

ABSTRACT METRIC SPACES AND CARISTI-NGUYEN-TYPE THEOREMS

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Abstract

In this paper we prove cone metric versions of common fixed point theorems for two and four mappings with the Caristi-Nguyen-type contractive conditions. Also, sufficient conditions for two or four mappings to have no common periodic points are deduced. Examples are given to distinguish these results from the known ones and to show that certain conditions cannot be omitted.

1 Introduction

As a generalization of metric spaces, cone metric spaces play an important role in Fixed Point Theory, Computer Science and some other research areas as well as in Functional Analysis. Huang and Zhang reintroduced in [8] such spaces replacing the set of real numbers with an ordered Banach space as the codomain for a metric (we note that such spaces were known earlier under the name of K -metric spaces, see [15]). They also discussed some properties of convergence of sequences and proved fixed point theorems for contractive mappings on cone metric spaces. Recently, some common fixed point theorems were proved for maps on cone metric spaces (see, e.g., [1]–[3], [9]–[11] and references therein).

Banach Contraction Principle in metric spaces has been generalized and extended in various ways. One of the important extensions was given by J. Caristi [5] who used contractivity condition of the form

$$d(x, Tx) \leq \phi(x) - \phi(Tx),$$

where ϕ was a lower semicontinuous function from the given complete metric space to the set of nonnegative real numbers. Nguyen H.D. in [14] made a further modification, considering a pair of (orbitally) continuous self-mappings S and T and using

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the combined Banach-Caristi condition

$$d(Sx, Ty) \leq \lambda d(x, y) + \sum_{i=1}^N (\phi_i(x) - \phi_i(Sx) + \phi_i(y) - \phi_i(Ty)),$$

where $\{\phi_i \mid i = 1, 2, \dots, N\}$ was a set of real functions on the given metric space. He also proved a version of this result for four mappings (the case first considered by B. Fisher [7]). M. Alimohammady et al. [3] adapted the first of mentioned results to the setting of cone metric spaces.

In this paper we prove cone metric versions of common fixed point theorems for two and four mappings with the Caristi-Nguyen-type contractive conditions. Also, sufficient conditions for two or four mappings to have no common periodic points are deduced. Examples are given to distinguish these results from the known ones and to show that certain conditions cannot be omitted.

2 Preliminaries

We repeat some definitions and results from [8], which will be needed in the sequel.

Let E be a real Banach space with θ as the zero element and let P be a subset of E with the interior $\text{int } P$. The subset P is called a *cone* if: (a) P is closed, nonempty and $P \neq \{\theta\}$; (b) $a, b \in \mathbb{R}$, $a, b \geq 0$, and $x, y \in P$ imply $ax + by \in P$; (c) $P \cap (-P) = \{\theta\}$. For the given cone P , a partial ordering \preceq with respect to P is introduced in the following way: $x \preceq y$ if and only if $y - x \in P$. We write $x \prec y$ to indicate that $x \preceq y$, but $x \neq y$. If $y - x \in \text{int } P$, we write $x \ll y$. If $\text{int } P \neq \emptyset$, the cone P is called *solid*.

The cone P is called *normal* if there is a number $K > 0$, such that, for all $x, y \in E$, $\theta \preceq x \preceq y$ implies $\|x\| \leq K\|y\|$ or, equivalently, if $(\forall n) x_n \preceq y_n \preceq z_n$, and $\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} z_n = x$ imply $\lim_{n \rightarrow \infty} y_n = x$ (for details see, e.g., [4, 6]).

The cone P is called *regular* if every nondecreasing sequence in E which is order-bounded from above is convergent, i.e., if whenever $\{x_n\}$ is a sequence in E such that $x_1 \preceq x_2 \preceq \dots \preceq x_n \preceq \dots \preceq y$ for some $y \in E$, then there exists $x \in E$ such that $x_n \rightarrow x$, $n \rightarrow \infty$. Every regular cone is normal [6].

Let X be a nonempty set. Suppose that the mapping $d : X \times X \rightarrow E$ satisfies: (d₁) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$; (d₂) $d(x, y) = d(y, x)$ for all $x, y \in X$; (d₃) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$. Then d is called a *cone metric* on X and (X, d) is called a *cone metric space* [8]. The concept of a cone metric space is obviously more general than that of a metric space.

Let (X, d) be a cone metric space, and $\{x_n\}$ a sequence in X . We say that $\{x_n\}$ converges to $x \in X$ if for every $c \in E$ with $\theta \ll c$, there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x) \ll c$ for all $n > n_0$. We write $\lim_{n \rightarrow \infty} x_n = x$, or $x_n \rightarrow x$, $n \rightarrow \infty$. If for every $c \in E$ with $\theta \ll c$, there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) \ll c$ for all $n, m > n_0$, then $\{x_n\}$ is called a Cauchy sequence in X . If every Cauchy sequence is convergent in X , then X is called a *complete cone metric space* (see [8]).

