Faculty of Mathematics, University of Belgrade, Serbia
Available at: `http://www.pmf.ni.ac.rs/filomat`

Filomat **25:3** (2011), 137–143 DOI: 10.2298/FIL1103137P

CHARACTERIZATIONS OF THE HARMONIC HARDY SPACE h¹ ON THE REAL BALL

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Abstract

We prove some characterizations of the space h^1 and use them to give new proofs of a theorem of Zygmund and a theorem of Kologorov and Smirnov.

1 Introduction

Throughout the paper, we denote by B the unit ball of $\mathbb{R}^n$ $(n \geq 2)$, by dV the normalized Lebesgue measure on B, and by $d\sigma$ the normalized surface measure on the sphere $S = \partial B$. Let $h(B)$ denote the class of all real-valued functions harmonic on B. The harmonic Hardy space $h^1(B)$ consists of those $u \in h(B)$ for which

$$\|u\|_1 := \sup_{0<r<1} I_1(r,u) < \infty, \quad \text{where } I_1(r,u) = \int_S |u(ry)|\, d\sigma(y).$$

We will consider $h^1(B)$ as a member of the family of Hardy-Orlicz spaces. Let ϕ be an Orlicz function, i.e. a non-negative convex function defined on $[0,\infty)$ such that $\phi(0) = 0$ and $\phi(t) > 0$ for some $t > 0$. The Hardy-Orlicz space $h_\phi(B)$ is defined to be the subclass of $h(B)$ consisting of those u for which

$$I_\phi(u) := \sup_{0<r<1} I_\phi(r,u) \ < \infty, \tag{1}$$

where

$$I_\phi(r,u) = \int_S \phi(|u(ry)|)\, d\sigma(y). \tag{2}$$

Since the function $\phi(|u|)$ is subharmonic, we have that $I_\phi(r,u)$ increases with r and so

$$I_\phi(u) = \lim_{r\to 1^-} I_\phi(r,u).$$

2010 *30D55, 31B05.*
Harmonic Hardy spaces, Orlicz spaces
Received: Required
Communicated by (name of the Editor, required)
Supported by MNTR Serbia, Project ON174017

If $\phi(t) = t^p$, $p \geq 1$, then $h_\phi(B)$ is denoted by $h^p(B)$ and is called the harmonic Hardy space. We are mainly concerned with the case $p = 1$. Our idea is very simple: use the obvious fact that $h^1 = h_\phi$, if $\limsup_{t\to\infty} \phi(t)/t < \infty$ and $\limsup_{t\to\infty} t/\phi(t) < \infty$, then choose a suitable ϕ and apply the Green formula to the function $\phi \circ |u|$ to get a Hardy-Stein type identity. In this way we get some surprising (at least for the author) results, e.g., that the condition

$$\int_B \frac{|\nabla u(x)|^2}{(1+|u(x)|)^\alpha}(1-|x|)\,dV(x) \ < \infty, \quad \alpha > 1, \tag{†}$$

is equivalent to $u \in h^1$ and is therefore independent of α. We also prove that (†) is equivalent to the apparently weaker condition

$$\int_{|u(x)|<1} |\nabla u(x)|^2 (1-|x|)\,dV(x) \ < \infty.$$

These results will be deduced (Section 3) from a Hardy-Stein type characterization of general Hardy-Orlicz spaces (Section 2). In Section 4 we give new proofs of theorems of Zygmund and Kolmogorov-Smirnov.

2 Characterizations of Hardy-Orlicz spaces

In this note we consider the class $\mathcal{F}$ that consists of those Orlicz functions ϕ for which the first derivative, ϕ', is absolutely continuous on $[0,\infty)$ and the second derivative, ϕ'', is continuous on $[0,\infty) \setminus A$, where A is a finite subset of $[0,\infty)$.

