Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Filomat **25:3** (2011), 163–173 DOI: 10.2298/FIL1103163P

ON AN INTEGRAL-TYPE OPERATOR FROM $\mathcal{Q}_K(p,q)$ SPACES TO α-BLOCH SPACES

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Abstract

Let $g \in H(\mathbb{D})$, n be a nonnegative integer and φ be an analytic self-map of $\mathbb{D}$. We study the boundedness and compactness of the integral operator $C^n_{\varphi,g}$, which is defined by

$$(C^n_{\varphi,g}f)(z) = \int_0^z f^{(n)}(\varphi(\xi))g(\xi)d\xi, \quad z \in \mathbb{D}, \quad f \in H(\mathbb{D}),$$

from $\mathcal{Q}_K(p,q)$ and $\mathcal{Q}_{K,0}(p,q)$ spaces to α-Bloch spaces and little α-Bloch spaces.

1 Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane and $H(\mathbb{D})$ the class of all analytic functions on $\mathbb{D}$. Let $\alpha > 0$. An $f \in H(\mathbb{D})$ is said to belong to the α-Bloch space, denoted by $\mathcal{B}^\alpha$, if

$$\|f\|_{\mathcal{B}^\alpha} = |f(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha |f'(z)| < \infty. \tag{1}$$

Under the above norm, $\mathcal{B}^\alpha$ is a Banach space. When $\alpha = 1$, $\mathcal{B}^1 = \mathcal{B}$ is the classical Bloch space. Let $\mathcal{B}^\alpha_0$ denote the subspace of $\mathcal{B}^\alpha$ consisting of those $f \in \mathcal{B}^\alpha$ for which $(1-|z|^2)^\alpha |f'(z)| \to 0$ as $|z| \to 1$. This space is called the little α-Bloch space.

Let $g(z,a)$ be the Green function with logarithmic singularity at a, i.e. $g(z,a) = \log \frac{1}{|\varphi_a(z)|}$ (φ_a is a conformal automorphism defined by $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$ for $a \in \mathbb{D}$). Let $p > 0$, $q > -2$, $K : [0,\infty) \to [0,\infty)$ be a nondecreasing continuous function. An $f \in H(\mathbb{D})$ is said to belong to $\mathcal{Q}_K(p,q)$ space if (see [9, 29])

$$\|f\| = \Big(\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^q K(g(z,a))dA(z)\Big)^{1/p} < \infty, \tag{2}$$

2010 *Mathematics Subject Classifications.* Primary 47B35, Secondary 30H05.
Key words and Phrases. $\mathcal{Q}_K(p,q)$ space, α-Bloch space, integral-type operator.
Received: March 07, 2011
Communicated by Dragana Cvetković Ilić
The author would like to thank the referee for his/her helpful comments.

where dA is the normalized Lebesgue area measure in $\mathbb{D}$. For $p \geq 1$, under the norm $\|f\|_{\mathcal{Q}_K(p,q)} = |f(0)| + \|f\|$, $\mathcal{Q}_K(p,q)$ is a Banach space. An $f \in H(\mathbb{D})$ is said to belong to $\mathcal{Q}_{K,0}(p,q)$ space if

$$\lim_{|a|\to 1} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^q K(g(z,a)) dA(z) = 0. \tag{3}$$

Throughout the paper we assume that (see [29])

$$\int_0^1 (1-r^2)^q K(-\log r) r dr < \infty, \tag{4}$$

since otherwise $\mathcal{Q}_K(p,q)$ consists only of constant functions.

Let $g \in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$. The composition operator C_φ is defined by $C_\varphi(f)(z) = f(\varphi(z)), f \in H(\mathbb{D})$. In [4], Li and Stević defined the generalized composition operator as follows

$$(C_\varphi^g f)(z) = \int_0^z f'(\varphi(\xi)) g(\xi) d\xi, \quad f \in H(\mathbb{D}), \quad z \in \mathbb{D}.$$

The generalized composition operator and its generalizations on various spaces were investigated in [4–7, 13, 14, 19, 21, 22, 24, 28, 30–33, 35, 36]. See, e.g., [1, 11] and the references therein for the study of the composition operator.

