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# AN INSTABILITY THEOREM FOR A CERTAIN FIFTH-ORDER DELAY DIFFERENTIAL EQUATION

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**Abstract**

Text of the abstract. The main purpose of this paper is to introduce a new instability theorem related to a fifth order nonlinear differential equation with a constant delay. By means of the Lyapunov-Krasovskii ([8], [13]) functional approach, we obtain a new result on the topic.

## 1 Introduction

It is well known that the problems relative to qualitative behaviors of solutions of higher order nonlinear differential equations are very important in the theory and applications of differential equations. See, in particular, the papers of Chlouverakis and Sprott [1], Eichhorn et al [2], Linz [12] and the references cited in these papers for some applications.

With respect to our observations from the literature, in the last three decades, some authors give attention to investigate the instability of solutions of fifth order nonlinear differential equations without delay of the form:

$$x^{(5)}(t) + A_1 x^{(4)}(t) + A_2 x'''(t) + A_3 x''(t) + A_4 x'(t) + A_5 x(t) = 0$$

where $x \in \Re$, $t \in \Re_+$, $\Re_+ = [0, \infty)$, $A_1$ , $A_2$ , $A_3$ , $A_4$ and $A_5$ are not necessarily constants. For a comprehensive treatment of the subject we refer the reader to the papers of Ezeilo [4-6], Li and Duan [10], Li and Yu [11], Sadek [14], Sun and Hou [15], Tiryaki [16], Tunç [17-19], Tunç and Erdogan [21], Tunç and Karta [22], Tunç and Şevli [23] and the references cited in these papers for some works performed on the subject, which include some fifth order nonlinear differential equations without delay.

It should be also noted that throughout all of these papers, based on Krasovskii's properties (seeKrasovskii [8]), the Lyapunov's [13]second (or direct) method has

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been used as a basic tool to prove the results established therein. On the other hand, it is crucial to obtain information on the instability of solutions of differential equations while we have no analytical expression for solutions. For this purpose, the theory of Lyapunov functions and functionals is a global and the most effective approach toward determining qualitative behaviors of solutions of higher order nonlinear differential equations. This theory became an important part of both mathematics and theoretical mechanics in twentieth century. However, construction of Lyapunov functions and functional remains as a general problem in the literature.

Meanwhile, in 1989, Tiryaki [16] gave an instability result for the fifth order nonlinear differential equation without delay

$$x^{(5)}+a_1x^{(4)}+k(x,x',x'',x''',x^{(4)})x'''+g(x')x''+h(x,x',x'',x'',x^{(4)})+f(x)=0. \quad (1)$$

In this paper, instead of Eq. (1), we consider nonlinear fifth order delay differential equation

$$x^{(5)}+a_1x^{(4)}+k(x,x',x'',x''',x^{(4)})x'''+g(x')x''+h(x,x',x'',x'',x^{(4)})+f(x(t-r))=0. \quad (2)$$

We write Eq. (2) in system form as

$$\begin{aligned} x' &= y \\ y' &= z \\ z' &= w \\ w' &= u \\ u' &= -a_1u - k(x,y,z,w,u)w - g(y)z - h(x,y,z,w,u) - f(x) \\ &+ \int_{t-r}^{t} f'(x(s))y(s)ds, \end{aligned} \quad (3)$$

where $a_1$ and $r$ are positive constants, $r$ is fixed delay, the primes in Eq. (2) denote differentiation with respect to $t$, $t \in \Re^+ = [0,\infty)$; $k$, $g$, $h$ and $f$ are continuous functions in their arguments on $\Re^5$, $\Re$, $\Re^5$ and $\Re$, respectively, and with $h(x,0,z,w,u) = f(0) = 0$. The continuity of these functions is a sufficient condition for the existence of the solution of Eq. (2) (see [3, pp.14]). It is also assumed as basic that the functions $k$, $g$, $h$ and $f$ satisfy a Lipschitz condition in their respective arguments. By this way, the uniqueness of solutions of Eq. (2) is guaranteed (see [3, pp.15]). We assume in what follows that $f$ is also differentiable, and $x(t)$, $y(t)$, $z(t)$, $w(t)$ and $u(t)$ are abbreviated as $x$, $y$, $z$, $w$ and $u$, respectively.

The motivation to write this paper comes from the foregoing papers done for ordinary differential equations without delay. Our purpose is to achieve the result established in[16] to nonlinear delay differential equation given in (2) for the instability of the trivial solution of this equation. Finally, to the best of our knowledge, we did not find any instability result in the literature for fifth order delay differential equations except that of Tunç [20]. The basic reason for the lack of any paper on this topic may be the difficulty of the construction or definition of appropriate

Lyapunov functionals for the instability problems relative to higher order delay differential equations. Here, by defining an appropriate Lyapunov functional we carry out our purpose. This paper is the second attempt and work on the topic.

