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ESTIMATION OF A CONDITION NUMBER RELATED TO $A^{(2)}_{T,S}$

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Abstract

In this paper we get estimation of the absolute condition number a Hilbert space operator, which is related with the outer generalized inverse of a given operator.

1 Introduction

In this paper X and Y denote arbitrary Hilbert spaces. We use $\mathcal{B}(X,Y)$ to denote the set of all linear bounded operators from X to Y. Set $\mathcal{B}(X)=\mathcal{B}(X,X)$.

Let $A\in\mathcal{B}(X,Y)$. We use $R(A)$ and $N(A)$, respectively, to denote the range and the null-space of A. If there exists some operator $A'\in\mathcal{B}(Y,X)$ satisfying $A'AA'=A'$, then A' is called the outer inverse of A [1]. If $T=R(A')$ and $S=N(A')$, then A' is well-known as the $A^{(2)}_{T,S}$ generalized inverse of A. It can easily be deduced that for given subspaces T of X and S of Y, there exists the generalized inverse $A^{(2)}_{T,S}$ of A if and only if the following is satisfied: T, S and $A(T)$ are closed complemented subspaces of X, Y and Y respectively, the reduction $A_1=A|_T:T\to A(T)$ is invertible and $A(T)\oplus S=Y$. In this case the generalized inverse $A^{(2)}_{T,S}$ is unique and the notation is justified. Moreover, the following holds $T=R(A^{(2)}_{T,S})=R(A^{(2)}_{T,S}A)$. Hence, we denote $T_1=N(A^{(2)}_{T,S}A)\subset X$ and $S_1=A(T)\subset Y$. Now we have

$$X=T\oplus T_1 \quad \text{and} \quad Y=S_1\oplus S.$$

The matrix form of A is as follows:

$$A=\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}:\begin{bmatrix} T \\ T_1 \end{bmatrix}\longrightarrow\begin{bmatrix} S_1 \\ S \end{bmatrix}, \tag{1}$$

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where $A_1 \in \mathcal{B}(T, S_1)$ is invertible. Now it is easy to verify that

$$A^{(2)}_{T,S} = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} S_1 \\ S \end{bmatrix} \longrightarrow \begin{bmatrix} T \\ T_1 \end{bmatrix}. \tag{2}$$

We use $\mathcal{B}(X, Y)_{T,S}$ to denote the set of all $A \in \mathcal{B}(X, Y)$, such that $A^{(2)}_{T,S}$ exists. Here we assume that T and S, respectively, are closed subsets of X and Y.

Let X and Y be equipped with the norms $\| \cdot \|_X$ and $\| \cdot \|_Y$. The P-norm for a vector $x \in X$, the Q-norm for a vector $y \in Y$ and the QP-norm for an operator $A \in \mathcal{B}(X, Y)$, respectively, are defined by (see [10]):

$$\|x\|_P = \sqrt{\|x_1\|_X^2 + \|x_2\|_X^2},$$

$$\|y\|_Q = \sqrt{\|y_1\|_Y^2 + \|y_2\|_Y^2},$$

$$\|A\|_{QP} = \sup_{\|x\|_P \leq 1} \|Ax\|_Q$$

where

$$x = x_1 + x_2,\ x_1 \in T,\ x_2 \in T_1,$$

$$y = y_1 + y_2,\ y_1 \in S_1,\ y_2 \in S.$$

Notice that we can also change the inner product in X in the following way:

$$\langle x, y \rangle_P = \langle x_1, y_1 \rangle_X + \langle x_2, y_2 \rangle_X$$

where

$$x = x_1 + x_2,\ y = y_1 + y_2,\ x_1,\ y_1 \in T,\ x_2,\ y_2 \in T_1.$$

Now, $\| \cdot \|_P$ is induced by $\langle \cdot, \cdot \rangle_P$. Similarly for $\langle \cdot, \cdot \rangle_Q$ and $\| \cdot \|_Q$ in Y.

Generalized inverses are frequently related with the system of equations

$$Ax = b,$$

with A and b given, and x is unknown. If A is invertible, then the condition number of A is defined as $\|A\|\|A^{-1}\|$. If A is singular, then we can use some generalized inverse of A instead of A^{-1}. Thus, the generalized condition number of A related with the generalized inverse $A^{(2)}_{T,S}$ (in the case when it exists), is denoted by $\kappa(A) = \|A\|\|A^{(2)}_{T,S}\|$.

