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SEVERAL NEW HARDY-HILBERT'S INEQUALITIES

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Abstract

In this paper, we obtain several extended analogues of Hardy-Hilbert's inequalities.

1 Introduction

If f, g are real measurable functions such that

$$0 < \int_0^\infty f^2(x)dx < \infty \quad \text{and } 0 < \int_0^\infty g^2(x)dx < \infty, \tag{1}$$

then we have the following well known Hilbert's integral inequality [3],

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \pi \left\{ \int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx \right\}^{1/2}$$

where the constant factor π is the best possible. Furthermore, we have also the following Hardy-Hilbert's type inequality [3, Th 341, Th342],

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max\{x,y\}} dxdy < 4 \left\{ \int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx \right\}^{1/2},$$

$$\int_0^\infty \int_0^\infty \frac{\log x - \log y}{x-y} f(x)g(y)dxdy < \pi^2 \left\{ \int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx \right\}^{1/2},$$

where the constant factors 4 and π^2 are both the best possible.

There are numerous papers which study the Hilbert's and Hardy-Hilbert's type inequalities from different directions [1, 6, 7, 8, 9, 11]. Recently, Li-Wu-He [5] obtained the following inequality: if (1) is satisfied, then we have

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y+\max\{x,y\}} dxdy < c \left\{ \int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx \right\}^{1/2}, \tag{2}$$

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where the constant factor $c = 1.7408\cdots$ is the best possible.

He-Qian-Li [4] have proved the following inequality: if (1) is satisfied, then we have

$$\int_0^\infty \int_0^\infty \frac{|\log x - \log y|}{x + y + \min\{x, y\}} f(x)g(y)dxdy < c\left\{\int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx\right\}^{1/2}, \tag{3}$$

where the constant factor $c = 6.88947\cdots$ is the best possible.

In this short paper, we will give several extended analogues of Hardy-Hilbert's inequalities.

2 Main results

Before giving our main results, we need to establish the following

Lemma 1. *Let γ, α, β be three non-negative real numbers. Then we have the following equations*

$$\begin{aligned}
&\int_0^\infty \frac{|\log x - \log y|^\gamma}{\alpha x + \beta y + \max\{x, y\}} \left(\frac{x}{y}\right)^{1/2} dy \\
=&\int_0^\infty \frac{|\log x - \log y|^\gamma}{\alpha x + \beta y + \max\{x, y\}} \left(\frac{y}{x}\right)^{1/2} dx \\
=&\int_0^1 \frac{2^{\gamma+1}|\log t|^\gamma}{t^2\alpha + (1+\beta)} dt + \int_0^1 \frac{2^{\gamma+1}|\log t|^\gamma}{t^2\beta + (1+\alpha)} dt \\
=:&A,
\end{aligned}$$

where $A := A(\gamma, \alpha, \beta) \in [0, \infty]$.

Proof. For any given y, let $y = tx$, then it follows that

$$\begin{aligned}
&\int_0^\infty \frac{|\log x - \log y|^\gamma}{\alpha x + \beta y + \max\{x, y\}} \left(\frac{x}{y}\right)^{1/2} dy \\
=&\int_0^\infty \frac{|\log t|^\gamma}{\alpha + t\beta + \max\{1, t\}} \left(\frac{1}{t}\right)^{1/2} dt \\
=&\int_1^\infty \frac{|\log t|^\gamma}{t(\beta+1) + \alpha} \left(\frac{1}{t}\right)^{1/2} dt + \int_0^1 \frac{|\log t|^\gamma}{t\beta + (1+\alpha)} \left(\frac{1}{t}\right)^{1/2} dt \\
=&\int_0^1 \frac{|\log t|^\gamma}{t\alpha + (1+\beta)} \left(\frac{1}{t}\right)^{1/2} dt + \int_0^1 \frac{|\log t|^\gamma}{t\beta + (1+\alpha)} \left(\frac{1}{t}\right)^{1/2} dt \\
=&\int_0^1 \frac{2^{\gamma+1}|\log t|^\gamma}{t^2\alpha + (1+\beta)} dt + \int_0^1 \frac{2^{\gamma+1}|\log t|^\gamma}{t^2\beta + (1+\alpha)} dt
\end{aligned}$$