Let $g \in H(\mathbb{D})$, n be a nonnegative integer and φ be an analytic self-map of $\mathbb{D}$. In [38], the author defined a new integral-type operator as follows:

$$(C_{\varphi,g}^n f)(z) = \int_0^z f^{(n)}(\varphi(\xi)) g(\xi) d\xi, \quad z \in \mathbb{D}, \quad f \in H(\mathbb{D}).$$

$C_{\varphi,g}^1$ is the generalized composition operator C_φ^g. When $n = 0$, then $C_{\varphi,g}^0$ is the Volterra composition operator defined by Li in [3], extended by Stević in the n-dimensional case in [16] and subsequently studied in [15, 17, 18, 20, 23, 25–27].

Here we characterized the boundedness and compactness of the operator $C_{\varphi,g}^n$ from $\mathcal{Q}_K(p,q)$ and $\mathcal{Q}_{K,0}(p,q)$ to α-Bloch and little α-Bloch spaces.

Throughout this paper, constants are denoted by C, they are positive and may differ from one occurrence to the other. The notation $A \asymp B$ means that there is a positive constant C such that $B/C \leq A \leq CB$.

2 Main results and proofs

In this section we give our main results and proofs. For this purpose, we need some auxiliary results. The following lemma can be proved in a standard way (see, e.g., [10]).

Lemma 1. *Let $\alpha, p > 0$, $q > -2$ and K be a nonnegative nondecreasing function on $[0, \infty)$. Assume that φ is an analytic self-map of $\mathbb{D}$ and n is a nonnegative integer. Then $C_{\varphi,g}^n : \mathcal{Q}_K(p,q)$(or $\mathcal{Q}_{K,0}(p,q)$) $\to \mathcal{B}^\alpha$ is compact if and only if $C_{\varphi,g}^n$:*

$\mathcal{Q}_K(p,q)$(or $\mathcal{Q}_{K,0}(p,q)$) $\to \mathcal{B}^\alpha$ is bounded and for any bounded sequence $(f_k)_{k\in\mathbb{N}}$ in $\mathcal{Q}_K(p,q)$(or $\mathcal{Q}_{K,0}(p,q)$) which converges to zero uniformly on compact subsets of $\mathbb{D}$, we have $\|C^n_{\varphi,g}f_k\|_{\mathcal{B}^\alpha} \to 0$ as $k \to \infty$.

The following lemma is essentially proved in [8], hence we omit its proof.

Lemma 2. *A closed set K in $\mathcal{B}^\alpha_0$ is compact if and only if it is bounded and satisfies*

$$\lim_{|z|\to 1^-} \sup_{f\in K} (1-|z|^2)^\alpha |f'(z)| = 0.$$

Lemma 3. [29] *Let $p > 0$, $q > -2$ and K is a nonnegative nondecreasing function on $[0,\infty)$. For $f \in \mathcal{Q}_K(p,q)$, we have $f \in \mathcal{B}^{\frac{q+2}{p}}$ and*

$$\|f\|_{\mathcal{B}^{\frac{q+2}{p}}} \le \|f\|_{\mathcal{Q}_K(p,q)}. \tag{5}$$

Lemma 4. [12] *Let $f \in \mathcal{B}^\alpha, 0 < \alpha < \infty$. Then*

$$|f(z)| \le \begin{cases} C\|f\|_{\mathcal{B}^\alpha} & , \quad 0<\alpha<1; \\ C\|f\|_{\mathcal{B}^\alpha} \ln \frac{e}{1-|z|} & , \quad \alpha = 1; \\ C\frac{\|f\|_{\mathcal{B}^\alpha}}{(1-|z|^2)^{\alpha-1}} & , \quad \alpha > 1. \end{cases}$$

Now we are in a position to state and prove the main results of this paper.

Theorem 1. *Let $\alpha, p > 0$, $q > -2$ and K be a nonnegative nondecreasing function on $[0,\infty)$ such that*

$$\int_0^1 K(-\log r)(1-r)^{\min\{-1,q\}} \Big(\log \frac{1}{1-r}\Big)^{\chi_{-1}(q)} r dr < \infty, \tag{6}$$

where $\chi_O(x)$ denote the characteristic function of the set O. Assume that φ is an analytic self-map of $\mathbb{D}$ and $n \in \mathbb{N}$. Then the following statements are equivalent.