The other approach to define the condition number of a linear system $Ax = b$, is connected with differentiable functions. Let $A \in \mathcal{B}(X, Y)_{T,S}$ and $b \in Y$. Define the mapping

$$F : \mathcal{B}(X, Y)_{T,S} \times Y \to X$$

as follows:

$$F(A, b) = A^{(2)}_{T,S} b.$$

The mapping F is differentiable, if the limit

$$\lim_{\epsilon\to 0}\frac{F(A+\epsilon E, b+\epsilon f)-F(A,b)}{\epsilon}=F'(A,b)|_{(E,f)}$$

exists for some perturbations $E\in\mathcal{B}(X,Y)$ of A and $f\in Y$ of b. We assume that $A+\epsilon E\in\mathcal{B}(X,Y)_{T,S}$ for small values of $\epsilon\in\mathbf{C}$. If we have this kind of differentiability, then

$$C(A,b)=\|F'(A,b)|_{(E,f)}\|$$

is the absolute condition number of the linear system $Ax=b$, related with the generalized inverse $A^{(2)}_{T,S}$ and perturbations E of A and f of b.

We can get easy the following useful result.

Theorem 1.1. *Suppose that for $A\in\mathcal{B}(X,Y)$ and for closed subspaces $T\subset X$ and $S\subset Y$, there exists the generalized inverse $A^{(2)}_{T,S}\in\mathcal{B}(Y,X)$. Let $B=A+E$, $R(E)\subseteq A(T)$ and $N(E)\supseteq T_1$. If $\|A^{(2)}_{T,S}\|_{PQ}\|E\|_{QP}<1$, then $B^{(2)}_{T,S}$ exists and*

$$B^{(2)}_{T,S}=[I+A^{(2)}_{T,S}E]^{-1}A^{(2)}_{T,S}=A^{(2)}_{T,S}[I+EA^{(2)}_{T,S}]^{-1}. \tag{3}$$

Proof. This result is analogy with the results in [18] for complex matrices. □

Higham [8] discussed different condition numbers of regular inverses and nonsinglar linear systems. Concerning generalized inverses and singular linear systems there are similar results on these problems. Papers [3, 7, 15, 16, 9, 2, 17] have some results when the generalized inverse is a Moore-Penrose inverse, Drazin inverse and generalized Bott-Duffin inverse, respectively. In [13], Y. Wei and H. Diao considered the condition number for the Drazin inverse and the Drazin inverse solution of singular linear system. X. Cui and H. Diao generalized the results of [13] and get the results of the condition number for the W-weighted Drazin inverse and the W-weighted Drazin inverse solution of a linear system in paper [4]. In [10], we extend the result obtained in [4] to linear bounded operators between Hilbert spaces. In [11], the authors established the condition number of the W–weighted Drazin inverse of a rectangular matrix by the Schur decomposition and the spectral norm. Because all generalized inverses belong to outer inverse $A^{(2)}_{T,S}$ with the prescribed range T and null space S, we are more interested in the condition numbers connected with the outer inverse $A^{(2)}_{T,S}$. In [5], H. Diao, M. Qin and Y. Wei investigated the condition number of the outer inverse $A^{(2)}_{T,S}$ and the outer inverse $A^{(2)}_{T,S}$ solution of a constrained linear system which extends the results in [13, 4]. They gave the explicit formula of the condition number for the outer inverse $A^{(2)}_{T,S}$ solution of a constrained linear system. The results obtained in [5] are generalized in [12] using the Schur decomposition and the spectral norm. In this paper we extend the result obtained in [5] to linear bounded operators between Hilbert spaces.

2 Absolute condition number of a linear system

First, we prove that the mapping F is differentiable if we assume some conditions.

Lemma 2.1. *The mapping $F : \mathcal{B}(X,Y) \times Y \to X$ is a differentiable function, if the perturbation (E,f) of (A,b) fulfils the following condition:*

$$AA^{(2)}_{T,S}E = E, \quad EA^{(2)}_{T,S}A = E, \quad \|A^{(2)}_{T,S}\|_{PQ}\|E\|_{QP} < 1. \tag{4}$$

Proof. From Theorem 1.1 follows that $(A+\epsilon E)^{(2)}_{T,S}$ exists and

$$\begin{aligned}(A+\epsilon E)^{(2)}_{T,S} &= [I + A^{(2)}_{T,S}\epsilon E]^{-1}A^{(2)}_{T,S} \\ &= [I - \epsilon A^{(2)}_{T,S}E + \epsilon^2(A^{(2)}_{T,S}E)^2 - ...]^{-1}A^{(2)}_{T,S} \\ &= A^{(2)}_{T,S} - \epsilon A^{(2)}_{T,S}EA^{(2)}_{T,S} + O(\epsilon^2)\end{aligned}$$