which implies the desired result. □

Theorem 1. *If f, g are real functions such that $0 < \int_0^\infty f^2(x)dx < \infty$ and $0 < \int_0^\infty g^2(x)dx < \infty$. Then we have*

$$\begin{aligned}\int_0^\infty \int_0^\infty & \frac{|\log x - \log y|^\gamma}{\alpha x + \beta y + \max\{x, y\}} f(x)g(y)dxdy \\ & < A \left(\int_0^\infty f^2(x)dx \right)^{1/2} \left(\int_0^\infty g^2(y)dy \right)^{1/2},\end{aligned} \tag{4}$$

where A is defined in Lemma 1 and is the best possible.

Proof. By Cauchy-Schwarz inequality and Lemma 1, we get

$$\begin{aligned}&\int_0^\infty \int_0^\infty \frac{|\log x - \log y|^\gamma}{\alpha x + \beta y + \max\{x, y\}} f(x)g(y)dxdy \\ \leq & \left\{ \int_0^\infty \left(\int_0^\infty \frac{|\log x - \log y|^\gamma}{\alpha x + \beta y + \max\{x, y\}} \left(\frac{x}{y}\right)^{1/2} dy \right) f^2(x)dx \right\}^{1/2} \\ & \times \left\{ \int_0^\infty \left(\int_0^\infty \frac{|\log x - \log y|^\gamma}{\alpha x + \beta y + \max\{x, y\}} \left(\frac{y}{x}\right)^{1/2} dx \right) g^2(y)dy \right\}^{1/2} \\ \leq & A \left(\int_0^\infty f^2(x)dx \right)^{1/2} \left(\int_0^\infty g^2(y)dy \right)^{1/2}.\end{aligned} \tag{5}$$

If the equality in (5) holds, then there exist two constant c and d, not both zero (without loss of generality, suppose that $c \neq 0$) and

$$c \frac{|\log x - \log y|^\gamma}{\alpha x + \beta y + \max\{x, y\}} \left(\frac{x}{y}\right)^{1/2} f^2(x) = d \frac{|\log x - \log y|^\gamma}{\alpha x + \beta y + \max\{x, y\}} \left(\frac{y}{x}\right)^{1/2} g^2(y), \quad a.e.$$

in $(0, \infty) \times (0, \infty)$. That is to say, we have

$$cxf^2(x) = dyg^2(y) = \text{constant}, \qquad a.e.$$

in $(0, \infty) \times (0, \infty)$. Thus

$$\int_0^\infty f^2(x)dx = \infty,$$

which contradicts the assumption $0 < \int_0^\infty f^2(x)dx < \infty$. Hence, the inequality (5) takes the form of strict inequality.

Assume that the constant A in the inequality (4) is not the best possible, then there exists a positive number K with $K < A$ and $a > 0$, such that

$$\begin{aligned}\int_a^\infty \int_0^\infty & \frac{|\log x - \log y|^\gamma}{\alpha x + \beta y + \max\{x, y\}} f(x)g(y)dxdy \\ & < K \left(\int_a^\infty f^2(x)dx \right)^{1/2} \left(\int_a^\infty g^2(y)dy \right)^{1/2}.\end{aligned} \tag{6}$$

For $0<\varepsilon<1$, setting

$$f_\varepsilon(x)=\begin{cases} x^{-\frac{\varepsilon+1}{2}}, & \text{for } x\in[b,\infty),\\ 0, & \text{for } x\in(0,b).\end{cases}\quad g_\varepsilon(y)=\begin{cases} y^{-\frac{\varepsilon+1}{2}}, & \text{for } x\in[b,\infty),\\ 0, & \text{for } x\in(0,b).\end{cases}$$