Let us recall that if P is a normal (*a fortiori* regular) cone then a sequence $\{x_n\}$ in X converges to $x \in X$ if and only if $d(x_n, x) \rightarrow \theta$, $n \rightarrow \infty$, i.e., if and only if

$\|d(x_n, x)\| \rightarrow 0, n \rightarrow \infty$. Further, $\{x_n\}$ in X is a Cauchy sequence if and only if $d(x_n, x_m) \rightarrow \theta, n, m \rightarrow \infty$ [8, Lemma 1], i.e., if and only if $\|d(x_n, x_m)\| \rightarrow 0, n, m \rightarrow \infty$.

A pair of self-mappings (F, G) on a cone metric space (X, d) is said to be *compatible* (see [10]) if for arbitrary sequence $\{x_n\}$ in X such that $\lim_{n \rightarrow \infty} Fx_n = \lim_{n \rightarrow \infty} Gx_n = t \in X$, and for arbitrary $c \in \text{int } P$, there exists $n_0 \in \mathbb{N}$ such that $d(FGx_n, GFx_n) \ll c$ whenever $n \geq n_0$. In the case of a normal cone P , we have that a pair (F, G) is compatible in the frame of cone metric spaces if and only if it is compatible in the sense of usual metric spaces (see, e.g., [13]).

3 Common fixed points of four mappings

Theorem 3.1. *Let (X, d) be a complete cone metric space over a regular cone P and let F, G, S and T be four continuous self-mappings on X . Suppose that*

- 1) $SX \subset GX, TX \subset FX$,
- 2) (S, F) and (T, G) are two pairs of compatible mappings,
- 3) *there exists a finite collection of functions $\{\phi_i : X \rightarrow P \mid i = 1, 2, \dots, N\}$ such that the inequality*

$$d(Sx, Ty) \preceq \lambda d(Fx, Gy) + \sum_{i=1}^N (\phi_i(Fx) - \phi_i(Sx) + \phi_i(Gy) - \phi_i(Ty)) \quad (3.1)$$

holds for some $\lambda \in [0, 1)$ and all $x, y \in X$. Then F, G, S and T have a unique common fixed point.

Proof. Let $x_0 \in X$ be chosen arbitrarily. There exists $x_1 \in X$ such that $Gx_1 = Sx_0 = z_0$ (because $SX \subset GX$). Since $TX \subset FX$, there exists $x_2 \in X$ such that $Fx_2 = Tx_1 = z_1$. We continue in this manner. In general, $x_{2n+1} \in X$ is chosen such that $Gx_{2n+1} = Sx_{2n} = z_{2n}$ and $x_{2n+2} \in X$ is chosen such that $Fx_{2n+2} = Tx_{2n+1} = z_{2n+1}$.

First we prove that $\{z_n\}_{n \geq 1}$ is a Cauchy sequence. We have the following two possible cases:

- For $n = 2\ell, \ell \in \mathbb{N}$, we have

$$\begin{aligned} d(z_{2\ell}, z_{2\ell+1}) &= d(Sx_{2\ell}, Tx_{2\ell+1}) \preceq \lambda d(Fx_{2\ell}, Gx_{2\ell+1}) \\ &\quad + \sum_{i=1}^N (\phi_i(Fx_{2\ell}) - \phi_i(Sx_{2\ell}) + \phi_i(Gx_{2\ell+1}) - \phi_i(Tx_{2\ell+1})) \\ &= \lambda d(z_{2\ell-1}, z_{2\ell}) + \sum_{i=1}^N (\phi_i(z_{2\ell-1}) - \phi_i(z_{2\ell}) + \phi_i(z_{2\ell}) - \phi_i(z_{2\ell+1})) \\ &= \lambda d(z_{2\ell-1}, z_{2\ell}) + \sum_{i=1}^N (\phi_i(z_{2\ell-1}) - \phi_i(z_{2\ell+1})). \end{aligned}$$

- For $n = 2\ell + 1$, $\ell \in \mathbb{N}$, we have

$$\begin{aligned}
d(z_{2\ell+1}, z_{2\ell+2}) &= d(Sx_{2\ell+2}, Tx_{2\ell+1}) \preceq \lambda d(Fx_{2\ell+2}, Gx_{2\ell+1}) \\
&\quad + \sum_{i=1}^N (\phi_i(Fx_{2\ell+2}) - \phi_i(Sx_{2\ell+2}) + \phi_i(Gx_{2\ell+1}) - \phi_i(Tx_{2\ell+1})) \\
&= \lambda d(z_{2\ell+1}, z_{2\ell}) + \sum_{i=1}^N (\phi_i(z_{2\ell+1}) - \phi_i(z_{2\ell+2}) + \phi_i(z_{2\ell}) - \phi_i(z_{2\ell+1})) \\
&= \lambda d(z_{2\ell}, z_{2\ell+1}) + \sum_{i=1}^N (\phi_i(z_{2\ell}) - \phi_i(z_{2\ell+2})).
\end{aligned}$$