Consider the existence of the limit

$$\begin{aligned} & \lim_{\epsilon\to 0}\frac{F(A+\epsilon E, b+\epsilon f) - F(A,b)}{\epsilon} \\ = \; & \lim_{\epsilon\to 0}\frac{(A+\epsilon E)^{(2)}_{T,S}(b+\epsilon f) - A^{(2)}_{T,S}b}{\epsilon} \\ = \; & \lim_{\epsilon\to 0}\frac{(A^{(2)}_{T,S} - \epsilon A^{(2)}_{T,S}EA^{(2)}_{T,S} + O(\epsilon^2))(b+\epsilon f) - A^{(2)}_{T,S}b}{\epsilon} \\ = \; & \lim_{\epsilon\to 0}\frac{\epsilon A^{(2)}_{T,S}f - \epsilon A^{(2)}_{T,S}EA^{(2)}_{T,S}(b+\epsilon f)}{\epsilon} \\ = \; & \lim_{\epsilon\to 0}(A^{(2)}_{T,S}f - A^{(2)}_{T,S}EA^{(2)}_{T,S}b - \epsilon A^{(2)}_{T,S}EA^{(2)}_{T,S}f) \\ = \; & -A^{(2)}_{T,S}Ex + A^{(2)}_{T,S}f.\end{aligned}$$

Hence,

$$F'(A,b)|_{(E,f)} = -A^{(2)}_{T,S}Ex + A^{(2)}_{T,S}f. \qquad \square$$

Let $A \in \mathcal{B}(X,Y)$, $b \in A(T)$ and let us consider the equation

$$Ax = b, \qquad x \in T. \tag{5}$$

If $A \in \mathcal{B}(X,Y)_{T,S}$, then the equation (5) have a unique solution if and only if $b \in A(T)$ and $T \cap N(A) = \{0\}$. Then the unique solution of the equation (5) is given by

$$x = A^{(2)}_{T,S}b. \tag{6}$$

α–alpha
β–beta

The norm on the data is the norm in $\mathcal{B}(X,Y) \times Y$ defined as

$$(A,b) \longmapsto \|[\alpha A, \beta b]\| = \sqrt{\alpha^2\|A\|^2_{QP} + \beta^2\|b\|^2_{Q}}.$$

Now, we prove the estimation of the absolute condition number of a linear system related to the generalized inverse $A^{(2)}_{T,S}$. The following result is a generalization of results from [5] and [10].

Theorem 2.1. *If the perturbation E in A fulfills the condition* (4)*, then the absolute condition number $C(A,b)$ of the generalized inverse $A^{(2)}_{T,S}$ solution of the constrained linear system, with the norm*

$$\|[\alpha A, \beta b]\| = \sqrt{\alpha^2\|A\|^2_{QP} + \beta^2\|b\|^2_Q}$$

on the data (A,b), and the norm $\|x\|_P$ on the solution, satisfies

$$C(A,b) \leq \|A^{(2)}_{T,S}\|_{PQ}\sqrt{\frac{1}{\beta^2} + \frac{\|x\|^2_P}{\alpha^2}}.$$

Let $(E_n)_n$ be a sequence of perturbations of A fulfilling the condition (4)*, and let $(f_n)_n$ be a sequence of perturbations of b. If $C(E_n, f_n)$ is the corresponding absolute condition number and $\|A^{(2)}_{T,S}\|_{PQ} < \alpha$, then*

$$C(E_n, f_n) \to \|A^{(2)}_{T,S}\|_{PQ}\sqrt{\frac{1}{\beta^2} + \frac{\|x\|^2_P}{\alpha^2}}, \qquad n \to \infty.$$

Hence, $\|A^{(2)}_{T,S}\|_{PQ}\sqrt{\frac{1}{\beta^2} + \frac{\|x\|^2_P}{\alpha^2}}$ is a sharp bound.

Proof. We know that $F(A,b) = A^{(2)}_{T,S}b$. Under the condition (4), F is a differentiable function and F' is defined as follows

$$F'(A,b)|_{(E,f)} = \lim_{\epsilon\to 0}\frac{(A+\epsilon E)^{(2)}_{T,S}(b+\epsilon f) - A^{(2)}_{T,S}b}{\epsilon},$$

where E is the perturbation of A and f is the perturbation of b.

