\mathbb{D}) : T(r,f) \leq C_f(1-r)^{-\alpha}, 0 \leq r < 1\}, \qquad \alpha \geq 0,$$

see [2], [6], [10] and [11]. Clearly $N_0^\infty = N$, where N is the classical Nevanlinna class. Let us recall that N consists of all functions admitting a representation in the form

$$f(z) = Cz^\lambda B(z, \{z_k\}) \exp\left(\int_{-\pi}^{\pi} \frac{d\mu(\theta)}{1 - ze^{-i\theta}}\right), \quad z \in \mathbb{D},$$

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where C is a complex constant, λ is a non-negative integer, $B(z, \{z_k\})$ is the classical Blaschke poduct with zeros z_k in $\mathbb{D}$ enumerated according to their multiplicities and satisfying the Blaschke density condition: $\sum_k (1 - |z_k|) < \infty$ and $\mu(\theta)$ is any real valued function of bounded variation on $[-\pi, \pi]$. Whenever we list zeros or poles of a meromorphic function into a sequence $\{a_k\}$, we assume $|a_k| \leq |a_{k+1}|$ and multiplicities are taken into account.

We will need Besov spaces on the unit circle $\mathbb{T}$ which we define using Besov spaces on the real line. Let $\Delta_h f(x) = f(x-h) - f(x)$ and we define an L^p modulus of continuity of f as $\omega_p^2(f,t) = \sup_{|h|\leq t} \|\Delta_h^2 f\|_p$. Let $s > 0$, set $s = n + \alpha$ where $0 < \alpha \leq 1$ and $n \in \mathbb{N}_0$. The Besov space $B_s^{p,q}(\mathbb{R})$, $0 < p, q < \infty$, consists of all functions f in the Sobolev space $W_p^n(\mathbb{R})$ (see [1]) satisfying

$$\int_0^\infty \left| \frac{\omega_p^2(f^{(n)}, t)}{t^\alpha} \right|^q \frac{dt}{t} < \infty,$$

with the usual modification to allow $q = +\infty$. For $1 \leq p < \infty$, $1 \leq q \leq \infty$ these spaces are Banach spaces with the following norm

$$\|f\|_{W_p^n(\mathbb{R})} + \left(\int_0^\infty \left| \frac{\omega_p^2(f^{(n)}, t)}{t^\alpha} \right|^q \frac{dt}{t} \right)^{1/q},$$

again with the usual modification for $q = +\infty$. These spaces on the real line have analogues on the unit circle, in fact it suffices to consider 2π periodic functions such that the above norm (or quasi-norm for $0 < p < \infty$, $0 < q \leq \infty$) is finite when $W_p^n(\mathbb{R})$ is replaced by $W_p^n(-\pi, \pi)$. We denote those spaces by $B_s^{p,q}(\mathbb{T})$.

Theorem A. (see [10]) *Let $\alpha > 0$ and $\beta > \alpha - 1$. Then the N_α^∞ class coincides with the set of all functions representable in the form*

$$f(z) = Cz^\lambda \prod_\beta (z, \{z_k\}) \exp\left(\int_{-\pi}^{\pi} \frac{\psi(e^{i\theta}) d\theta}{(1 - ze^{-i\theta})^{\beta+2}} \right), \qquad z \in \mathbb{D}, \tag{1}$$

where C is a constant, λ is a non-negative integer, $\prod_\beta (z, \{z_k\})$ is a Weierstrass-type product:

$$\prod_\beta (z, \{z_k\}) = \prod_{k=1}^\infty \left(1 - \frac{z}{z_k}\right) \exp\left(\frac{\beta+1}{\pi} \int_{\mathbb{D}} \frac{(1-|\xi|^2)^\beta \ln|1 - \xi/z_k|}{(1 - \bar{\xi} z)^{\beta+2}} dm_2(\xi) \right),$$

which converges locally absolutely and uniformly in $\mathbb{D}$ to an analytic function f with zeros z_k in $\mathbb{D}$ satisfying condition

$$n_f(r) \leq \frac{c}{(1-r)^{\alpha+1}}$$

for some $c > 0$ and ψ is a real valued function in $B_{\beta-\alpha+1}^{1,\infty}(\mathbb{T})$.

Let S^p_α, $0 < p < \infty$, $\alpha > -1$, be a class of holomorphic functions defined by

$$S^p_\alpha = \{f \in H(\mathbb{D}) : \|f\|^p_{S^p_\alpha} = \int_0^1 (1-r)^\alpha T^p(r,f)dr < \infty\}.$$

The following theorem, proved in [9], is an analogue of Theorem A for S^p_α classes.