(i) $C^n_{\varphi,g} : \mathcal{Q}_K(p,q) \to \mathcal{B}^\alpha$ is bounded;

(ii) $C^n_{\varphi,g} : \mathcal{Q}_{K,0}(p,q) \to \mathcal{B}^\alpha$ is bounded;

(iii)

$$M_1 := \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\alpha |g(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q-p}{p}+n}} < \infty. \tag{7}$$

Proof. $(iii) \Rightarrow (i)$. Suppose that (7) holds. First it is easy to see that $(C^n_{\varphi,g}f)(0) = 0$ and $(C^n_{\varphi,g}f)'(z) = f^{(n)}(\varphi(z))g(z)$ for every $f \in H(\mathbb{D})$. For any

$z \in \mathbb{D}$ and $f \in \mathcal{Q}_K(p,q)$, by Lemma 3 we have

$$\begin{aligned}(1-|z|^2)^\alpha |(C^n_{\varphi,g} f)'(z)| &= (1-|z|^2)^\alpha |f^{(n)}(\varphi(z))g(z)| \\ &\le \frac{(1-|z|^2)^\alpha |g(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q-p}{p}+n}} \|f\|_{\mathcal{B}^{\frac{q+2}{p}}} \\ &\le \frac{(1-|z|^2)^\alpha |g(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q-p}{p}+n}} \|f\|_{\mathcal{Q}_K(p,q)}, \qquad (8)\end{aligned}$$

where we have used the following well-known characterization for α-Bloch functions (see, e.g., [34])

$$\sup_{z\in\mathbb{D}} (1-|z|^2)^\alpha |f'(z)| \asymp |f'(0)| + \cdots + |f^{(n-1)}(0)| + \sup_{z\in\mathbb{D}} (1-|z|^2)^{n+\alpha-1} |f^{(n)}(z)|.$$

Taking the supremum in (8) for $z \in \mathbb{D}$, then employing (7) we obtain that $C^n_{\varphi,g} : \mathcal{Q}_K(p,q) \to \mathcal{B}^\alpha$ is bounded.

$(i) \Rightarrow (ii)$. It is clear.

$(ii) \Rightarrow (iii)$. Suppose that $C^n_{\varphi,g} : \mathcal{Q}_{K,0}(p,q) \to \mathcal{B}^\alpha$ is bounded, i.e. there exists a constant C such that $\|C^n_{\varphi,g} f\|_{\mathcal{B}^\alpha} \le C\|f\|_{\mathcal{Q}_K(p,q)}$ for all $f \in \mathcal{Q}_{K,0}(p,q)$. Taking the function $f(z) \equiv z^n$, which belongs to $\mathcal{Q}_{K,0}(p,q)$, we get

$$\sup_{z\in\mathbb{D}} (1-|z|^2)^\alpha |g(z)| < \infty. \qquad (9)$$

For $w \in \mathbb{D}$, let $f_w(z) = \frac{1-|w|^2}{(1-z\overline{w})^{\frac{q+2}{p}}}$. Using the condition (6), we see that $f_w \in \mathcal{Q}_{K,0}(p,q)$, for each $w \in \mathbb{D}$ (see [2]), moreover there is a positive constant C such that $\sup_{w\in\mathbb{D}} \|f_w\|_{\mathcal{Q}_K(p,q)} \le C$ and

$$|f_w^{(n)}(w)| = \prod_{j=0}^{n-1} \Big(\frac{q+2}{p} + j\Big) \frac{|w|^n}{(1-|w|^2)^{\frac{q+2-p}{p}+n}}.$$

Hence,

$$\begin{aligned}\infty &> C\|C^n_{\varphi,g}\|_{\mathcal{Q}_{K,0}(p,q)\to\mathcal{B}^\alpha} \ge \|C^n_{\varphi,g} f_{\varphi(\lambda)}\|_{\mathcal{B}^\alpha} \\ &\ge \prod_{j=0}^{n-1} \Big(\frac{q+2}{p} + j\Big) \frac{(1-|\lambda|^2)^\alpha |g(\lambda)||\varphi(\lambda)|^n}{(1-|\varphi(\lambda)|^2)^{\frac{q+2-p}{p}+n}} \qquad (10)\end{aligned}$$

for each $\lambda \in \mathbb{D}$.