Since E satisfies the condition (4), we have

$$(A+\epsilon E)^{(2)}_{T,S} = A^{(2)}_{T,S} - \epsilon A^{(2)}_{T,S}EA^{(2)}_{T,S} + O(\epsilon^2),$$

and then we can easily get that

$$F'(A,b)|_{(E,f)} = -A^{(2)}_{T,S}EA^{(2)}_{T,S}b + A^{(2)}_{T,S}f = -A^{(2)}_{T,S}Ex + A^{(2)}_{T,S}f.$$

Then

$$\begin{aligned}\|F'(A,b)|_{(E,f)}\|_P &= \|A^{(2)}_{T,S}(Ex - f)\|_P \\ &\leq \|A^{(2)}_{T,S}\|_{PQ}(\|E\|_{QP}\|x\|_P + \|f\|_Q).\end{aligned}$$

The norm of a linear map $(E,f) \mapsto F'(A,b)|_{(E,f)}$ is the supermum of $\|F'(A,b)|_{(E,f)}\|_P$ on the unit ball of $\mathcal{B}(X,Y)\times Y$. Since

$$\|[\alpha E, \beta f]\|^2 = \alpha^2\|E\|^2_{QP} + \beta^2\|f\|^2_Q$$

we get

$$\begin{aligned}
&\|F'(A,b)|_{(E,f)}\| \\
&\leq \sup_{\alpha^2\|E\|_{QP}^2+\beta^2\|f\|_Q^2\leq 1} \|A_{T,S}^{(2)}\|_{PQ}(\|E\|_{QP}\|x\|_P+\|f\|_Q) \\
&= \sup_{\alpha^2\|E\|_{QP}^2+\beta^2\|f\|_Q^2\leq 1} \|A_{T,S}^{(2)}\|_{PQ}\left(\alpha\|E\|_{QP}\frac{\|x\|_P}{\alpha}+\beta\|f\|_Q\frac{1}{\beta}\right) \\
&= \|A_{T,S}^{(2)}\|_{PQ} \sup_{\alpha^2\|E\|_{QP}^2+\beta^2\|f\|_Q^2\leq 1} (\alpha\|E\|_{QP},\beta\|f\|_Q)\cdot\left(\frac{\|x\|_P}{\alpha},\frac{1}{\beta}\right)
\end{aligned}$$

where $(\alpha\|E\|_{QP},\beta\|f\|_Q)$ and $\left(\frac{\|x\|_P}{\alpha},\frac{1}{\beta}\right)$ can be consider as vectors in R^2, and the previous line contains the inner product in R^2.

Therefore, from the Cauchy–Schwarz inequality, we get:

$$\|F'(A,b)|_{(E,f)}\| \leq \|A_{T,S}^{(2)}\|_{PQ}\sqrt{\frac{\|x\|_P^2}{\alpha^2}+\frac{1}{\beta^2}}.$$

Next, we show the other part of the theorem. Recall the matrix forms (1) and (2). There exists a sequence $(u_n)_n$ in S_1 satisfying $\|u_n\|=1$ and $\lim_{n\to\infty}\|A_1^{-1}u_n\|=\|A_1^{-1}\|$. So, there exists a sequence $(v_n)_n$ in T, $\left(v_n=\frac{A_1^{-1}}{\|A_1^{-1}\|}u_n\right)$, such that $\|v_n\|\leq 1$, $\lim_{n\to\infty}\|v_n\|=1$ and, for all $n\in N$,

$$A_1^{-1}u_n=\|A_1^{-1}\|v_n=\|A_{T,S}^{(2)}\|_{PQ}v_n.$$

The last equality follows from

$$\begin{aligned}
\|A_{T,S}^{(2)}\|_{PQ} &= \sup_{\|x\|_Q\leq 1}\|A_{T,S}^{(2)}x\|_P \\
&= \sup_{\sqrt{\|x_1\|^2+\|x_2\|^2}\leq 1}\left\|\begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0\end{bmatrix}\begin{bmatrix} x_1 \\ x_2\end{bmatrix}\right\|_P \\
&= \sup_{\|x_1\|\leq 1}\left\|\begin{bmatrix} A_1^{-1}x_1 \\ 0\end{bmatrix}\right\|_P \\
&= \sup_{\|x_1\|\leq 1}\|A_1^{-1}x_1\| \\
&= \|A_1^{-1}\|
\end{aligned}$$