Theorem B. (see [9]) *Let $p \geq 1$ and $\beta > \frac{\alpha+1}{p}$. Then $f \in S^p_\alpha$ if and only if f admits representation* (1), where C is a constant, λ is a non-negative integer, z_k is a sequence in $\mathbb{D}$ such that

$$\int_0^1 (1-r)^{\alpha+p} n_f(r)^p dr < \infty$$

and $\psi \in B^{1,p}_s(\mathbb{T})$, where $s = \beta - \frac{\alpha+1}{p}$.

These two theorems give parametric representations of the spaces N^∞_α and S^p_α. One of the goals of this paper is to obtain similar parametric representations of new analytic Nevanlinna type classes defined as follows:

$$(NA)_{p,\gamma,\delta} = \left\{ f \in H(\mathbb{D}) : \int_0^1 \left[\sup_{0<\tau<r} T(\tau,f)(1-\tau)^\gamma \right]^p (1-r)^\delta dr < \infty \right\}, \quad (2)$$

where $\gamma \geq 0$, $\delta \geq 0$ and $0 < p < \infty$. It turns out that complete analogues of the results we cited above exist for the $(NA)_{p,\gamma,\delta}$ classes. Note that the limiting case $p = +\infty$, $\gamma = 0$ and $\delta = 0$ gives the extensively studied Nevanlinna class $N = (NA)_{\infty,0,0}$. Of course, the zero sets and parametric representation of functions in class N are classical results, see [3], [4], [5], [6], [10], [11] and references therein. Thus it is natural to extend these important results to all $(NA)_{p,\gamma,\delta}$ classes. The zero set description problem can be stated in the following abstract, but simple, form:

Given a subspace X of $H(\mathbb{D})$ find a class Y of sequences in $\mathbb{D}$ such that the zero set of any $f \in X$, arranged in a sequence z_k, is in Y and conversely, given a sequence z_k in Y there is an $f \in X$ such that $f(z_k) = 0$ for all k.

For many classical spaces X this problem is still open, for example for the weighted Bergman spaces A^p_α, see [7] and references therein. On the other hand, a complete description of the zero sets of the classes N^p_α, S^p_α is well known, see [10] and [11]. One of the goals of this paper is to describe zero sets of functions in some new Nevanlinna type analytic classes, including the $(NA)_{p,\gamma,\delta}$ classes and then to use this description to obtain parametric representation of functions in these classes.

It is easy to verify that all the above classes of analytic functions are topological vector spaces with complete invariant metrics. We emphasize that the above mentioned problem of zero sets description and parametric representations have numerous applications and are important in function theory (see [2], [3], [4], [5] and [6]). Some related sharp results on analogous classes of meromorphic functions will be presented as well.

We follow an usual convenction: a constant C can change its value from one appearance to the next one.

2 Preliminary results

Here we collect some results that are needed in the main section of the paper. First we introduce a Weierstrass type infinite product (see [6], Chapter 1) and give an estimate of such a product which is essential in this work.

Proposition A. (see [2]) *Let $\{z_k\}_{k=1}^{\infty}$ be a sequence in the unit disk satisfying condition $\sum_{k=1}^{\infty}(1-|z_k|)^{t+2} < \infty$ for some $t > -1$. Then for such t the infinite product*

$$\prod_t(z,\{z_k\}) = \prod_{k=1}^{\infty}\left(1-\frac{z}{z_k}\right)\exp\left(\frac{-(t+1)}{\pi}\int_{\mathbb{D}}\frac{(1-|\xi|^2)^t\ln|1-\frac{\xi}{z_k}|}{(1-\overline{\xi}z)^{t+2}}dm_2(\xi)\right) \quad (3)$$

converges locally uniformly and absolutely in $\mathbb{D}$ where it presents an analytic function with zeros $\{z_k\}$.

Proposition B. (see [2]) *Let $\{z_k\}_{k=1}^{\infty}$ be a sequence in the unit disk satisfying condition $\sum_{k=1}^{\infty}(1-|z_k|)^{t+2} < \infty$ for some $t > -1$. Then the following estimate holds for the infinite product $\prod_t(z,\{z_k\})$ introduced in the above proposition:*

$$\ln^+|\prod_t(z,\{z_k\})| \le C_t\sum_{k=1}^{\infty}\frac{(1-|z_k|^2)^{t+2}}{|1-z\overline{z}_k|^{t+2}}, \qquad z\in\mathbb{D},$$

where C_t is a constant depending only on t.

Let us define, for $\alpha > -1$, $z,\xi\in\mathbb{D}$:

$$W_\alpha(z,\xi) = \sum_{k=1}^{\infty}\frac{\Gamma(\alpha+k+2)}{\Gamma(\alpha+2)\Gamma(k+1)}\left((\overline{\xi}z)^k\int_{|\xi|}^1\frac{(1-x)^\alpha dx}{x^{k+1}} - \frac{z^k}{\xi^k}\int_0^{|\xi|}(1-x)^\alpha x^{k-1}dx\right).$$

The next proposition introduces another infinite product which we need in the next section.