Then

$$K\left(\int_a^\infty f_\varepsilon^2(x)dx\right)^{1/2}\left(\int_a^\infty g_\varepsilon^2(y)dy\right)^{1/2}=K\frac{1}{\varepsilon a^\varepsilon}.$$

Let $y=tx$, we get

$$\begin{aligned}&\int_a^\infty\int_0^\infty \frac{|\log x-\log y|^\gamma}{\alpha x+\beta y+\max\{x,y\}}f_\varepsilon(x)g_\varepsilon(y)dxdy\\ =&\int_a^\infty\int_b^\infty \frac{|\log x-\log y|^\gamma}{\alpha x+\beta y+\max\{x,y\}}x^{-\frac{\varepsilon+1}{2}}y^{-\frac{\varepsilon+1}{2}}dxdy\\ =&\int_a^\infty\int_{b/x}^\infty \frac{|\log t|^\gamma}{\alpha+t\beta+\max\{1,t\}}x^{-(\varepsilon+1)}t^{-\frac{\varepsilon+1}{2}}dtdx\end{aligned}\tag{7}$$

Letting $b\to 0^+$, by (6) and Fatou's lemma, we have

$$\begin{aligned}&\int_a^\infty\int_0^\infty \frac{|\log t|^\gamma}{\alpha+t\beta+\max\{1,t\}}x^{-(\varepsilon+1)}t^{-\frac{\varepsilon+1}{2}}dtdx\\ =&\frac{1}{\varepsilon a^\varepsilon}\int_0^\infty \frac{|\log t|^\gamma}{\alpha+t\beta+\max\{1,t\}}t^{-\frac{\varepsilon+1}{2}}dt\le K\frac{1}{\varepsilon a^\varepsilon},\end{aligned}$$

which yields

$$\lim_{\varepsilon\to0^+}\int_0^\infty \frac{|\log t|^\gamma}{\alpha+t\beta+\max\{1,t\}}t^{-\frac{\varepsilon+1}{2}}dt=A\le K.$$

The contradiction implies that the constant A is the best possible. □

Theorem 2. *Suppose* $f\ge 0$ *and* $0<\int_0^\infty f^2(x)dx<\infty$. *Then*

$$\int_0^\infty\left(\int_0^\infty \frac{|\log x-\log y|^\gamma}{\alpha x+\beta y+\max\{x,y\}}f(x)dx\right)^2dy<A^2\int_0^\infty f^2(x)dx\tag{8}$$

Proof. Let

$$g(y)=\int_0^\infty \frac{|\log x-\log y|^\gamma}{\alpha x+\beta y+\max\{x,y\}}f(x)dx,$$

then by (5), we have

$$\begin{aligned}0<&\int_0^\infty g^2(y)dy\\ =&\int_0^\infty\left(\int_0^\infty \frac{|\log x-\log y|^\gamma}{\alpha x+\beta y+\max\{x,y\}}f(x)dx\right)^2dy\\ =&\int_0^\infty\int_0^\infty \frac{|\log x-\log y|^\gamma}{\alpha x+\beta y+\max\{x,y\}}f(x)g(y)dxdy\\ \le&A\left(\int_0^\infty f^2(x)dx\right)^{1/2}\left(\int_0^\infty g^2(y)dy\right)^{1/2},\end{aligned}\tag{9}$$

which yields

$$0 < \int_0^\infty g^2(y)dy \le A^2 \int_0^\infty f^2(x)dx < \infty. \tag{10}$$

By (4), both (9) and (10) take the form of strict inequality, so we have the inequality (8). On the other hand, suppose that (8) is valid. Again, you use the Cauchy-Schwarz inequality, we get

$$\begin{aligned}&\int_0^\infty \int_0^\infty \frac{|\log x - \log y|^\gamma}{\alpha x + \beta y + \max\{x, y\}} f(x)g(y)dxdy \\ =&\int_0^\infty \left(\int_0^\infty \frac{|\log x - \log y|^\gamma}{\alpha x + \beta y + \max\{x, y\}} f(x)dx\right) g(y)dy \\ <&A\left(\int_0^\infty f^2(x)dx\right)^{1/2}\left(\int_0^\infty g^2(y)dy\right)^{1/2}\end{aligned}$$

which is the inequality (4). □

Remark 1. If we take $\gamma = \alpha = \beta = 1$, then the inequality (3) can be induced by the inequality (4).