Taking, for all $n\in N$,

$$\hat{u}_n=\begin{bmatrix} u_n \\ 0\end{bmatrix}\in\begin{bmatrix} S_1 \\ S\end{bmatrix},\quad \hat{v}_n=\begin{bmatrix} v_n \\ 0\end{bmatrix}\in\begin{bmatrix} T \\ T_1\end{bmatrix},$$

we obtain

$$\begin{aligned} A^{(2)}_{T,S}\hat{u}_n &= \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_n \\ 0 \end{bmatrix} = \begin{bmatrix} A_1^{-1}u_n \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \|A_1^{-1}\|v_n \\ 0 \end{bmatrix} = \|A_1^{-1}\| \begin{bmatrix} v_n \\ 0 \end{bmatrix} \\ &= \|A^{(2)}_{T,S}\|_{PQ}\hat{v}_n. \end{aligned}$$

It is easy to check that $\|\hat{u}_n\|_Q = 1$ and $\|\hat{v}_n\|_P \leq 1$, for all $n \in N$.

Let $u \in S_1$ and $v \in T$. Define $S_{u,v} \in \mathcal{B}(T, S_1)$ as follows: if $x \in T$, then

$$S_{u,v}(x) \stackrel{\text{def}}{=} \langle x, v\rangle u.$$

For all $T \in \mathcal{B}(S_1, T)$ we have

$$TS_{u,v}(x) = T(u)\langle x, v\rangle.$$

Now we choose, for $n = 1, 2, 3, \ldots,$

$$\eta = \sqrt{\frac{\|x\|_P^2}{\alpha^2} + \frac{1}{\beta^2}}, \quad f_n = \frac{1}{\beta^2\eta}\hat{u}_n,$$

$$E_n = -\frac{1}{\alpha^2\eta} \begin{bmatrix} S_{u_n,x} & 0 \\ 0 & 0 \end{bmatrix}.$$

Then, for a fixed n, we can verify that E_n fulfills the first equation of the condition (4):

$$\begin{aligned} AA^{(2)}_{T,S}E_n &= -\frac{1}{\alpha^2\eta} \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} S_{u_n,x} & 0 \\ 0 & 0 \end{bmatrix} \\ &= -\frac{1}{\alpha^2\eta} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} S_{u_n,x} & 0 \\ 0 & 0 \end{bmatrix} \\ &= -\frac{1}{\alpha^2\eta} \begin{bmatrix} S_{u_n,x} & 0 \\ 0 & 0 \end{bmatrix} \\ &= E_n. \end{aligned}$$

In the same way, we have

$$\begin{aligned} E_nA^{(2)}_{T,S}A &= -\frac{1}{\alpha^2\eta} \begin{bmatrix} S_{u_n,x} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \\ &= -\frac{1}{\alpha^2\eta} \begin{bmatrix} S_{u_n,x} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \\ &= -\frac{1}{\alpha^2\eta} \begin{bmatrix} S_{u_n,x} & 0 \\ 0 & 0 \end{bmatrix} \\ &= E_n \end{aligned}$$

and

$$\begin{aligned}
\|A^{(2)}_{T,S}\|_{PQ}\|E_n\|_{QP} &= \left\|\begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}\right\|_{PQ}\left\|-\frac{1}{\alpha^2\eta}\begin{bmatrix} S_{u_n,x} & 0 \\ 0 & 0 \end{bmatrix}\right\|_{QP} \\
&= \frac{1}{\alpha^2\eta}\|A_1^{-1}\|\|S_{u_n,x}\| \\
&= \frac{1}{\alpha^2\eta}\|A_1^{-1}\|\sup_{\|z\|\leq 1}\|S_{u_n,x}z\| \\
&= \frac{1}{\alpha^2\eta}\|A_1^{-1}\|\sup_{\|z\|\leq 1}\|u_n\langle z,x\rangle\| \\
&\leq \frac{1}{\alpha^2\eta}\|A_1^{-1}\|\|u_n\|\|x\| \\
&= \frac{\|x\|}{\alpha^2\eta}\|A_1^{-1}\| \\
&< \frac{\|A_1^{-1}\|}{\alpha} \\
&= \frac{\|A^{(2)}_{T,S}\|_{PQ}}{\alpha} \\
&< 1.
\end{aligned}$$

Thus E_n fulfills the condition (4), for all $n \in N$. Now we want to verify that the perturbation (E_n, f_n) satisfies $\alpha^2\|E_n\|^2_{QP} + \beta^2\|f_n\|^2_Q \leq 1$.