Proposition C. (see [3], [4], [6]) *Let $\alpha > -1$ and let $\{z_k\}_{k=1}^{\infty}$ be a sequence in $\mathbb{D}$ such that $\sum_{k=1}^{\infty}(1-|z_k|)^{\alpha+2} < \infty$. Then the infinite product*

$$B_\alpha(z,\{z_k\}) = \prod_{k=1}^{\infty}\left(1-\frac{z}{z_k}\right)\exp(-W_\alpha(z,z_k)), \quad z\in\mathbb{D}$$

converges locally uniformly and absolutely in $\mathbb{D}$ and presents an analytic function in $\mathbb{D}$ with zeros only at points z_k.

In the remaining part of this section we deal with meromorphic functions. Let $f(z)$ be a meromorphic function in $\mathbb{D}$, let $f(z) = \sum_{k=m}^{\infty}C_kz^k$, $C_m \ne 0$, be the Laurent expansion of f at $z=0$. In this situation we denote by $\{a_k\}$ and $\{b_k\}$ be the sequence of zeros, respectively poles, of f listed in increasing order of their modulus and counted according to their multiplicity. Let us recall the Poisson-Jensen formula (see [3], [4], [6]):

$$\begin{aligned}\ln|f(z)| &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\ln|f(\rho e^{i\theta})|\frac{\rho^2-r^2}{\rho^2-2r\rho\cos(\theta-\phi)+r^2}d\theta+m\ln\frac{|z|}{\rho}\\ &+ \sum_{0<|a_\nu|<\rho}\ln\left|\frac{\rho(z-a_\nu)}{\rho^2-\overline{a}_\nu z}\right|-\sum_{0<|b_\nu|<\rho}\ln\left|\frac{\rho(z-b_\nu)}{\rho^2-\overline{b}_\nu z}\right|\end{aligned}$$

for $z=re^{i\phi}\in\mathbb{D}$, $(r<\rho<1)$, where m is the multiplicity of zero or pole of f at $z=0$. For $z=0$ we get the classical Jensen's formula which can be written in a symmetrical form:

$$\begin{aligned}&\tfrac{1}{2\pi}\textstyle\int_{-\pi}\pi\ln^+|f(\rho e^{i\theta})|d\theta+\sum_{0<|b_\nu|<\rho}\ln\frac{\rho}{|b_\nu|}\\ =\ &\tfrac{1}{2\pi}\textstyle\int_{-\pi}^{\pi}\ln^+\frac{1}{f(\rho e^{i\theta})}d\theta+\sum_{0<|a_\nu|<\rho}\ln\frac{\rho}{|a_\nu|}+m\ln\rho+\ln|C_m|\end{aligned}$$

This will be needed in the proof of our Theorem 1; we also need the Nevanlinna characteristics of a meromorphic function f which can be expressed in the following form:

$$\begin{aligned}T(r,f) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\ln^+|f(\rho e^{i\theta})|d\theta+\sum_{|b_\nu|<\rho}\ln\frac{\rho}{|b_\nu|}\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\ln^+|f(\rho e^{i\theta})|d\theta+N_f(r).\end{aligned} \tag{4}$$

The function $T(r,f)$ is increasing on $(0,1)$, we say that a meromorphic function f is of bounded characteristics if $T(1,f)=\lim_{r\to 1-0}T(r,f)<\infty$. The class of all such functions coincides with the class of all meromorphic functions f such that

$$f(z)=Cz^\lambda\frac{B(z,\{a_k\})}{B(z,\{b_k\})}\exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}d\psi(\theta)\right),\qquad z\in\mathbb{D},$$

where C is a consant, $\lambda\in\mathbb{Z}$, ψ is a real valued function of bounded variation and a_k and b_k are sequences in $\mathbb{D}$ satisfying the Blaschke condition. In the next section we show that results of this type can be obtained for certain larger scales of meromorphic functions in the unit disk. Let us mention that such results are known for certain classes of meromorphic functions, we give new results of this type below. Note that analytic versions of Theorems C-E can be found in [2], [3], [4]. The extension to the meromorphic case can be achieved by standard arguments, presented at the end of the paper (see also [12]).