From these two inequalities, one can deduce that, for each $n \geq 1$,

$$\sum_{\ell=1}^n d(z_{2\ell}, z_{2\ell+1}) \preceq \lambda \sum_{\ell=1}^n d(z_{2\ell-1}, z_{2\ell}) + \sum_{i=1}^N \sum_{\ell=1}^n (\phi_i(z_{2\ell-1}) - \phi_i(z_{2\ell+1})) \quad (3.2)$$

and

$$\sum_{\ell=1}^n d(z_{2\ell+1}, z_{2\ell+2}) \preceq \lambda \sum_{\ell=1}^n d(z_{2\ell}, z_{2\ell+1}) + \sum_{i=1}^N \sum_{\ell=1}^n (\phi_i(z_{2\ell}) - \phi_i(z_{2\ell+2})). \quad (3.3)$$

From (3.2) and (3.3), we have

$$\sum_{j=2}^{2n+1} d(z_j, z_{j+1}) \preceq \lambda \sum_{j=1}^{2n} d(z_j, z_{j+1}) + \sum_{i=1}^N (\phi_i(z_1) - \phi_i(z_{2n+1}) + \phi_i(z_2) - \phi_i(z_{2n+2})).$$

Thus,

$$\begin{aligned}
\sum_{j=1}^{2n+1} d(z_j, z_{j+1}) &\preceq d(z_1, z_2) + \lambda \sum_{j=1}^{2n+1} d(z_j, z_{j+1}) \\
&\quad + \sum_{i=1}^N (\phi_i(z_1) - \phi_i(z_{2n+1}) + \phi_i(z_2) - \phi_i(z_{2n+2})) \\
&\preceq d(z_1, z_2) + \lambda \sum_{j=1}^{2n+1} d(z_j, z_{j+1}) + \sum_{i=1}^N (\phi_i(z_1) + \phi_i(z_2)).
\end{aligned}$$

Since $\lambda \in [0, 1)$, we obtain that for all $n \geq 1$

$$\sum_{j=1}^{2n+1} d(z_j, z_{j+1}) \preceq \frac{1}{1-\lambda} d(z_1, z_2) + \frac{1}{1-\lambda} \sum_{i=1}^N (\phi_i(z_1) + \phi_i(z_2)). \quad (3.4)$$

The right-hand side of the inequality (3.4) is an element of P . Therefore, the sequence $\{\sum_{j=1}^n d(z_j, z_{j+1})\}_{n \geq 1}$ is an increasing sequence which is bounded from

above in the regular cone P , so $\{\sum_{j=1}^n d(z_j, z_{j+1})\}_{n \geq 1}$ is convergent, i.e., the series $\sum_{j=1}^{\infty} d(z_j, z_{j+1})$ converges. Hence, for $n < m$,

$$d(z_n, z_m) \preceq \sum_{j=n}^{m-1} d(z_j, z_{j+1}) \rightarrow \theta, \quad n \rightarrow \infty.$$

This gives that $\{z_n\}_{n \geq 1}$ is a Cauchy sequence in X .

Completeness of X implies the existence of $z \in X$ such that $\lim_{n \rightarrow \infty} z_n = z$, i.e.,

$$\lim_{n \rightarrow \infty} Gx_{2n+1} = \lim_{n \rightarrow \infty} Sx_{2n} = \lim_{n \rightarrow \infty} Fx_{2n} = \lim_{n \rightarrow \infty} Tx_{2n+1} = z.$$

Let us prove that $Sz = Fz$.

Using the continuity and compatibility of S and F , we have that

$$\theta \preceq d(Sz, Fz) \preceq d(Sz, SFx_{2n}) + d(SFx_{2n}, FSx_{2n}) + d(FSx_{2n}, Fz) \rightarrow \theta, \quad n \rightarrow \infty,$$

wherefrom it follows that $d(Sz, Fz) = \theta$, i.e., $Sz = Fz$.

By the same arguments as above (for mappings T and G), we conclude that $d(Tz, Gz) = \theta$, i.e., $Tz = Gz$. Let us prove that $Sz = Tz$. If we suppose that $\theta \prec d(Sz, Tz)$, then, for $\lambda \in [0, 1)$,

$$\begin{aligned} d(Sz, Tz) &\preceq \lambda d(Fz, Gz) + \sum_{i=1}^N (\phi_i(Fz) - \phi_i(Sz) + \phi(Gz) - \phi_i(Tz)) \\ &= \lambda d(Sz, Tz) + \sum_{i=1}^N (\phi_i(Sz) - \phi_i(Sz) + \phi_i(Tz) - \phi_i(Tz)) \\ &= \lambda d(Sz, Tz) \prec d(Sz, Tz), \end{aligned}$$

which is not possible. Thus, $d(Sz, Tz) = \theta$, so $Sz = Tz$. We can conclude that z is a coincidence point of S, F, T and G .