3 Several special inequalities

In this section, by choosing different γ, α, β, we establish several special inequalities. In what follows, assume that (1) is satisfied.

(1) If $\gamma = 0, \alpha = \beta = 1$, then

$$\begin{aligned}&\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x + y + \max\{x, y\}} dxdy \\ &\qquad < A\left(\int_0^\infty f^2(x)dx\right)^{1/2}\left(\int_0^\infty g^2(y)dy\right)^{1/2},\end{aligned} \tag{11}$$

where

$$A = 4\int_0^1 \frac{1}{t^2+2}dt = 2\sqrt{2}\arctan(\sqrt{2}/2).$$

(2) If $\alpha = \beta = 0$, $\gamma = 1$, then

$$\begin{aligned}&\int_0^\infty \int_0^\infty \frac{|\log x - \log y|}{\max\{x, y\}} f(x)g(y)dxdy \\ &\qquad < A\left(\int_0^\infty f^2(x)dx\right)^{1/2}\left(\int_0^\infty g^2(y)dy\right)^{1/2},\end{aligned} \tag{12}$$

where

$$A = -8\int_0^1 \log t dt = 7.99988\cdots.$$

(3) If $\alpha = 0$, $\beta = \gamma = 1$, then

$$\int_0^\infty \int_0^\infty \frac{|\log x - \log y|}{y + \max\{x, y\}} f(x)g(y)dxdy < A \left(\int_0^\infty f^2(x)dx \right)^{1/2} \left(\int_0^\infty g^2(y)dy \right)^{1/2}, \tag{13}$$

where

$$A = -2 \int_0^1 \log t dt - 4 \int_0^1 \frac{\log t}{1 + t^2} dt = 5.66377 \cdots .$$

(4) If $\gamma = 2$, $\alpha = \beta = 1$, then

$$\int_0^\infty \int_0^\infty \frac{|\log x - \log y|^2}{x + y + \max\{x, y\}} f(x)g(y)dxdy < A \left(\int_0^\infty f^2(x)dx \right)^{1/2} \left(\int_0^\infty g^2(y)dy \right)^{1/2}, \tag{14}$$

where

$$A = 16 \int_0^1 \frac{|\log t|^2}{t^2 + 2} dt = 15.72916 \cdots .$$

(5) If $\gamma = \alpha = 0$, $\beta = 1$, then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{y + \max\{x, y\}} dxdy < A \left(\int_0^\infty f^2(x)dx \right)^{1/2} \left(\int_0^\infty g^2(y)dy \right)^{1/2}, \tag{15}$$

$$A = 1 + 2 \int_0^1 \frac{1}{t^2 + 1} dt = 1 + \pi/2.$$

(6) If $\gamma = \beta = \alpha = 0$, then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max\{x, y\}} dxdy < A \left(\int_0^\infty f^2(x)dx \right)^{1/2} \left(\int_0^\infty g^2(y)dy \right)^{1/2}, \tag{16}$$

where

$$A = 4.$$

4 Further discussions

In this section, we give two new Hardy-Hilbert's inequalities. Before our works, the following result need be mentioned.