In the following theorems, we give basic idea of the method about the instability of solutions of ordinary and delay differential equations. The following theorem, due to the Russian mathematician N. G. Cetaev's (see LaSalle and Lefschetz [9]).

**Theorem A(**Instability Theorem of Cetaev's)**.** Let $\Omega$ be a neighborhood of the origin. Let there be given a function $V(x)$ and region $\Omega_1$ in $\Omega$ with the following properties:

**(i)** $V(x)$ has continuous first partial derivatives in $\Omega_1$.
**(ii)** $V(x)$ and $\dot{V}(x)$ are positive in $\Omega_1$.
**(iii)** At the boundary points of $\Omega_1$ inside $\Omega$, $V(x) = 0$.
**(iv)** The origin is a boundary point of $\Omega_1$.
Under these conditions the origin is unstable.

Let $r \geq 0$ be given, and let $C = C([-r, 0],\ \Re^n)$ with

$$\|\phi\| = \max_{-r \leq s \leq 0} |\phi(s)|\,,\ \phi \in C.$$

For $H > 0$ define $C_H \subset C$ by

$$C_H = \{\phi \in C : \|\phi\| < H\}.$$

If $x : [-r,\ a] \to \Re^n$ is continuous, $0 < A \leq \infty$, then, for each $t$ in $[0, A)$, $x_t$ in $C$ is defined by

$$x_t(s) = x(t + s), -r \leq s \leq 0, t \geq 0.$$

Let $G$ be an open subset of $C$ and consider the general autonomous delay differential system with finite delay

$$\dot{x} = F(x_t), x_t = x(t + \theta), -r \leq \theta \leq 0, t \geq 0,$$

where $F : G \to \Re^n$ is a continuous and maps closed and bounded sets into bounded sets. It follows from these conditions on $F$ that each initial value problem

$$\dot{x} = F(x_t), x_0 = \phi \in G$$

has a unique solution defined on some interval $[0, A)$, $0 < A \leq \infty$. This solution will be denoted by $x(\phi)(.)$ so that $x_0(\phi) = \phi$.

**Definition .** The zero solution, $x = 0$, of $\dot{x} = F(x_t)$ is stable if for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\|\phi\| < \delta$ implies that $|x(\phi)(t)| < \varepsilon$ for all $t \geq 0$. The zero solution is said to be unstable if it is not stable.

**Theorem B.** Suppose there exists a Lyapunov function $V : G \to \Re_+$ such that $V(0) = 0$ and $V(x) > 0$ if $x \neq 0$. If either

**(i)** $\dot{V}(\phi) > 0$ for all $\phi$ in $G$ for which

$$V[\phi(0)] = \max_{-s \le t \le 0} V[\phi(s)] > 0$$

or

**(ii)** $\dot{V}(\phi) > 0$ for all $\phi$ in $G$ for which

$$V[\phi(0)] = \min_{-s \le t \le 0} V[\phi(s)] > 0,$$

then the solution $x = 0$ of $\dot{x} = F(x_t)$ is unstable (see Haddock and Zhao [7]).

# 2 Main result

Our main result is the following theorem.

**Theorem 1.** In addition to the basic assumptions imposed on the functions $k$, $g$, $h$ and $f$ appearing in Eq. (2), we assume that there exist constants $\delta > 0$ and $a_5$ ($\neq 0$) such that the following conditions hold:

$$f(0) = 0, f(x) \neq 0 \text{ for all } x \neq 0, |f'(x)| \le |a_5|, h(x, 0, z, w, u) = 0,$$

$$\frac{h(x, y, z, w, u)}{y} - \frac{1}{4}k^2(x, y, z, w, u) \ge \delta \text{ for all } x\ ,\ y\ (\neq 0),\ z,\ w \text{ and } u.$$

Then the trivial solution $x = 0$ of Eq. (2) is unstable provided that $r < \frac{\delta}{|a_5|}$.

**Remark 1.** In order to prove this theorem, it is sufficient to show that there exists a continuous Lyapunov functional $V(x_t, y_t, z_t, w_t, u_t)$ which satisfies the following Krasovskii properties (see Krasovskii [8])**:**

$(K_1)$ In every neighborhood of $(0, 0, 0, 0, 0)$, there exists a point $(\xi, \eta, \zeta, \mu, \rho)$ such that $V(\xi, \eta, \zeta, \mu, \rho) > 0$ ;

$(K_2)$ the time derivative $\frac{d}{dt}V(x_t, y_t, z_t, w_t, u_t)$ along solution paths of (3) is positive semi-definite;

$(K_3)$ the only solution $(x, y, z, w, u) = (x(t), y(t), z(t), w(t), u(t))$ of (3) which satisfies $\frac{d}{dt}V(x_t, y_t, z_t, w_t, u_t)$ =0 is the trivial solution $(0, 0, 0, 0, 0)$.