From (10), we have

$$\begin{aligned}\sup_{|\varphi(\lambda)|>\frac{1}{2}} \frac{(1-|\lambda|^2)^\alpha |g(\lambda)|}{(1-|\varphi(\lambda)|^2)^{\frac{q+2-p}{p}+n}} &\le 2^n \sup_{|\varphi(\lambda)|>\frac{1}{2}} \frac{(1-|\lambda|^2)^\alpha |g(\lambda)||\varphi(\lambda)|^n}{(1-|\varphi(\lambda)|^2)^{\frac{q+2-p}{p}+n}} \\ &\le C\|C^n_{\varphi,g}\|_{\mathcal{Q}_{K,0}(p,q)\to\mathcal{B}^\alpha} < \infty. \qquad (11)\end{aligned}$$

Inequality (9) gives

$$\sup_{|\varphi(\lambda)|\le\frac12}\frac{(1-|\lambda|^2)^\alpha|g(\lambda)|}{(1-|\varphi(\lambda)|^2)^{\frac{q+2-p}{p}+n}} \le \frac{4^{\frac{q+2-p}{p}+n}}{3^{\frac{q+2-p}{p}+n}}\sup_{|\varphi(\lambda)|\le\frac12}(1-|\lambda|^2)^\alpha|g(\lambda)|<\infty, \quad (12)$$

where we used the assumption $(q+2-p)/p+n>0$. Therefore, (7) follows from (11) and (12). This completes the proof of Theorem 1. □

Theorem 2. *Let $\alpha, p>0$, $q>-2$ and K be a nonnegative nondecreasing function on $[0,\infty)$ such that (6) holds. Assume that φ is an analytic self-map of $\mathbb{D}$ and $n\in\mathbb{N}$. Then the following statements are equivalent.*
(i) $C^n_{\varphi,g}:\mathcal{Q}_K(p,q)\to\mathcal{B}^\alpha$ is compact;
(ii) $C^n_{\varphi,g}:\mathcal{Q}_{K,0}(p,q)\to\mathcal{B}^\alpha$ is compact;
(iii) $C^n_{\varphi,g}:\mathcal{Q}_K(p,q)\to\mathcal{B}^\alpha$ is bounded and

$$\lim_{|\varphi(z)|\to1}\frac{(1-|z|^2)^\alpha|g(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q-p}{p}+n}}=0. \quad (13)$$

Proof. *(iii)* ⇒ *(i)*. Suppose that $C^n_{\varphi,g}:\mathcal{Q}_K(p,q)\to\mathcal{B}^\alpha$ is bounded and (13) holds. Let $(f_k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{Q}_K(p,q)$ such that $\sup_{k\in\mathbb{N}}\|f_k\|_{\mathcal{Q}_K(p,q)}\le C$ and f_k converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $k\to\infty$. By the assumption, for any $\varepsilon>0$, there exists a $\delta\in(0,1)$ such that

$$\frac{(1-|z|^2)^\alpha|g(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q-p}{p}+n}}<\varepsilon \quad (14)$$

when $\delta<|\varphi(z)|<1$. Since $C^n_{\varphi,g}:\mathcal{Q}_K(p,q)\to\mathcal{B}^\alpha$ is bounded, then from the proof of Theorem 1 we have

$$M_2:=\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|g(z)|<\infty. \quad (15)$$

Let $\Omega=\{z\in\mathbb{D}:|\varphi(z)|\le\delta\}$. Then, we have

$$\begin{aligned}
&\|C^n_{\varphi,g}f_k\|_{\mathcal{B}^\alpha}=\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|(C^n_{\varphi,g}f_k)'(z)|\\
\le\ & \sup_{\Omega}(1-|z|^2)^\alpha|g(z)||f_k^{(n)}(\varphi(z))|+\sup_{\mathbb{D}\setminus\Omega}(1-|z|^2)^\alpha|g(z)||f_k^{(n)}(\varphi(z))|\\
\le\ & \sup_{\Omega}(1-|z|^2)^\alpha|g(z)||f_k^{(n)}(\varphi(z))|+C\sup_{\mathbb{D}\setminus\Omega}\frac{(1-|z|^2)^\alpha|g(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q-p}{p}+n}}\|f_k\|_{\mathcal{Q}_K(p,q)}\\
\le\ & M_2\sup_{|w|\le\delta}|f_k^{(n)}(w)|+C\varepsilon\|f_k\|_{\mathcal{Q}_K(p,q)}. \quad (16)
\end{aligned}$$

¿From Cauchy's estimate and the assumption that $f_k\to0$ as $k\to\infty$ on compact subsets of $\mathbb{D}$, we see that $f_k^{(n)}\to0$ as $k\to\infty$ on compact subsets of $\mathbb{D}$. Letting

$k \to \infty$ in (16) and using the fact that ε is an arbitrary positive number, we obtain $\lim_{k\to\infty} \|C^n_{\varphi,g} f_k\|_{\mathcal{B}^\alpha} = 0$. Applying Lemma 1, the result follows.