Lemma 2. *[2] Let f be a nonnegative integrable function. Define*

$$F(x) = \int_0^x f(t)dt.$$

Then

$$\int_0^\infty \left(\frac{F(x)}{x}\right)^p dx < \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx, \quad p > 1.$$

Theorem 3. *Let $f, g \geq 0$,*

$$F(x) = \int_0^x f(t)dt, \quad G(x) = \int_0^x g(t)dt.$$

Furthermore assume that $0 < \int_0^\infty f^2(x)dx < \infty$ and $0 < \int_0^\infty g^2(x)dx < \infty$ and let $A \in (0, \infty)$, then we have

$$\begin{aligned}\int_0^\infty \int_0^\infty & \frac{|\log x - \log y|^\gamma}{\alpha x + \beta y + \max\{x,y\}} \frac{F(x)}{x} \frac{G(y)}{y} dxdy \\ & < 4A \left(\int_0^\infty f^2(x)dx\right)^{1/2} \left(\int_0^\infty g^2(y)dy\right)^{1/2}.\end{aligned} \tag{17}$$

Proof. By Hölder's inequality, Lemma 1 and Lemma 2, we have

$$\begin{aligned}&\int_0^\infty \int_0^\infty \frac{|\log x - \log y|^\gamma}{\alpha x + \beta y + \max\{x,y\}} \frac{F(x)}{x} \frac{G(y)}{y} dxdy \\ \leq & \left\{\int_0^\infty \left(\int_0^\infty \frac{|\log x - \log y|^\gamma}{\alpha x + \beta y + \max\{x,y\}} \left(\frac{x}{y}\right)^{1/2} dy\right) \left(\frac{F(x)}{x}\right)^2 dx\right\}^{1/2} \\ & \times \left\{\int_0^\infty \left(\int_0^\infty \frac{|\log x - \log y|^\gamma}{\alpha x + \beta y + \max\{x,y\}} \left(\frac{y}{x}\right)^{1/2} dx\right) \left(\frac{G(y)}{y}\right)^2 dy\right\}^{1/2} \\ < & 4A \left(\int_0^\infty f^2(x)dx\right)^{1/2} \left(\int_0^\infty g^2(y)dy\right)^{1/2}.\end{aligned}$$

The proof of the theorem can be completed. □

Theorem 4. *Let $f, g \geq 0$,*

$$F(x) = \int_0^x f(t)dt, \quad G(x) = \int_0^x g(t)dt.$$

Furthermore assume that $p, q > 1$, $\alpha, \beta, s, t, \mu, \nu > 0$, such that

$$\frac{1}{p} + \frac{1}{q} = 1, sp > \beta q + 1, \ tq > \alpha p + 1$$

and

$$(\beta + \mu - s)p + 1 = 0, \quad (\alpha + \nu - t)q + 1 = 0.$$

Then we have

$$\int_0^\infty \int_0^\infty \frac{x^\alpha y^\beta F^\mu(x) G^\nu(y)}{(x+y)^{s+t}} dxdy$$
$$< \kappa \left(\frac{p\mu}{p\mu - 1}\right)^\mu \left(\frac{q\nu}{q\nu - 1}\right)^\nu \left(\int_0^\infty f^{p\mu}(x)dx\right)^{1/p} \left(\int_0^\infty g^{q\nu}(y)dy\right)^{1/q},$$

where

$$\kappa = B^{1/p}(\beta p + 1, sp - (\beta p + 1)) B^{1/q}(\alpha p + 1, tq - (\alpha p + 1))$$

and $B(\cdot,\cdot)$ denotes the Beta function.

Proof. By Hölder's inequality, it is easy to see

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \frac{x^\alpha y^\beta F^\mu(x) G^\nu(y)}{(x+y)^{s+t}} dxdy \\
= &\int_0^\infty \int_0^\infty \frac{x^\alpha F^\mu(x)}{(x+y)^s} \frac{y^\beta G^\nu(y)}{(x+y)^t} dxdy \\
\le &\left(\int_0^\infty \int_0^\infty \frac{y^{\beta p} F^{p\mu}(x)}{(x+y)^{sp}} dxdy\right)^{1/p} \left(\int_0^\infty \int_0^\infty \frac{x^{\alpha q} G^{q\nu}(y)}{(x+y)^{tq}} dxdy\right)^{1/q} \\
= &P^{1/p} Q^{1/q}.
\end{aligned}$$