Theorem C. (see [2], [3], [4], [12]) *Let f be a meromorphic function in $\mathbb{D}$ such that $\int_0^1(1-r)^\alpha T(r,f)dr<\infty$. Let $f(z)=C_\lambda z^\lambda+\cdots$, $C_\lambda\neq 0$ be its Laurent expansion at $z=0$. Then for every $z\in\mathbb{D}$ we have:*

$$f(z)=K_\alpha\overline{C}_\lambda\frac{\prod_\alpha(z,\{a_k\})}{\prod_\alpha(z,\{b_k\})}\exp\left(\frac{2(\alpha+1)}{\pi}\int_0^1\int_{-\pi}^{\pi}(1-\rho^2)^\alpha\frac{\ln|f(\rho e^{i\theta})|\rho d\rho d\theta}{(1-z\rho e^{-i\theta})^{\alpha+2}}\right),$$

where

$$K_\alpha = \exp\left(\lambda(\alpha+1)\int_0^1 (1-\rho)^\alpha \ln\frac{1}{\rho}d\rho\right).$$

It is known, (see [2], [3], [4]), that sequences a_k and b_k can be represented as zeros and poles of a function f in $MS_\alpha = \{f \in M(\mathbb{D}) : \int_0^1 (1-r)^\alpha T(r,f)dr < \infty\}$ if and only if

$$\sum_k (1-|a_k|)^{\alpha+2} < \infty, \qquad \sum_k (1-|b_k|)^{\alpha+2} < \infty. \tag{5}$$

Now we mention another result on parametric representation. First, we describe some classes of meromorphic functions introduced and studied in [3] and [4]. For $f \in M(\mathbb{D})$, $\alpha > -1$ and $0 < r < 1$ we set

$$m_\alpha(r,f) = \frac{r^{-\alpha}}{2\pi}\int_{-\pi}^{\pi}\left(\int_0^r (r-t)^\alpha \ln|f(te^{i\phi}|dt\right)^+ d\phi.$$

Next, let $\tilde{n}(t)$ denotes the number of poles, counting multiplicity, of f in $t\mathbb{D} = \{z : |z| < t\}$ and set

$$T_\alpha(r,f) = m_\alpha(r,f) + \frac{r^{-\alpha-1}}{\Gamma(\alpha+2)}\exp\int_0^r \frac{(r-t)^{\alpha+1}}{t}(\tilde{n}(t)-n(0))dt + \frac{n(0)}{\Gamma(\alpha+2)}\ln r.$$

Finally, we define $MN_\alpha = \{f \in M(\mathbb{D}) : \sup_{0<r<1} T_\alpha(r,f) < \infty\}$.

Theorem D. (see [3], [4], [12]) *The class MN_α coincides with the class of all meromorphic functions f in $\mathbb{D}$ admitting representation*

$$f(z) = C_\lambda z^\lambda \frac{B_\alpha(z,\{a_k\})}{B_\alpha(z,\{b_k\})}\exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{2}{(1-e^{-i\theta}z)^{\alpha+2}} - 1\right)d\psi(\theta)\right)$$

where C_λ is a complex number, λ is an integer, sequences a_k and b_k satisfy condition (5) and ψ is a real function of bounded variation.

Now we present a related result obtained in [9], see also [10]. Let, for $0 < p < \infty$ and $\alpha > -1$,

$$MS_\alpha^p = \{f \in M(\mathbb{D}) : \int_0^1 T^p(r,f)(1-r)^\alpha dr < \infty\}.$$

Theorem E. (see [9], [12]) *Let $0 < p < \infty$, $\alpha > -1$ and $\beta > \frac{\alpha+1}{p}$. Then a function $f \in M(\mathbb{D})$ belongs to SM_α^p if and only if it admits representation*

$$f(z) = C_\lambda z^\lambda \frac{\prod_\beta(z,\{a_k\})}{\prod_\beta(z,\{b_k\})}\exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\psi(e^{i\theta})}{(1-e^{-i\theta}z)^{\alpha+2}}d\psi(\theta)\right)$$

where $\{a_k\}$ *and* $\{b_k\}$ *are sequences satisfying*

$$\sum_k \frac{n_k^p}{2^{k(p+1+\alpha)}} < \infty,$$

$\psi \in B^s_{1,p}$, *where* $s = \beta - \frac{\alpha+1}{p}$, C_λ *is a complex number and* λ *is an integer.*

We are going to give a different parametric representation of these classes, using Weierstrass type infinite products. Note that similar results are valid for classes defined by weight functions ω (see [10]):

$$MS^\infty_{\omega,\alpha} = \left\{ f \in M(\mathbb{D}) : T(r,f) \leq C\frac{\omega(1-r)}{(1-r)^\alpha} \right\}, \qquad \alpha > 0.$$

3 Main results on $(NA)_{p,\gamma,\delta}$ spaces of analytic and meromorphic functions

In this section we give complete analogues for the $(NA)_{p,\gamma,\delta}$ classes of results described above. We denote the more general class, with the same quasinorm as in (2), but consisting of meromorphic functions by $(NM)_{p,\gamma,\delta}$. The following theorem gives a complete description of the zero sets of the $(NA)_{p,\gamma,\delta}$ class.