$(i) \Rightarrow (ii)$. This implication is obvious.

$(ii) \Rightarrow (iii)$. Suppose that $C^n_{\varphi,g} : \mathcal{Q}_{K,0}(p,q) \to \mathcal{B}^\alpha$ is compact. Then it is clear that $C^n_{\varphi,g} : \mathcal{Q}_{K,0}(p,q) \to \mathcal{B}^\alpha$ is bounded and from Theorem 1 we see that $C^n_{\varphi,g} : \mathcal{Q}_K(p,q) \to \mathcal{B}^\alpha$ is bounded. Let $(z_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_k)| \to 1$ as $k \to \infty$ (if such a sequence does not exist then condition (13) is vacuously satisfied). Let $f_k(z) = \frac{1-|\varphi(z_k)|^2}{(1-\overline{\varphi(z_k)}z)^{\frac{q+2}{p}}}$. Then, $f_k \in \mathcal{Q}_{K,0}(p,q)$, $\sup_{k\in\mathbb{N}} \|f_k\|_{\mathcal{Q}_K(p,q)} < \infty$ and f_k converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$. Since $C^n_{\varphi,g} : \mathcal{Q}_{K,0}(p,q) \to \mathcal{B}^\alpha$ is compact, by Lemma 1 we have

$$\lim_{k\to\infty} \|C^n_{\varphi,g} f_k\|_{\mathcal{B}^\alpha} = 0. \tag{17}$$

On the other hand, from (10) we have

$$\|C^n_{\varphi,g} f_k\|_{\mathcal{B}^\alpha} \geq \prod_{j=0}^{n-1} \Big(\frac{q+2}{p} + j\Big) \frac{(1-|z_k|^2)^\alpha |g(z_k)||\varphi(z_k)|^n}{(1-|\varphi(z_k)|^2)^{\frac{2+q-p}{p}+n}}$$

which together with (17) implies that

$$\lim_{|\varphi(z_k)|\to 1} \frac{(1-|z_k|^2)^\alpha |g(z_k)||\varphi(z_k)|^n}{(1-|\varphi(z_k)|^2)^{\frac{2+q-p}{p}+n}} = \lim_{k\to\infty} \frac{(1-|z_k|^2)^\alpha |g(z_k)||\varphi(z_k)|^n}{(1-|\varphi(z_k)|^2)^{\frac{2+q-p}{p}+n}} = 0, \tag{18}$$

from which (13) easily follows. □

Theorem 3. *Let $\alpha, p > 0$, $q > -2$ and K be a nonnegative nondecreasing function on $[0,\infty)$ such that (6) holds. Assume that φ is an analytic self-map of $\mathbb{D}$ and $n \in \mathbb{N}$. Then $C^n_{\varphi,g} : \mathcal{Q}_{K,0}(p,q) \to \mathcal{B}^\alpha_0$ is bounded if and only if $C^n_{\varphi,g} : \mathcal{Q}_{K,0}(p,q) \to \mathcal{B}^\alpha$ is bounded and*

$$\lim_{|z|\to 1} (1-|z|^2)^\alpha |g(z)| = 0. \tag{19}$$

Proof. Suppose that $C^n_{\varphi,g} : \mathcal{Q}_{K,0}(p,q) \to \mathcal{B}^\alpha_0$ is bounded. It is obvious that $C^n_{\varphi,g} : \mathcal{Q}_{K,0}(p,q) \to \mathcal{B}^\alpha$ is bounded. Taking the function $f(z) = z^n$, and employing the boundedness of $C^n_{\varphi,g} : \mathcal{Q}_{K,0}(p,q) \to \mathcal{B}^\alpha_0$ we see that (19) holds.