Let us prove now that z is a fixed point for S . Using (3.1), we have

$$\begin{aligned} d(Sz, z_{2n+1}) &= d(Sz, Tx_{2n+1}) \preceq \lambda d(Fz, Gx_{2n+1}) \\ &\quad + \sum_{i=1}^N (\phi_i(Fz) - \phi_i(Sz) + \phi_i(Gx_{2n+1}) - \phi_i(Tx_{2n+1})) \\ &\preceq \lambda d(Sz, Gx_{2n+1}) \\ &\quad + \sum_{i=1}^N (\phi_i(Sz) - \phi_i(Sz) + \phi_i(Gx_{2n+1}) - \phi_i(Tx_{2n+1})) \\ &= \lambda d(Sz, z_{2n}) + \sum_{i=1}^N (\phi_i(z_{2n}) - \phi_i(z_{2n+1})), \end{aligned}$$

and

$$\begin{aligned} d(Sz, z_{2n}) &= d(Tz, z_{2n}) = d(Sx_{2n}, Tz) \preceq \lambda d(Fx_{2n}, Gz) \\ &\quad + \sum_{i=1}^N (\phi_i(Fx_{2n}) - \phi_i(Sx_{2n}) + \phi_i(Gz) - \phi_i(Tz)) \\ &\preceq \lambda d(Sz, z_{2n-1}) + \sum_{i=1}^N (\phi_i(z_{2n-1}) - \phi_i(z_{2n})). \end{aligned}$$

From these inequalities, one can deduce that, for each $n \geq 1$,

$$\sum_{j=1}^n d(Sz, z_j) \preceq \lambda \sum_{j=0}^n d(Sz, z_j) + \sum_{i=1}^N \sum_{j=0}^n (\phi_i(z_j) - \phi_i(z_{j+1})),$$

so

$$(1 - \lambda) \sum_{j=1}^n d(Sz, z_j) \preceq \lambda d(Sz, z_0) + \sum_{i=1}^N \phi_i(z_0).$$

Since $\lambda \in [0, 1)$, we obtain that

$$\sum_{j=1}^n d(Sz, z_j) \preceq \frac{\lambda}{1 - \lambda} d(Sz, z_0) + \frac{1}{1 - \lambda} \sum_{i=1}^N \phi_i(z_0). \quad (3.5)$$

The right-hand side of the previous inequality is an element of P . Therefore, the sequence $\{\sum_{j=1}^n d(Sz, z_j)\}_{n \geq 1}$ is an increasing sequence which is bounded from above in the regular cone P . So $\{\sum_{j=1}^n d(Sz, z_j)\}_{n \geq 1}$ is convergent, i.e., the series $\sum_{j=1}^{\infty} d(Sz, z_j)$ converges. Hence, $d(Sz, z_n) \rightarrow \theta$, $n \rightarrow \infty$ and, therefore, $\lim_{n \rightarrow \infty} z_n = Sz$. Since the limit of a convergent sequence in a cone metric space is unique, and $\lim_{n \rightarrow \infty} z_n = z$, we have $Sz = z$. We can conclude that z is a common fixed point of S, T, F and G .

In order to prove uniqueness of the fixed point, we suppose that $\tilde{z} \neq z$ is another fixed point for S, T, F and G . From (3.1),

$$\begin{aligned} d(z, \tilde{z}) &= d(Sz, T\tilde{z}) \preceq \lambda d(Fz, G\tilde{z}) + \sum_{i=1}^N (\phi_i(Fz) - \phi_i(Sz) + \phi_i(G\tilde{z}) - \phi_i(T\tilde{z})) \\ &= \lambda d(z, \tilde{z}) + \sum_{i=1}^N (\phi_i(z) - \phi_i(z) + \phi_i(\tilde{z}) - \phi_i(\tilde{z})) \\ &= \lambda d(z, \tilde{z}), \end{aligned}$$

where $\lambda \in [0, 1)$. It follows that $d(z, \tilde{z}) = \theta$, i.e., $z = \tilde{z}$. \square

Remark 3.2. The sequence $\{z_n\}$, constructed in the previous proof, is usually called a *Jungck sequence* for the mappings S, T, F, G . It follows from the previous proof that each Jungck sequence of the given mappings converges to the same point, the unique common fixed point of these mappings.

The following result is an immediate consequence of Theorem 3.1.

Corollary 3.3. *Let (X, d) be a complete cone metric space over a regular cone P and let F, G, S and T be four continuous self-mappings on X . Suppose that*

- 1) $SX \subset GX, TX \subset FX,$
- 2) (S, F) and (T, G) are two pairs of compatible mappings,
- 3) there exists $\phi : X \rightarrow P$ such that the inequality

$$d(Sx, Ty) \preceq \phi(Fx) - \phi(Sx) + \phi(Gy) - \phi(Ty)$$

holds for all $x, y \in X$.

Then F, G, S and T have a unique common fixed point.