Theorem 1. *For any* $0 < p < \infty$, $\gamma \geq 0$ *and* $\delta \geq 0$ *the following conditions are equivalent:*

a) $\{z_k\} \in Z((NA)_{p,\gamma,\delta})$.

b)

$$\sum_{k=1}^\infty \frac{n_k^p}{2^{k((\gamma+1)p+\delta+1)}} < \infty. \tag{6}$$

If the condition (6) is satisfied, then $\prod_t(z,\{z_k\}) \in (NA)_{p,\gamma,\delta}$ *for* $0 < p \leq 1$, $t > \frac{\delta+1}{p} + \gamma - 1$ *and for* $p > 1$, $t > \frac{\delta}{p} + \gamma$.

In fact, we can derive implication a) $\Rightarrow$ b) from a more general statement. Let us denote by S the class of all positive measurable functions $\omega(t)$ on $(0,1]$ such that there are constants $m = m_\omega$, $M = M_\omega$ and $q = q_\omega$ satisfying: $0 < m, q < 1$ and

$$m \leq \frac{\omega(\lambda r)}{\omega(r)} \leq M, \qquad 0 < r < 1, \;\; q \leq \lambda \leq 1,$$

see [8] for detailed study of such functions. The following lemma was proved in [9] in the case $p = \gamma$, but the same proof applies in the general case.

Lemma 1. *Let* $\omega \in S$, $p > 0$, $\gamma \geq 0$ *and let* $\{z_k\}$ *be a sequence in* $\mathbb{D}$, $|z_k| \leq |z_{k+1}|$. *Let* $n(r)$ *be the number of elements of this sequence in the disc* $|z| \leq r$, $n_k = n(1-2^{-k})$. *Then*

$$\int_0^1 \omega(1-r) n^p(r)(1-r)^\gamma dr < +\infty$$

if and only if

$$\sum_k \frac{n_k^p \omega(2^{-k})}{2^{k(\gamma+1)}} < +\infty.$$

We define $(NA)_{p,\gamma,\omega}$ as the class of all analytic functions on the unit disc such that

$$\int_0^1 \left[\sup_{0<r<R} T(r,f)(1-r)^\gamma\right]^p \omega(1-R)dR < +\infty.$$

Proposition 1. *Let $\{z_k\}$ be the sequence of zeroes of a function $f \in (NA)_{p,\gamma,\omega}$ where $\omega \in S$. Then we have*

$$\sum_k \frac{n_k^p \omega(1/2^k)}{2^{k((\gamma+1)p+1)}} < +\infty.$$

The next theorem gives a new parametric representation of the $(NA)_{p,\gamma,\delta}$ classes and is based on the previous theorem.

Theorem 2. *For $0 < p < \infty$, $\gamma > 0$ and $\delta > 0$ the $(NA)_{p,\gamma,\delta}$ class coincides with the set of functions allowing representation*

$$\begin{aligned} f(z) &= C_\lambda z^\lambda \prod_k \left(1 - \frac{z}{z_k}\right) \exp\left(\frac{t+1}{\pi}\int_0^1\int_{-\pi}^{\pi} \frac{(1-\rho^2)\ln\left|1-\frac{\rho e^{i\phi}}{z_k}\right|\rho d\rho d\phi}{(1-\rho e^{-i\phi}z)^{t+2}}\right) \\ &\quad \exp(h(z)), \end{aligned}$$

where $0 < p \le 1$, $t > \frac{\delta+1}{p} + \gamma - 1$ or $p > 1$, $t > \frac{\delta}{p} + \gamma$, C_λ is complex number, $\lambda \ge 0$, the sequence $\{z_k\}$ satisfies condition (6) and h is a holomorphic function in the unit disk satisfying the following condition:

$$\int_0^1 \left[\sup_{0<\tau<R} \int_{\mathbb{T}} |h(\tau\xi)|d\xi(1-\tau)^\gamma\right]^p (1-R)^\delta dR < +\infty.$$

Proof of Proposition 1. We can assume that $f(0) = 1$. We start as in the proof of Theorem 1 from [9], [10], but with more careful examination of estimates in Jensen's formula. By the classical Jensen's inequality we have

$$\int_0^1 \left[\sup_{0<\tau<R} \left(\int_0^\tau \frac{n(u)}{u}du\right)(1-\tau)^\gamma\right]^p \omega(1-R)dR < +\infty.$$