Theorem 1. *Let $\phi \in \mathcal{F}$, $\phi'(0) = 0$, and $u \in h(B)$. Then u belongs to $h_\phi(B)$ if and only if*

$$\int_B \phi''(|u(x)|)\,|\nabla u(x)|^2 G_n(x)\,dV(x) \ < \infty, \tag{3}$$

and we have

$$I_\phi(u) = \phi(|u(0)|) + \int_B \phi''(|u(x)|)\,|\nabla u(x)|^2 G_n(x)\,dV(x), \tag{4}$$

where

$$G_n(x) = \begin{cases} \dfrac{1}{2}\log\dfrac{1}{|x|}, & n = 2 \\[2mm] \dfrac{|x|^{2-n}-1}{n(n-2)}, & n \geq 3. \end{cases}$$

In the case where $\phi(t) = t^p$, $p > 1$, this theorem was proved by P. Stein [5] (for the case $n \geq 3$ see [3]). Analogous results for analytic Hardy-Orlicz spaces were considered by Stoll [6] ($n = 1$), and Ouyang and Riihentaus [2], and by Stoll [7] ($n \geq 2$). Our proof completely differs from those in these papers.

Remark 1. Although $\phi''(x)$ does not exist for $x \in A$, the integral in (3) is defined because the measure of the set $\cup_{x\in A}\{t \in B : u(t) = x\}$ is equal to zero.

Proof. We extend ϕ to $\mathbb{R}$ by $\phi(-t) = \phi(t)$. Assume first that ϕ is of class $C^2(\mathbb{R})$ (which implies $\phi'(0) = 0$). Then we use the Green formula

$$\int_S v(ry)\,d\sigma(y) - v(0) = \int_{rB} \Delta v(x)\,G_n(x,r)\,dV(x), \quad v \in C^2(B),\ 0 < r < 1, \tag{5}$$

where

$$G_n(x,r) = \begin{cases} \dfrac{1}{2}\log\dfrac{r}{|x|}, & n = 2 \\ \dfrac{|x|^{2-n} - r^{2-n}}{n(n-2)}, & n \geq 3. \end{cases} \tag{6}$$

Taking $v(x) = \phi(|u(x)|) = \phi(u(x))$ and using the formula

$$\Delta v = \phi''(|u|)|\nabla u|^2 \tag{7}$$

we get

$$I_\phi(r,\phi) - \phi(|u(0)|) = \int_B \phi''(|u|)\,|\nabla u|^2 G_n(\cdot,r)\,dV, \quad 0 < r < 1. \tag{8}$$

Next we consider the case where $\phi \in \mathcal{F}$ is such that ϕ'' is bounded. In this case we define the sequence ϕ_k by

$$\phi_n''(t) = k\int_{-1/k}^{1/k} \omega(ks)\phi''(s+t)\,dt \quad \text{and } \phi_k'(0) = \phi_k(0) = 0, \tag{9}$$

where ω is an even nonnegative function of class $C^\infty(\mathbb{R})$ with $\operatorname{supp}\omega \subset (-1,1)$, and $\int_{\mathbb{R}} \omega(t)\,dt = 1$. It is well known and easy to see that $\phi_k \in C^\infty(\mathbb{R})$ and

$$\lim_{k\to\infty} \phi_k''(t) = \phi''(t) \quad \text{if } \phi'' \text{ is continuous at } t.$$

Since ϕ_n are Orlicz functions, we can appeal to the preceding case to get

$$I_{\phi_k}(r,\phi_k) - \phi_k(|u(0)|) = \int_{rB} \phi_k''(|u|)\,|\nabla u|^2 G_k(\cdot,r)\,dV \tag{10}$$

Since $\sup_{\mathbb{R}} |\phi_k| \leq \sup_{\mathbb{R}} |\phi|$, by (9), and $\int_{rB} G_k(x,r)\,dV(x) < \infty$, we see that the sequence $\phi_k''(|u|)\,|\nabla u|^2 G_k(\cdot,r)$ has an integrable dominant, so we can apply the dominated convergence theorem to (10) to get the result.

Finally assume that ϕ'' is not bounded. Then we consider the functions ψ_k ($k \in \mathbb{N}$) defined by

$$\psi_k''(t) = g_k(t) := \min\{k, \phi''(t)\}\ (t > 0) \text{ and } \psi_k(0^+) = \psi_k'(0) = 0.$$

Applying (10) to ψ_n we get

$$\int_{rB} g_k(|u|)\,|\nabla u|^2 G_k(\cdot,r)\,dV = \int_S \psi_k(|u(ry)|)\,d\sigma(y) - \psi_k(|u(0)|).$$

Since $0 \leq g_k \uparrow \phi''$ and $0 \leq \phi_k \uparrow \phi$ ($k \to \infty$) we can apply the monotone convergence theorem to obtain (8) in the general case.