Conversely, assume that $C^n_{\varphi,g} : \mathcal{Q}_{K,0}(p,q) \to \mathcal{B}^\alpha$ is bounded and (19) holds. Then, for each polynomial $p(z)$, we have that

$$(1-|z|^2)^\alpha |(C^n_{\varphi,g} p)'(z)| \leq (1-|z|^2)^\alpha |g(z)| \|p^{(n)}\|_\infty,$$

from which it follows that $C^n_{\varphi,g} p \in \mathcal{B}^\alpha_0$. Since the set of all polynomials is dense in $\mathcal{Q}_{K,0}(p,q)$ (see [2]), we have that for every $f \in \mathcal{Q}_{K,0}(p,q)$ there is a sequence of polynomials $(p_k)_{k\in\mathbb{N}}$ such that $\|f - p_k\|_{\mathcal{Q}_K(p,q)} \to 0$, as $k \to \infty$. Hence

$$\|C^n_{\varphi,g} f - C^n_{\varphi,g} p_k\|_{\mathcal{B}^\alpha} \leq \|C^n_{\varphi,g}\|_{\mathcal{Q}_{K,0}(p,q)\to\mathcal{B}^\alpha} \|f - p_k\|_{\mathcal{Q}_K(p,q)} \to 0$$

as $k \to \infty$. Since $\mathcal{B}_0^\alpha$ is closed subset of $\mathcal{B}^\alpha$, we obtain $C_{\varphi,g}^n(\mathcal{Q}_{K,0}(p,q)) \subset \mathcal{B}_0^\alpha$. Therefore $C_{\varphi,g}^n : \mathcal{Q}_{K,0}(p,q) \to \mathcal{B}_0^\alpha$ is bounded. □

Theorem 4. *Let $\alpha, p > 0$, $q > -2$ and K be a nonnegative nondecreasing function on $[0,\infty)$ such that (6) holds. Assume that φ is an analytic self-map of $\mathbb{D}$ and $n \in \mathbb{N}$. Then the following statements are equivalent.*

(i) $C_{\varphi,g}^n : \mathcal{Q}_K(p,q) \to \mathcal{B}_0^\alpha$ is compact;

(ii) $C_{\varphi,g}^n : \mathcal{Q}_{K,0}(p,q) \to \mathcal{B}_0^\alpha$ is compact;

(iii)

$$\lim_{|z|\to 1} \frac{(1-|z|^2)^\alpha |g(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q-p}{p}+n}} = 0. \tag{20}$$

Proof. $(iii) \Rightarrow (i)$. Assume that (20) holds. Let $f \in \mathcal{Q}_K(p,q)$. By the proof of Theorem 1 we have

$$(1-|z|^2)^\alpha |(C_{\varphi,g}^n f)'(z)| \le C \frac{(1-|z|^2)^\alpha |g(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q-p}{p}+n}} \|f\|_{\mathcal{Q}_K(p,q)}. \tag{21}$$

Taking the supremum in (21) over all $f \in \mathcal{Q}_K(p,q)$ such that $\|f\|_{\mathcal{Q}_K(p,q)} \le 1$, then letting $|z| \to 1$, we get

$$\lim_{|z|\to 1} \sup_{\|f\|_{\mathcal{Q}_K(p,q)}\le 1} (1-|z|^2)^\alpha |(C_{\varphi,g}^n f)'(z)| = 0.$$

¿From which by Lemma 2 we see that $C_{\varphi,g}^n : \mathcal{Q}_K(p,q) \to \mathcal{B}_0^\alpha$ is compact.

$(i) \Rightarrow (ii)$. This implication is obvious.

$(ii) \Rightarrow (iii)$. Suppose that $C_{\varphi,g}^n : \mathcal{Q}_{K,0}(p,q) \to \mathcal{B}_0^\alpha$ is compact. Then $C_{\varphi,g}^n : \mathcal{Q}_{K,0}(p,q) \to \mathcal{B}_0^\alpha$ is bounded and by Theorem 3 we get

$$\lim_{|z|\to 1} (1-|z|^2)^\alpha |g(z)| = 0. \tag{22}$$

If $\|\varphi\|_\infty < 1$, from (22), we obtain that

$$\lim_{|z|\to 1} \frac{(1-|z|^2)^\alpha |g(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q-p}{p}+n}} \le \frac{1}{(1-\|\varphi\|_\infty^2)^{\frac{2+q-p}{p}+n}} \lim_{|z|\to 1} (1-|z|^2)^\alpha |g(z)| = 0,$$

from which the result follows in this case.