Also, we have the following estimates where $1/3 < R < 1$ and $\tilde{R} = \frac{3R-1}{2} < R$:

$$\begin{aligned} \sup_{0<\tau<R} \int_0^\tau \frac{n(u)}{u}du(1-\tau)^\gamma &\ge C \sup_{1/3<\tau<R} \int_{\tau-\frac{1-\tau}{2}}^\tau \frac{n(u)}{u}du(1-\tau)^\gamma \\ &\ge C \sup_{1/3<\tau<R} n\left(\frac{3\tau-1}{2}\right)(1-\tau)(1-\tau)^\gamma \end{aligned}$$

$$\geq \quad C \sup_{0<\rho<\tilde{R}} n(\rho)(1-\rho)^{\gamma+1}$$

Therefore, it is easily seen that we have

$$\int_{1/3}^{1} \omega(1-R) \sup_{0<\rho<\tilde{R}} n(\rho)^p (1-\rho)^{(\gamma+1)p} dR < +\infty$$

and, using defining properties of $\omega \in S$ we obtain, with $\tau_0 = \max(1/3, q_\omega)$:

$$\begin{aligned} +\infty \quad &> \quad \int_{\tau_0}^{1} \omega(1-R)(1-\tilde{R})^{(\gamma+1)p} n(\tilde{R})^p dR \\ &= \quad \int_{\tau_0}^{1} \omega(2(1-\tilde{R})/3)(1-\tilde{R})^{(\gamma+1)p} n(\tilde{R})^p d\,R \\ &\geq \quad c_\omega \int_{\tau_0}^{1} \omega(1-\tilde{R}) n(\tilde{R})^p (1-\tilde{R})^{(\gamma+1)p} d\,\tilde{R} \end{aligned}$$

Hence $\int_0^1 \omega(1-\tilde{R})(1-\tilde{R})^{(\gamma+1)p} n(\tilde{R})^p d\,\tilde{R} < +\infty$. Now it remains to apply Lemma 1 to finish the proof of the proposition.

Proof of Theorem 1. The implication a) $\Rightarrow$ b) follows from Proposition 1 applied to $\omega(t) = t^\delta$. To prove the converse statement, we fix a $t > -1$ satisfying conditions of Proposition A and Proposition B. Then we have

$$\int_{-\pi}^{\pi} \left| \ln \left| \prod_t (z, \{z_k\}) \right| \right| d\phi \leq C \sum_k (1-|z_k|)^{t+2} \int_{-\pi}^{\pi} \frac{d\phi}{|1-\tau\tau_k e^{i\phi}|^{t+2}},$$

where $z = \tau e^{i\phi}$, $z_k = \tau_k \xi_k$, $\tau_k = |z_k|$. Hence for $0 < p \leq 1$ and $t > \frac{\delta+1}{p} + \gamma - 1$

$$\begin{aligned} \| \prod_t \|^p_{(NA)_{p,\gamma,\delta}} \quad &\leq \quad C \int_0^1 \left[\sum_k \frac{(1-|z_k|)^{t+2}}{(1-R|z_k|)^{t+1-\gamma}} \right]^p (1-R)^\delta dR \\ &\leq \quad C \int_0^1 (1-R)^\delta \left[\int_0^1 \frac{(1-s)^{t+1} n(s) ds}{(1-Rs)^{t+1-\gamma}} \right]^p d\,R \\ &\leq \quad C \int_0^1 (1-R)^\delta \sum_k \frac{n_k^p 2^{-k[(t+1)p+p]}}{[1-(1-\frac{1}{2^{k+1}})R]^{(t+1-\gamma)p}} d\,R \\ &\leq \quad C \sum_k n_k^p \frac{2^{-k(t+1)p} 2^{-kp}}{2^{-k[(t+1-\gamma)p-\delta-1]}} \\ &\leq \quad C \sum_k \frac{n_k^p}{2^{k((\gamma+1)p+\delta+1)}} \end{aligned}$$

since one can easily verify that

$$\sum_k \frac{(1-|z_k|)^{t+2}}{(1-R|z_k|)^{t+1-\gamma}} \quad = \quad \int_0^1 \frac{(1-s)^{t+2} d\,u(s)}{(1-Rs)^{t+1-\gamma}}$$

$$\begin{aligned}
&= \frac{(1-s)^{t+2}n(s)}{(1-Rs)^{t+1-\gamma}}|_0^1 - \int_0^1 n(s)\left(\frac{(1-s)^{t+2}}{(1-Rs)^{t+1-\gamma}}\right)' d\,s \\
&= \int_0^1 \frac{n(s)(t+2)(1-s)^{t+1}d\,s}{(1-Rs)^{t+1-\gamma}} \\
&- \int_0^1 \frac{n(s)(1-s)^{t+2}}{(1-Rs)^{t+2-\gamma}}(t+1-\gamma)Rd\,s \\
&\leq C\int_0^1 \frac{n(s)(1-s)^{t+1}}{(1-Rs)^{t+1-\gamma}}(t+2)d\,s,
\end{aligned}$$

and

$$\begin{aligned}
\left[\int_0^1 \frac{n(s)(1-s)^{t+1}}{(1-Rs)^{t+1-\gamma}}d\,s\right]^p &\leq C\left[\sum_k \frac{n(1-2^{-k-1})2^{-k(t+1)}2^{-k}}{(1-\rho_k R)^{t+1-\gamma}}\right]^p \\
&\leq C\sum_k \frac{n_k^p 2^{-k(t+1)}2^{-kp}}{(1-\rho_k R)^{(t+1-\gamma)p}}.
\end{aligned}$$