Now (4) is obtained by application of the monotone convergence theorem ($r \to 1^-$) (observe that $G_k(x,r)$ increases with r). □

We can write (8) as

$$I_\phi(r,u) - \phi(|u(0)|) = \frac{1}{n}\int_0^r \rho^{1-n}\,d\rho \int_{\rho B} \phi''(|u|)\,|\nabla u|^2\,dV, \tag{11}$$

from which we get:

Theorem 2. *Under the hypotheses of Theorem 1, the function* $r \mapsto I_\phi(r,u)$ $(0 < r < 1)$ *is of class* C^1 *and we have*

$$\frac{d}{dr} I_\phi(r,u) = \frac{r^{1-n}}{n}\int_{rB} \phi''(|u|)\,|\nabla u|^2\,dV, \quad 0 < r < 1. \tag{12}$$

If $\phi'(0) > 0$, then an application of Theorem 1 to the function $\phi_1(t) = \phi(t) - \phi'(0)t$ yields the following.

Theorem 3. *If* $\phi \in \mathcal{F}$ *and* $u \in h(B)$, *then*

$$I_\phi(u) - I_\phi(|u(0)|) = \phi'(0)(I_1(u) - |u(0)|) + \int_B |\phi''(|u|)\,|\nabla u|^2\,G_n\,dV,$$

where

$$I_1(u) = \sup_{0<r<1} \int_S |u(ry)|\,d\sigma(y).$$

Corollary 1. *Let* $\phi \in \mathcal{F}$ *and* $\phi'(0) > 0$. *Then* (4) *holds if and only if* u *is of constant sign on* B.

Proof. By the theorem, (4) holds if and only if $I_1(u) = |u(0)|$. If u is of constant sign, then $I_1(u) = |u(0)|$, by the mean value property of u. Conversely, if $I_1(u) = |u(0)|$, then $\int_S |u(ry)|\,d\sigma(y) = |u(0)|$ for all $r \in (0,1)$, by the sub-mean-value property of $|u|$, whence

$$\begin{aligned}\int_B |u|\,dV &= n\int_0^1 r^{n-1}\,dr \int_S |u(ry)|\,d\sigma(y) \\ &= |u(0)| \\ &= \Big|\int_B u\,dV\Big|.\end{aligned}$$

This implies that u is of constant sign. □

3 Characterizations of $h^1(B)$

Since, by (8),

$$\int_{(1/2)B} \phi''(|u(x)|)|\nabla u(x)|^2 G_n(x)\,dV(x) \ < \infty$$

for all $u \in h(B)$, and $G_n(x,r) \asymp (1-|x|)$, $1/2 < |x| < 1$, we have, as a consequence of Theorem 1:

Theorem 4. *Let $\phi \in \mathcal{F}$, $\phi'(0) = 0$, and $u \in h(B)$. Then u is in $h_\phi(B)$ if and only if*

$$\int_B \phi''(|u(x)|)|\nabla u(x)|^2(1-|x|)\,dV(x) \; < \infty.$$

Now we will apply this theorem to characterize h^1.

Theorem 5. *Let $g : [0,\infty) \mapsto [0,\infty)$ be a function continuous on $[0,\infty) \setminus A$, where A is a finite subset of $[0,\infty)$ and $0 < \int_0^\infty g(t)\,dt < \infty$. Then, a function $u \in h(B)$ is in $h^1(B)$ if and only if*

$$\int_B g(|u(x)|)\,|\nabla u(x)|^2(1-|x|)\,dV(x) \; < \infty.$$

Proof. Define the function ϕ by $\phi'' = g$, and $\phi(0) = \phi'(0) = 0$. From this and the hypotheses on g it follows that there are positive constants t_0 and C such that $t/C \le \phi(t) \le Ct$ for $t > t_0$, which implies $h^1(B) = h_\phi(B)$. Now the conclusion follows from Theorem 4. □

Positive harmonic functions belong to $h^1(B)$. Hence:

Corollary 2. *If g is as above and $u \in h(B)$ is positive, then*

$$\int_B g(|u(x)|)\,|\nabla u(x)|^2(1-|x|)\,dV(x) \; < \infty.$$

In particular this holds in the case where u is the Poisson kernel,

$$u(x) = \frac{1-|x|^2}{|x-y|^n} \quad (y \in S \text{ is fixed}).$$

The last fact can certainly be verified by direct computation.