**Remark 2.** If we take $k(x, x(t-r), ..., u, u(t-r))$ and $h(x, x(t-r), ..., u, u(t-r))$ in Eq. (2) instead of the functions $k(x, y, z, w, u)$ and $h(x, y, z, w, u)$, respectively, then the result of the above theorem remains valid.

**Proof.** We define a Lyapunov functional $V = V(x_t, y_t, z_t, w_t, u_t)$ as:

$$V = \frac{1}{2}a_1 z^2 + zw - yu - a_1 yw - \int_0^y g(\eta)\eta d\eta - \int_0^s f(s)ds - \lambda \int_{-r}^0 \int_{t+s}^t y^2(\theta)d\theta ds, \quad (4)$$

where $s$ is a real variable such that the integral $\int\limits_{-r}^{0}\int\limits_{t+s}^{t} y^2(\theta)d\theta ds$ is non-negative, and $\lambda$ is a positive constant which will be determined later in the proof.

It is clear that

$$V(0,0,\varepsilon^2,\varepsilon,0)= \frac{1}{2}a_1\varepsilon^4+\varepsilon^3>0$$

for all sufficiently small $\varepsilon>0$, which verifies the property $(K_1)$ of Krasovskii [8].

Using the Lyapunov functional $V$ and (3), the time derivative of $V$ yields that

$$\begin{aligned}\frac{d}{dt}V(x_t,y_t,z_t,w_t,u_t) &= w^2+k(x,y,z,w,u)yw+h(x,y,z,w,u)y \\ &\quad -y\int\limits_{t-r}^{t} f'(y(s)z(s)ds-\lambda ry^2+\lambda\int\limits_{t-r}^{t} y^2(s)ds.\end{aligned} \tag{5}$$

Utilizing the assumptions of the theorem and applying the relation $2\,|mn|\le m^2+n^2$, one can easily get the following inequality for fourth term included in (5):

$$\begin{aligned}-y\int\limits_{t-r}^{t} f'(x(s))y(s)ds &\ge -|y|\int\limits_{t-r}^{t} |f'(x(s))|\ \ |y(s)|\,ds \\ &\ge -\tfrac{1}{2}\,|a_5|\,ry^2-\tfrac{1}{2}\,|a_5|\int\limits_{t-r}^{t} y^2(s)ds.\end{aligned}$$

Then, we have

$$\begin{aligned}\frac{d}{dt}V(x_t,y_t,z_t,w_t,u_t) &\ge w^2+k(x,y,z,w,u)yw+h(x,y,z,w,u)y \\ &\quad -\tfrac{1}{2}\,|a_5|\,ry^2-\lambda ry^2+\left(\lambda-\tfrac{1}{2}\,|a_5|\right)\int\limits_{t-r}^{t} y^2(s)ds \\ &= [w+2^{-1}yk(x,y,z,w,u)]^2 \\ &\quad +[h(x,y,z,w,u)y^{-1}-4^{-1}k^2(x,y,z,w,u)]y^2 \\ &\quad -\tfrac{1}{2}\,|a_5|\,ry^2-\lambda ry^2+\left(\lambda-\tfrac{1}{2}\,|a_5|\right)\int\limits_{t-r}^{t} y^2(s)ds.\end{aligned}$$

Let $\lambda=\frac{1}{2}\,|a_5|$ . Hence

$$\begin{aligned}\frac{d}{dt}V(x_t,y_t,z_t,w_t,u_t) &\ge [h(x,y,z,w,u)y^{-1}-4^{-1}k^2(x,y,z,w,u)-|a_5|\,r]y^2 \\ &\ge (\delta-|a_5|\,r)y^2>0\end{aligned}$$

provided that $r<\frac{\delta}{|a_5|}$, which verifies the property $(K_2)$ of Krasovskii [8].

On the other hand $\frac{d}{dt}V(x_t,y_t,z_t,w_t,u_t)=0$ if and only if $y=0$, which implies that

$$y = z = w = u = 0.$$

Moreover, by $f(x) \neq 0$ for all $x \neq 0$, it follows that the only invariant set of (3) for which $y = 0$ is $x = 0$, which verifies the property $(K_3)$ of Krasovskii [8]. By the foregoing discussion, we conclude that the zero solution of Eq. (2) is unstable.

The proof of Theorem 1 is now completed.

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