We also used that for $\mu = (t+1-\gamma)p - (\delta+1) > 0$ and $\delta > -1$ we have

$$\int_0^1 \frac{(1-R)^\delta}{(1-\rho_k R)^{(t+1-\gamma)p}}d\,R \leq C\left(\frac{1}{2^{-k}}\right)^\mu,$$

where $\rho_k = 1-2^{-k}$, $k \geq 0$. Now let $p > 1$. Let q be the conjugate exponent and choose $t > \gamma - 1$. Then we have

$$\begin{aligned}
\|\prod_t\|^p_{(NA)_{p,\gamma,\delta}} &\leq C\int_0^1\left[\int_0^1 \frac{n(s)(1-s)^{t+1}d\,s}{(1-Rs)^{t+1-\gamma}}\right]^p (1-R)^\delta d\,R \\
&= C\left(\int_0^1 (1-R)^{\delta/p}\psi(R)\left(\int_0^1 \frac{n(s)(1-s)^{t+1}d\,s}{(1-Rs)^{t+1-\gamma}}\right)\right)^p \\
&= C[I_1+I_2]^p,
\end{aligned}$$

for a suitably chosen non-negative $\psi \in L^q(0,1)$, $\|\psi\|_q = 1$, where

$$\begin{aligned}
I_1 &= \int_0^1 \frac{n(s)(1-s)^{t+1}}{(1-Rs)^{t+1-\gamma}}\left(\int_0^s \psi(R)(1-R)^{\delta/p}d\,R\right)d\,s \\
I_2 &= \int_0^1 \frac{n(s)(1-s)^{t+1}}{(1-Rs)^{t+1-\gamma}}\left(\int_s^1 \psi(R)(1-R)^{\delta/p}d\,R\right)d\,s.
\end{aligned}$$

Further, using Hardy and Hölder inequalities, we obtain

$$\begin{aligned}
I_1 &\leq C\int_0^1 n(s)(1-s)^{t+1}\int_0^s \frac{\psi(R)(1-R)^{\delta/p}}{(1-R)^{t+1-\gamma}}d\,Rd\,s \\
&\leq C\int_0^1 \frac{n(s)(1-s)^{t+1}}{(1-s)^{t-\gamma-\frac{\delta}{p}}}\int_0^s \frac{\psi(R)}{1-R}d\,Rd\,s
\end{aligned}$$

$$\begin{aligned}
&\leq C\left(\int_0^1 n(s)^p(1-s)^{\gamma p+p+\delta}d\,s\right)^{1/p}\\
&\asymp \left(\sum_k \frac{n_k^p}{2^{k((\gamma+1)p+\delta+1)}}\right)^{1/p}
\end{aligned}$$

for $t>\gamma+\delta/p$. Using again Hardy and Hölder inequalities we estimate the second integral:

$$\begin{aligned}
I_2 &= \int_0^1 n(s)(1-s)^{t+1}d\,s\int_s^1 \frac{(1-R)^{\delta/p}\psi(R)}{(1-Rs)^{t+1-\gamma}}d\,R\\
&\leq C\int_0^1 n(s)\left(\int_s^1 \frac{(1-R)^{\delta/p}\psi(R)}{(1-Rs)^{-\gamma}}d\,R\right)d\,s\\
&\leq C\int_0^1 n(s)(1-s)^{\frac{\delta}{p}+\gamma}\int_0^{1-s}\psi(1-u)d\,u d\,s\\
&\leq C\left[\int_0^1 n(s)^p(1-s)^{\gamma p+p+\delta}d\,s\right]^{1/p}\left[\int_0^1\left(\frac{1}{1-s}\int_0^{1-s}\psi(1-u)d\,u\right)^q d\,s\right]^{1/q}\\
&\leq C\|\psi\|_q\left[\int_0^1 n(s)^p(1-s)^{\gamma p+p+\delta}d\,s\right]^{1/p}\\
&\asymp \left(\sum_k \frac{n_k^p}{2^{k((\gamma+1)p+\delta+1)}}\right)^{1/p}.
\end{aligned}$$

The last asymptotic relation follows from a decomposition of $(0,1)$ into subintervals $[1-2^{-k},1-2^{-k-1}]$, $k\geq 0$.

As in the proof of Theorem 1 from [9], [10], it remains to show that the infinite product $\prod_t$ converges for the considered values of t. In fact, if (6) holds, then $\sum_k(1-|z_k|)^{t+2}<\infty$. This can be proved along the lines of [9] and [10] and we omit details.