Remark 3.4. For mappings $S, T : X \rightarrow X$ on a cone metric space (X, d) , it is said that they satisfy Caristi-type condition with respect to a finite collection of functions $\{\phi_i : X \rightarrow P \mid i = 1, 2, \dots, N\}$ if, for all $x, y \in X$, the inequality

$$d(Sx, Ty) \preceq \sum_{i=1}^N (\phi_i(x) - \phi_i(Sx) + \phi_i(y) - \phi_i(Ty))$$

holds. In the case when $i = 1$, we have the condition

$$d(Sx, Ty) \preceq \phi(x) - \phi(Sx) + \phi(y) - \phi(Ty).$$

By taking $F = G = I_X$ in the condition 3) of Corollary 3.3, we obtain that the mappings S and T satisfy Caristi-type condition in a complete cone metric space (X, d) over a regular cone P .

Remark 3.5. By taking $F = G = I_X$ in the condition 3) of Theorem 3.1, we obtain the contractive condition of [3, Theorem 2.3] (which is a combination of Banach and Caristi-type condition). Hence our Theorem 3.1 generalizes [3, Theorem 2.3].

By taking $E = \mathbb{R}$ and $P = [0, +\infty)$ in Theorem 3.1, we obtain [14, Theorem 2.1] as a consequence.

We also note that a Caristy-type fixed point result was proved in [2] as well, but with the stronger condition of the so called strong minihedralness of the cone.

The following example shows that mappings may not satisfy conditions of Fisher's theorem for four mappings [7], but still satisfy all conditions of Theorem 3.1. Hence, this theorem is a proper generalization of Fisher's theorem.

Example 3.6. Let $X = \{1, 2, 3\}$. Consider the mappings

$$T : \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \end{pmatrix}, S : \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}, F : \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \end{pmatrix}, G : \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}.$$

Let the metric $d : X \times X \rightarrow \mathbb{R}$ be defined in the following way

$$d(1, 2) = 5, \quad d(2, 3) = 3, \quad d(3, 1) = 7, \quad d(x, y) = d(y, x), \quad d(x, x) = 0,$$

for all $x, y \in X$. It is easy to see that $SX = \{1\}$, $TX = \{1, 3\}$, $FX = \{1, 3\}$, $GX = \{1, 2, 3\}$ and $SX \subset FX$, $TX \subset GX$. Since $d(S2, T2) = d(1, 3) = 7 > 0 = d(3, 3) = d(F2, G2)$, i.e., $d(S2, T2) > d(F2, G2)$, these mappings do not satisfy the conditions of Fisher's theorem.

Take the function $\phi : \begin{pmatrix} 1 & 2 & 3 \\ 0 & 7 & 2 \end{pmatrix}$. We will show that S, T, F and G satisfy the contractive condition of Theorem 3.1. The inequalities

$$(1) \quad d(S1, T1) \leq \lambda d(F1, G1) + \phi(1) - \phi(S1) + \phi(1) - \phi(T1);$$

$$(2) \quad d(S1, T2) \leq \lambda d(F1, G2) + \phi(1) - \phi(S1) + \phi(2) - \phi(T2);$$

$$(3) \quad d(S2, T1) \leq \lambda d(F2, G1) + \phi(2) - \phi(S2) + \phi(1) - \phi(T1);$$

$$(4) \quad d(S2, T2) \leq \lambda d(F2, G2) + \phi(2) - \phi(S2) + \phi(2) - \phi(T2),$$

hold for $\lambda \geq \frac{2}{7}$. Indeed,

$$(1) \quad d(1, 1) \leq \lambda d(1, 1) + 0 - 0 + 0 - 0 \text{ holds for all } \lambda \in [0, 1];$$

$$(2) \quad d(1, 3) \leq \lambda d(1, 3) + 0 - 0 + \phi(2) - \phi(3), \text{ i.e., } 7 \leq 7\lambda + 5 \text{ is true for } \lambda \geq \frac{2}{7};$$

$$(3) \quad d(1, 1) \leq \lambda d(3, 3) + 7 - 0 + 7 - 2 \text{ holds for all } \lambda \in [0, 1];$$

$$(4) \quad d(1, 3) \leq \lambda d(3, 3) + 7 - 0 + 7 - 2 \text{ holds for all } \lambda \in [0, 1].$$

Thus, for $\frac{2}{7} \leq \lambda < 1$ all the conditions of Theorem 3.1 are satisfied. We obtain that S, T, F and G have a unique fixed point $z = 1$.

Using the previous construction, one can easily obtain a respective example for a cone metric space over a regular cone P . Indeed, take $E = \mathbb{R}^2$, $P = \{(x, y) \mid x \geq 0, y \geq 0\}$, and the cone metric $d_c : X \times X \rightarrow P$ defined by $d_c(a, b) = (2d(a, b), 3d(a, b))$ for $a, b \in X$, where $d(a, b)$ is defined above. All conclusions for the mappings S, T, F and G remain the same.

Remark 3.7. Using Example 3.6, we have

$$d(S1, T2) = d(1, 3) = 7 > 5 = \phi(1) - \phi(S1) + \phi(2) - \phi(T2).$$

and the mappings S and T do not satisfy the Caristi-type condition, either.