Next by Lemma 2, we obtain

$$\begin{aligned}
P &= \int_0^\infty \left(\frac{F(x)}{x}\right)^{p\mu} dx \int_0^\infty \frac{y^{\beta p} x^{p\mu}}{(x+y)^{sp}} dy \\
&= \int_0^\infty \left(\frac{F(x)}{x}\right)^{p\mu} dx \int_0^\infty \frac{\left(\frac{y}{x}\right)^{\beta p} x^{-1}}{\left(1 + \frac{y}{x}\right)^{sp}} dy \\
&= \int_0^\infty \left(\frac{F(x)}{x}\right)^{p\mu} dx \int_0^\infty \frac{u^{\beta p}}{(1+u)^{sp}} du \\
&< B(\beta p + 1, sp - (\beta p + 1)) \left(\frac{p\mu}{p\mu - 1}\right)^{p\mu} \int_0^\infty f^{p\mu}(x) dx.
\end{aligned}$$

Similarly, it can be shown that

$$\begin{aligned}
Q &= \int_0^\infty \left(\frac{G(y)}{y}\right)^{q\nu} dy \int_0^\infty \frac{x^{\alpha q} y^{q\nu}}{(x+y)^{tq}} dx \\
&< B(\alpha q + 1, tq - (\alpha q + 1)) \left(\frac{q\nu}{q\nu - 1}\right)^{q\nu} \int_0^\infty g^{q\nu}(y) dy
\end{aligned}$$

which implies the desired result. □

Remark 2. Let $\mu = \nu = 1$, $\alpha = 1/p$, $\beta = 1/q$, $s = t = 2$, then we have the following special inequality (see [10]),

$$\int_0^\infty \int_0^\infty \frac{x^{1/p} y^{1/q} F(x) G(y)}{(x+y)^4} dxdy$$
$$\leq B^{1/p}(p,p) B^{1/q}(q,q) \frac{p}{p-1} \frac{q}{q-1} \left(\int_0^\infty f^p(x)dx \right)^{1/p} \left(\int_0^\infty g^q(y)dy \right)^{1/q}.$$

References

[1] H. X. Du and Y. Miao, *Several analogues of Hilbert inequalities.* Demonstratio Math. XLII (2) (2009), 297–302.

[2] G. H. Hardy, *Note on a theorem of Hilbert.* Math. Z. 6(3-4) (1920), 314–317.

[3] G. H. Hardy, J. E. Littlewood and G. Polya, *Inequalities.* Cambridge University Press, 1934.

[4] B. He, Y. Qian and Y. Li, *On analogues of Hilbert's inequalities.* Commun. Math. Anal. 4(2) (2008), 47–53.

[5] Y. Li, J. Wu and B. He, *A new Hilbert-type integral inequality and the equivalent form.* Int. J. Math. Math. Sci., (2006), Art. ID 45378, 6 pp.

[6] Y. Miao and H. X. Du, *A note of Hilbert type integral inequality.* Inequality Theory & Applications. 6 (2010), 261–267.

[7] D. S. Mitrinović, J. Pečarić and A. M. Fink, *Inequalities involving functions and their integrals and derivatives.* Kluwer Academic, Boston, 1991.

[8] D. S. Mitrinović, J. Pečarić and A. M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht, 1993.

[9] W. T. Sulaiman, *Four inequalities similar to Hardy-Hilbert's integral inequality.* JIPAM. J. Inequal. Pure Appl. Math. 7(2) (2006), Article 76, 8 pp.

[10] W. T. Sulaiman, *On three inequalities similar to Hardy-Hilbert's integral inequality.* Acta Math. Univ. Comenian. (N.S.), 76(2) (2007), 273–278.

[11] B. Yang, *On the norm of an integral operator and applications.* J. Math. Anal. Appl. 321 (2006), 182–192.

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