Proof of Theorem 2. Here we follow arguments from [9], [10] and [11] for $(NA)_{p,\gamma,\delta}$ classes. It is important to note that if $f,g\in(NA)_{p,\gamma,\delta}$ and $Z(f)\supset Z(g)$, then $f/g\in(NA)_{p,\gamma,\delta}$. Here $Z(f)$ denotes the set of zeros of a function f, counting multiplicities. Note that for t as in the theorem we have $\prod_t(z,\{z_k\})\in(NA)_{p,\gamma,\delta}$. Therefore

$$\psi(z)=\frac{f(z)}{C_\lambda z^\lambda\prod_t(z,\{z_k\})}\in(NA)_{p,\gamma,\delta}.$$

Using

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\ln|\psi(re^{i\phi})|d\,\phi=\ln|\psi(0)|$$

and

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|\ln|\psi(re^{i\phi})||d\,\phi=\frac{1}{2\pi}\int_{-\pi}^{\pi}\ln^+|\psi(re^{i\phi})|d\,\phi-\ln|\psi(0)|$$

we obtain

$$\int_0^1 \left(\sup_{r<R} \int_{\mathbb{T}} |\ln|\psi(re^{i\phi})||d\,\phi (1-r)^\gamma \right)^p (1-R)^\delta d\,R < \infty.$$

It remains to set $h(z) = \lg \psi(z)$, $z \in \mathbb{D}$ where we choose the main branch of the logarithm. The reverse implication follows from Theorem 1 and Theorem 2 is proved.

Now we turn to the meromorphic classes $(NM)_{p,\gamma,\delta}$. We give a complete analogue of Theorems C, D and E for these classes.

Theorem 3. *The $(NM)_{p,\gamma,\delta}$ class coincides with the class of all meromorphic functions having representation*

$$f(z) = \frac{g(z)}{\prod_\alpha(z, \{b_k\})},$$

where $g \in (NA)_{p,\gamma,\delta}$, $\{b_k\}$ is a sequence in $\mathbb{D}$ satisfying condition (6), $\alpha > \frac{\delta+1}{p}$ for $p \le 1$ and $\alpha > \frac{\delta}{p} + \gamma$ for $p > 1$. In this representation $\{b_k\}$ is the sequence of poles of f, counting orders of poles.

An immediate consequence of the above theorem is the following result.

Corollary 1. *Two sequences $\{a_k\}$ and $\{b_k\}$ in $\mathbb{D}$ are sequences of zeros and poles of a function in $(NM)_{p,\gamma,\delta}$ if and only if both sequences satisfy condition (6).*

Proof of Theorem 3. If f is in $(NM)_{p,\gamma,\delta}$, then by definition we have

$$\int_0^1 \left(\sup_{0<r<R} T(r,f)(1-r)^\gamma \right)^p (1-R)^\delta d\,R < \infty.$$

It follows, using (4), that

$$\int_0^1 \left(\sup_{0<r<R} N(r)(1-r)^\gamma \right)^p (1-R)^\delta d\,R < \infty.$$

The same reasoning as in the proof of Theorem 1 shows that sequence $\{b_k\}$ satisfies condition (6). Therefore $\prod_\alpha(z, \{b_k\}) \in (NA)_{p,\gamma,\delta}$ and hence $g = f \prod_\alpha(z, \{b_k\}) \in (NA)_{p,\gamma,\delta}$ as desired. It is clear that any function admitting the above representation is in $(NM)_{p,\gamma,\delta}$.

Theorem 4. *Let $0 < p < \infty$, $\gamma \ge 0$ and $\delta \ge 0$. Then a meromorphic function f belongs to $(NM)_{p,\gamma,\delta}$ if and only if it admits a representation*

$$f(z) = C_\lambda z^\lambda \frac{\prod_\alpha(z, \{a_k\})}{\prod_\alpha(z, \{b_k\})} \exp h(z), \qquad z \in \mathbb{D}$$

where C_λ is a complex number, $\lambda \in \mathbb{Z}$, sequences $\{a_k\}$ and $\{b_k\}$ satisfy condition (6), $\alpha > \frac{\delta+1}{p} + \gamma - 1$ for $0 < p \le 1$ and $\alpha > \frac{\delta}{p} + \gamma$ for $p > 1$ and function $h \in H(\mathbb{D})$ satisfies

$$\int_0^1 \left[\sup_{0<r<R} \int_{-\pi}^{\pi} |h(re^{i\phi})| d\phi (1-r)^\gamma \right]^p (1-R)^\delta dR < \infty.$$

This theorem follows easily from Theorem 2 and Theorem 3 and is an analogue of Theorems C, D and E for the classes we investigate in this paper.

Remark 1. It is possible to obtain certain uncomplete parametric representations of functions in $(NA)_{p,\gamma,\delta}$ classes involving $B_\alpha(z, \{z_k\})$ products from Proposition C by using simple embeddings between spaces we mentioned in the previous section, for which parametric representations already exist (see [12]), and our $(NA)_{p,\gamma,\delta}$ and $(NM)_{p,\gamma,\delta}$ spaces.

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