The following example shows the importance of the condition 2) from Theorem 3.1, i.e., that there are mappings S, T, F and G which are not compatible in pairs, and satisfy all other conditions of Theorem 3.1 including the contractive one, but do not have a common fixed point. Hence, when generalizing [14, Theorem 1.2] to the case of four functions, additional condition of compatibility is necessary.

Example 3.8. Let $X = \mathbb{R}$, $E = \mathbb{R}^2$, and $P = \{(x, y) \mid x \geq 0, y \geq 0\}$. We define $d : X \times X \rightarrow E$ as $d(x, y) = (|x - y|, 0)$. It is easy to see that (X, d) is a cone metric space and P is a regular cone. Now, we consider the mappings $S, T, F, G : X \rightarrow X$ defined by $Sx = Tx = 2 - x$, $Fx = Gx = 2x$, $x \in X$. Let $\phi : X \rightarrow P$ be defined by

$\phi(x) = (1, 1)$, for all $x \in X$, and let $\lambda = \frac{1}{2}$. We have that $Sx = Fx$ if and only if $x = \frac{2}{3} \in X$. Since $FS\frac{2}{3} = F\frac{4}{3} = \frac{8}{3}$ and $SF\frac{2}{3} = S\frac{4}{3} = \frac{2}{3}$, then S, F are not weakly compatible mappings, and therefore they are not compatible. It is easy to check that all the conditions of Theorem 3.1, except 2), are satisfied. In particular, for all $x, y \in X$, the contractive condition

$$d(Sx, Ty) \preceq \lambda d(Fx, Gy) + \phi(Fx) - \phi(Sx) + \phi(Gy) - \phi(Ty)$$

holds, but these mappings do not have a common fixed point.

In the following theorem we use another version of the Caristi-type condition. It is an extension of [14, Theorem 2.2] to the cone metric setting.

Theorem 3.9. *Let (X, d) be a complete cone metric space over a regular cone P , let S, F be compatible continuous self-mappings on X such that $SX \subset FX$, and let $\phi : X \rightarrow P$. Let $z \in X$ satisfy*

$$d(Sx, z) \preceq \lambda d(Fx, z) + \phi(Fx) - \phi(Sx)$$

for some $\lambda \in [0, 1)$ and all $x \in X$. Then z is a common fixed point of S and F .

Proof. Let us choose $x_0 \in X$ arbitrarily. Since $SX \subset FX$, there exists $x_1 \in X$ such that $Fx_1 = Sx_0$. Also, there exists $x_2 \in X$ such that $Fx_2 = Sx_1$. We continue in this manner. In general, $x_n \in X$ is chosen such that $Fx_n = Sx_{n-1}$.

Since, for any $k \in \mathbb{N}$, we have

$$d(Fx_k, z) = d(Sx_{k-1}, z) \preceq \lambda d(Fx_{k-1}, z) + \phi(Fx_{k-1}) - \phi(Fx_k),$$

we obtain

$$\sum_{k=1}^n d(Fx_k, z) \preceq \lambda \sum_{k=1}^n d(Fx_{k-1}, z) + \sum_{k=1}^n (\phi(Fx_{k-1}) - \phi(Fx_k)),$$

i.e.,

$$\sum_{k=1}^n d(Fx_k, z) \preceq \frac{\lambda}{1-\lambda} d(Fx_0, z) + \frac{1}{1-\lambda} \phi(Fx_0).$$

By the same arguments as in the proof of Theorem 3.1, we have that $d(Fx_n, z) \rightarrow \theta$ as $n \rightarrow \infty$, that is, $\lim_{n \rightarrow \infty} Fx_n = z$.

Using the continuity and the compatibility of S and F , we have

$$\theta \preceq d(Sz, Fz) \preceq d(Sz, SFx_n) + d(SFx_n, FSx_n) + d(FSx_n, Fz) \rightarrow \theta, \quad n \rightarrow \infty,$$

wherefrom it follows that $d(Sz, Fz) = \theta$, i.e., $Sz = Fz$.

Let us, now, prove that z is a fixed point of S . Suppose that $Sz \neq z$. Then,

$$d(Sz, z) \preceq \lambda d(Fz, z) + \phi(Fz) - \phi(Sz) = \lambda d(Sz, z) \prec d(Sz, z),$$

which gives a contradiction. Therefore, $Sz = Fz = z$.

In order to prove uniqueness, we suppose that $\tilde{z} \neq z$ is a fixed point of S and F . Then

$$\begin{aligned} d(z_1, z) &= d(Sz_1, z) \preceq \lambda d(Fz_1, z) + \phi(Fz_1) - \phi(Sz_1) \\ &= \lambda d(z_1, z) + \phi(z_1) - \phi(z_1) = \lambda d(z_1, z) \prec d(z_1, z). \end{aligned}$$

Hence, $z_1 = z$, which completes the proof of the theorem. \square

4 Mappings having no periodic points

In what follows, we will denote the set of all fixed points of a self-mapping T by $\Phi(T)$, i.e., $\Phi(T) = \{z \in X \mid Tz = z\}$.