Corollary 3. *Let $\alpha > 1$ and $u \in h(B)$. Then u is in $h^1(B)$ if and only if*

$$\int_B \frac{|\nabla u(x)|^2}{(1+|u(x)|)^\alpha}(1-|x|)\,dV(x) \; < \infty. \tag{13}$$

Proof. Take $g(t) = (1+t)^{-\alpha}$ and apply the theorem. □

Corollary 4. *A function $u \in h(B)$ belongs to h^1 if and only if*

$$\int_{|u(x)|<1} |\nabla u(x)|^2(1-|x|)\,dV(x) \; < \infty.$$

Proof. In this case we take $g(t) = 1$ for $0 \le t \le 1$ and $g(t) = 0$ for $t > 1$. □

In view of Corollary 3, it is natural to ask whether the condition (13) corresponds to some Hardy-Orlicz space for $\alpha = 1$. The answer is affirmative:

Corollary 5. *Let $\phi(t) = t\log(1+t)$, $u \in h(B)$. Then u belongs to $h_\phi(B)$ if and only if*

$$\int_B \frac{|\nabla u(x)|^2}{1+|u(x)|}(1-|x|)\,dV(x) \ < \infty.$$

Proof. We have

$$\begin{aligned}\phi''(t) &= \frac{1}{1+t} + \frac{1}{(1+t)^2} \\ &\asymp \frac{1}{1+t}, \quad t > 1.\end{aligned}$$

The result follows. □

Remark 2. We write $\phi_1(t) \asymp \phi_2(t)$, $t > 1$, to indicate that $1/C \le \phi_1(t)/\phi_2(t) \le C$, for $t > 1$, where C is a constant independent of t. It is easy to check that if $\phi_1 \asymp \phi_2$, then $h_{\phi_1} = h_{\phi_2}$.

Corollary 6. *Let $p > 1$, $u \in h(B)$. Then $u \in h^p(B)$ if and only if*

$$\int_B (1+|u(x)|)^{p-2}|\nabla u(x)|^2(1-|x|)\,dV(x) < \ \infty.$$

Proof. It is enough to take $\phi(t) = (1+t)^p - 1$ and observe that $h^p = h_\phi$. □

4 Theorems of Zygmund and Kolmogorov-Smirnov

The preceding results can be used to prove two well-known theorems. In order to state them we denote by $B = B_{2m}$ the unit ball in $\mathbb{C}^m = \mathbb{R}^{2m}$, by $H(B)$ the class of all functions analytic in B, and by $H^p(B)$ $(0 < p < \infty)$ the ordinary Hardy space,

$$H^p(B) = \{f \in H(B) : \sup_{0<r<1} \int_B |f(r\zeta)|^p\,d\sigma(\zeta) \ < \infty\}.$$

Theorem A (Zygmund [8]) *If $f \in H(B)$ and $\operatorname{Re} f \in h_\phi(B)$, where $\phi(t) = t\log(1+t)$, then $f \in H^1(B)$.*

Theorem B (Kolmogorov-Smirnov [1, 4]) *If If $f \in H(B)$ and $\operatorname{Re} f \in h^1$, then $f \in H^p(B)$ for all $p < 1$.*

In order to prove these theorems we need the following characterization of Hardy spaces.

Theorem 6. *Let $0 < p < \infty$. A function $f \in H(B)$ belongs to $H^p(B)$ if and only if*

$$\int_B (1+|f(z)|)^{p-2}|Df(z)|^2(1-|z|)\,dV(z) \ < \infty,$$

where

$$|Df(z)| = \Big(\sum_{j=1}^{n} \Big|\frac{\partial f}{\partial z_j}\Big|^2\Big)^{1/2}.$$

Proof. This can easily be deduced from formula (5), with $n = 2m$ and $v = (1 + |f|^2)^{p/2} - 1$, and the formulas $|\nabla|f|^2| = 2|f|\,|Df|$, $\Delta(|f|^2) = 4|Df|^2$. □

Proof of Theorem A. This theorem follows from Corollary 5 and Theorem 6 ($p = 1$) together with the relations $1/(1 + |\operatorname{Re} f|) \geq 1/(1 + |f|)$ and $|\nabla(\operatorname{Re} f)| = |Df|$. □

Proof of Theorem B. In this case we use Corollary 3 ($\alpha = 2 - p$, $p < 1$) and Theorem 6. □

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