$$\begin{aligned}
\alpha^2\|E_n\|^2_{QP} + \beta^2\|f_n\|^2_Q &= \frac{1}{\alpha^2\eta^2}\left\|\begin{bmatrix} S_{u_n,x} & 0 \\ 0 & 0 \end{bmatrix}\right\|^2_{QP} + \frac{1}{\beta^2\eta^2}\|\hat{u}_n\|^2_Q \\
&= \frac{1}{\alpha^2\eta^2}\|S_{u_n,x}\|^2 + \frac{1}{\beta^2\eta^2} \\
&\leq \frac{1}{\alpha^2\eta^2}\|u_n\|^2\|x\|^2_P + \frac{1}{\beta^2\eta^2} \\
&= \frac{1}{\eta^2}\left(\frac{\|x\|^2_P}{\alpha^2} + \frac{1}{\beta^2}\right) \\
&= 1.
\end{aligned}$$

The inner product $\langle\cdot,\cdot\rangle_P$ in T is the same as the inner product $\langle\cdot,\cdot\rangle$. Thus, we have, for $x = A^{(2)}_{T,S}b$,

$$\begin{aligned}
F'(A,b)|_{(E_n,f_n)} &= -A^{(2)}_{T,S}E_nx + A^{(2)}_{T,S}f_n \\
&= \frac{1}{\alpha^2\eta}\begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} S_{u_n,x} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ 0 \end{bmatrix} + \frac{1}{\beta^2\eta}A^{(2)}_{T,S}\hat{u}_n \\
&= \frac{1}{\alpha^2\eta}\begin{bmatrix} A_1^{-1}S_{u_n,x} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ 0 \end{bmatrix} + \frac{1}{\beta^2\eta}A^{(2)}_{T,S}\hat{u}_n \\
&= \frac{1}{\alpha^2\eta}\begin{bmatrix} A_1^{-1}\langle x,x\rangle u_n \\ 0 \end{bmatrix} + \frac{1}{\beta^2\eta}A^{(2)}_{T,S}\hat{u}_n \\
&= \frac{1}{\alpha^2\eta}\begin{bmatrix} \|x\|_P^2 A_1^{-1}u_n \\ 0 \end{bmatrix} + \frac{1}{\beta^2\eta}A^{(2)}_{T,S}\hat{u}_n \\
&= \frac{1}{\alpha^2\eta}\|x\|_P^2\begin{bmatrix} \|A_1^{-1}\|v_n \\ 0 \end{bmatrix} + \frac{1}{\beta^2\eta}A^{(2)}_{T,S}\hat{u}_n \\
&= \frac{1}{\alpha^2\eta}\|x\|_P^2\|A_1^{-1}\|\begin{bmatrix} v_n \\ 0 \end{bmatrix} + \frac{1}{\beta^2\eta}\|A^{(2)}_{T,S}\|_{PQ}\hat{v}_n \\
&= \frac{1}{\alpha^2\eta}\|x\|_P^2\|A^{(2)}_{T,S}\|_{PQ}\hat{v}_n + \frac{1}{\beta^2\eta}\|A^{(2)}_{T,S}\|_{PQ}\hat{v}_n \\
&= \frac{\|A^{(2)}_{T,S}\|_{PQ}}{\eta}\left(\frac{\|x\|_P^2}{\alpha^2} + \frac{1}{\beta^2}\right)\hat{v}_n \\
&= \|A^{(2)}_{T,S}\|_{PQ}\eta\hat{v}_n.
\end{aligned}$$

So

$$\|F'(A,b)|_{(E_n,f_n)}\|_P \to \|A^{(2)}_{T,S}\|_{PQ}\sqrt{\frac{\|x\|_P^2}{\alpha^2} + \frac{1}{\beta^2}} \qquad (n \to \infty).$$

Knowing $\alpha^2\|E_n\|_{QP}^2 + \beta^2\|f_n\|_Q^2 \le 1$, we get

$$\|F'(A,b)|_{(E_n,f_n)}\| \to \|A^{(2)}_{T,S}\|_{PQ}\sqrt{\frac{\|x\|_P^2}{\alpha^2} + \frac{1}{\beta^2}}, \qquad (n \to \infty)$$

and we complete the proof. □

3 Concluding remarks

In this paper, we consider the absolute condition number of a operator between Hilbert spaces, which is related with the outer generalized inverse of a given operator. In [5, 12] our Theorem 2.1 is proved for complex matrices. In [10] the author proved Theorem 2.1 considering the weighted Drazin inverse in Hilbert spaces. It is of interest to extend our results to the outer inverse of a operator between Banach spaces.

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