Remark 4.1. It can be easily verified that if z is a fixed point of T , then z is also a fixed point of T^n , $n = 1, 2, \dots$, that is, $\Phi(T) \subset \Phi(T^n)$ if $\Phi(T) \neq \emptyset$. The converse statement is not valid. Indeed, the mapping $T : \mathbb{R} \rightarrow \mathbb{R}$ defined by $Tx = \frac{1}{2} - x$ has a unique fixed point, i.e., $\Phi(T) = \{\frac{1}{4}\}$, but every $x \in \mathbb{R}$ is a fixed point for T^2 . If $\Phi(T) = \Phi(T^n)$, $n = 1, 2, \dots$, then we say that T has no periodic points (for details see [12]).

In what follows we will obtain conditions for two, resp. four mappings to have no common periodic points, i.e., that

$$\begin{aligned} \Phi(T) \cap \Phi(S) &= \Phi(T^n) \cap \Phi(S^n), \quad \text{or} \\ \Phi(T) \cap \Phi(S) \cap \Phi(G) \cap \Phi(F) &= \Phi(T^n) \cap \Phi(S^n) \cap \Phi(G^n) \cap \Phi(F^n) \end{aligned}$$

holds for each n , respectively.

Theorem 4.2. *Let (X, d) be a complete cone metric space over a regular cone P and let $\{\phi_i : X \rightarrow P \mid i = 1, 2, \dots, N\}$ be a finite collection of functions. Suppose that S and T are two continuous self-mappings on X and that there exists $\lambda \in [0, 1)$ such that*

$$d(Sx, Ty) \preceq \lambda d(x, y) + \sum_{i=1}^N (\phi_i(x) - \phi_i(Sx) + \phi_i(y) - \phi_i(Ty))$$

for all $x, y \in X$. Then $\Phi(S) \cap \Phi(T) = \Phi(S^n) \cap \Phi(T^n)$ for each $n \in \mathbb{N}$.

Proof. Taking $F = G = I_X$ in Theorem 3.1, we obtain that $\Phi(S) \cap \Phi(T) = \{z\}$. Since $\Phi(S) \cap \Phi(T) \subset \Phi(S^n) \cap \Phi(T^n)$, we only have to prove that $\Phi(S) \cap \Phi(T) \supset \Phi(S^n) \cap \Phi(T^n)$. Let $v \in \Phi(S^n) \cap \Phi(T^n)$ for some fixed $n > 1$. In order to prove that $v = z$, we suppose that $v \neq z$. We have

$$\begin{aligned} d(z, v) &= d(S^n z, T^n v) = d(SS^{n-1}z, TT^{n-1}v) \\ &\preceq \lambda d(S^{n-1}z, T^{n-1}v) \\ &\quad + \sum_{i=1}^N (\phi_i(S^{n-1}z) - \phi_i(SS^{n-1}z) + \phi_i(T^{n-1}v) - \phi_i(TT^{n-1}v)) \end{aligned}$$

$$\begin{aligned}
 &= \lambda d(SS^{n-2}z, TT^{n-2}v) \\
 &\quad + \sum_{i=1}^N (\phi_i(S^{n-1}z) - \phi_i(S^n z) + \phi_i(T^{n-1}v) - \phi_i(T^n v)) \\
 &\leq \lambda^2 d(S^{n-2}z, T^{n-2}v) \\
 &\quad + \sum_{i=1}^N (\phi_i(S^{n-1}z) - \phi_i(S^n z) + \phi_i(T^{n-1}v) - \phi_i(T^n v)) \\
 &\quad + \phi_i(S^{n-2}z) - \phi_i(S^{n-1}z) + \phi_i(T^{n-2}v) - \phi_i(T^{n-1}v) \\
 &= \lambda^2 d(SS^{n-3}z, TT^{n-3}v) \\
 &\quad + \sum_{i=1}^N (\phi_i(S^{n-2}z) - \phi_i(z) + \phi_i(T^{n-2}v) - \phi_i(v)) \\
 &\quad \vdots \\
 &\leq \lambda^n d(z, v) + \sum_{i=1}^N (\phi_i(z) - \phi_i(z) + \phi_i(v) - \phi_i(v)) = \lambda^n d(z, v).
 \end{aligned}$$

Thus, $d(z, v) \leq \lambda^n d(z, v)$. Since $\lambda^n \in [0, 1)$, we have that $d(z, v) = \theta$, i.e., $z = v$, which is a contradiction.

Hence, $\Phi(S) \cap \Phi(T) = \Phi(S^n) \cap \Phi(T^n)$. \square

Theorem 4.3. *Let (X, d) be a complete cone metric space over a regular cone P , and $\phi : X \rightarrow P$. Suppose that S and F are commuting continuous self-mappings on X such that $SX \subset FX$. Let $z \in X$ satisfy*

$$d(Sx, z) \leq \lambda d(Fx, z) + \phi(Fx) - \phi(Sx)$$

for some $\lambda \in [0, 1)$ and all $x \in X$. Then $\Phi(S) \cap \Phi(F) = \Phi(S^n) \cap \Phi(F^n)$.

Proof. From Theorem 3.9, we obtain that $\Phi(S) \cap \Phi(F) = \{z\}$. We already have $\Phi(S) \cap \Phi(F) \subset \Phi(S^n) \cap \Phi(F^n)$ (see Remark 4.1). Let $v \in \Phi(S^n) \cap \Phi(F^n)$, for some fixed $n > 1$, and suppose that $v \neq z$. We have

$$\begin{aligned}
 d(v, z) &= d(S^n v, z) = d(SS^{n-1}v, z) \\
 &\leq \lambda d(FS^{n-1}v, z) + \phi(FS^{n-1}v) - \phi(SS^{n-1}v) \\
 &= \lambda d(SS^{n-2}Fv, z) + \phi(S^{n-1}Fv) - \phi(v) \\
 &\leq \lambda^2 d(FS^{n-2}Fv, z) + \phi(FS^{n-2}Fv) - \phi(S^{n-1}Fv) + \phi(S^{n-1}Fv) - \phi(v) \\
 &= \lambda^2 d(SS^{n-3}F^2v, z) + \phi(S^{n-2}F^2v) - \phi(v) \\
 &\quad \vdots \\
 &\leq \lambda^n d(F^n v, z) + \phi(F^n v) - \phi(v) = \lambda^n d(v, z) \prec d(v, z),
 \end{aligned}$$

which is a contradiction. Thus, $\Phi(S) \cap \Phi(F) = \Phi(S^n) \cap \Phi(F^n)$. \square

Theorem 4.4. *Let S, T, F and G be four mappings satisfying all the conditions of Theorem 3.1. If $\{S, F\}$ and $\{T, G\}$ are commuting pairs of mappings, then*

$$\Phi(S) \cap \Phi(T) \cap \Phi(F) \cap \Phi(G) = \Phi(S^n) \cap \Phi(T^n) \cap \Phi(F^n) \cap \Phi(G^n).$$

Proof. From Theorem 3.1, we have that $\Phi(S) \cap \Phi(T) \cap \Phi(F) \cap \Phi(G) = \{z\}$. It is obvious that $\Phi(S) \cap \Phi(T) \cap \Phi(F) \cap \Phi(G) \subset \Phi(S^n) \cap \Phi(T^n) \cap \Phi(F^n) \cap \Phi(G^n)$. Let $v \in \Phi(S^n) \cap \Phi(T^n) \cap \Phi(F^n) \cap \Phi(G^n)$, for some fixed $n > 1$, and suppose that $v \neq z$. Using the commutativity of the corresponding pairs of mappings, we have

$$\begin{aligned} d(z, v) &= d(S^n z, T^n v) = d(SS^{n-1}z, TT^{n-1}v) \\ &\leq \lambda d(FS^{n-1}z, GT^{n-1}v) \\ &\quad + \sum_{i=1}^N (\phi_i(FS^{n-1}z) - \phi_i(SS^{n-1}z) + \phi_i(GT^{n-1}v) - \phi_i(TT^{n-1}v)) \\ &= \lambda d(S^{n-1}Fz, T^{n-1}Gv) \\ &\quad + \sum_{i=1}^N (\phi_i(S^{n-1}Fz) - \phi_i(S^n z) + \phi_i(T^{n-1}Gv) - \phi_i(T^n v)) \\ &= \lambda d(SS^{n-2}z, TT^{n-2}Gv) + \sum_{i=1}^N (\phi_i(T^{n-1}Gv) - \phi_i(v)) \\ &\leq \lambda^2 d(FS^{n-2}z, GT^{n-2}Gv) + \sum_{i=1}^N (\phi_i(T^{n-1}Gv) - \phi_i(v) + \phi_i(FS^{n-2}z) \\ &\quad - \phi_i(S^{n-1}z) + \phi_i(GT^{n-2}Gv) - \phi_i(T^{n-1}Gv)) \\ &= \lambda^2 d(S^{n-2}Fz, T^{n-2}G^2v) + \sum_{i=1}^N (\phi_i(T^{n-2}G^2v) - \phi_i(v)) \\ &= \lambda^2 d(SS^{n-3}z, TT^{n-3}G^2v) + \sum_{i=1}^N (\phi_i(T^{n-2}G^2v) - \phi_i(v)) \\ &\quad \vdots \\ &\leq \lambda^n d(z, G^n v) + \sum_{i=1}^N (\phi_i(G^n v) - \phi_i(v)) = \lambda^n d(z, v) \prec d(z, v), \end{aligned}$$

which is a contradiction.

Hence, $\Phi(S) \cap \Phi(T) \cap \Phi(F) \cap \Phi(G) = \Phi(S^n) \cap \Phi(T^n) \cap \Phi(F^n) \cap \Phi(G^n)$. \square

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