Assume that $\|\varphi\|_\infty = 1$. Let $(\varphi(z_k))_{k\in\mathbb{N}}$ be a sequence such that $\lim_{k\to\infty} |\varphi(z_k)| = 1$. From the compactness of $C_{\varphi,g}^n : \mathcal{Q}_{K,0}(p,q) \to \mathcal{B}_0^\alpha$ we see that the operator $C_{\varphi,g}^n : \mathcal{Q}_{K,0}(p,q) \to \mathcal{B}^\alpha$ is compact. ¿From Theorem 2 we get

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\alpha |g(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q-p}{p}+n}} = 0 \tag{23}$$

From (23), we have that for every $\varepsilon > 0$, there exists an $r \in (0,1)$ such that

$$\frac{(1-|z|^2)^\alpha |g(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q-p}{p}+n}} < \varepsilon \tag{24}$$

when $r < |\varphi(z)| < 1$. From (22), there exists a $\sigma \in (0,1)$ such that

$$(1-|z|^2)^\alpha |g(z)| \le \varepsilon (1-r^2)^{\frac{2+q-p}{p}+n} \tag{25}$$

when $\sigma < |z| < 1$.

Therefore, when $\sigma < |z| < 1$ and $r < |\varphi(z)| < 1$, we have

$$\frac{(1-|z|^2)^\alpha |g(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q-p}{p}+n}} < \varepsilon. \tag{26}$$

On the other hand, if $\sigma < |z| < 1$ and $|\varphi(z)| \le r$, we obtain

$$\frac{(1-|z|^2)^\alpha |g(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q-p}{p}+n}} < \frac{1}{(1-r^2)^{\frac{2+q-p}{p}+n}}(1-|z|^2)^\alpha |g(z)| < \varepsilon. \tag{27}$$

¿From (26) and (27) we get (20), as desired. The proof is completed. □

Next, we consider the case $n = 0$.

Theorem 5. *Let $\alpha, p > 0$, $q > -2$ such that $q+2 \ge p$. Let K be a nonnegative nondecreasing function on $[0,\infty)$ such that (6) holds. Assume that φ is an analytic self-map of $\mathbb{D}$. Then the following statements are equivalent.*

(i) $C^0_{\varphi,g} : \mathcal{Q}_K(p,q) \to \mathcal{B}^\alpha$ is bounded;

(ii) $C^0_{\varphi,g} : \mathcal{Q}_{K,0}(p,q) \to \mathcal{B}^\alpha$ is bounded;

(iii)

$$\begin{cases} \sup_{z\in\mathbb{D}} (1-|z|^2)^\alpha |g(z)| \ln \frac{e}{1-|\varphi(z)|^2} < \infty & , \quad q+2 = p; \\ \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\alpha |g(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q-p}{p}}} < \infty & , \quad q+2 > p. \end{cases}$$

Proof. $(ii) \Rightarrow (iii)$. Assume that $C^0_{\varphi,g} : \mathcal{Q}_{K,0}(p,q) \to \mathcal{B}^\alpha$ is bounded. For $w \in \mathbb{D}$, let

$$f_w(z) = \begin{cases} \ln \frac{e}{1-z\overline{w}} & , \quad q+2 = p; \\ \frac{1-|w|^2}{(1-z\overline{w})^{\frac{q+2}{p}}} & , \quad q+2 > p. \end{cases}$$

Then $f_w \in \mathcal{Q}_{K,0}(p,q)$ (see [2]). The other proof is similar to the proof of Theorem 1 and hence we omit it.

$(i) \Rightarrow (ii)$ is obvious.

$(iii) \Rightarrow (i)$. Using Lemma 4, similar to the proof of Theorem 1, the implication follows. We omit the details of the proofs.

Let $(z_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_k)| \to 1$ as $k \to \infty$. Taking the test function

$$f_{\varphi(z_k)}(z) = \begin{cases} \ln \frac{e}{1-z\overline{\varphi(z_k)}} & , \quad q+2 = p; \\ \frac{1-|\varphi(z_k)|^2}{(1-z\overline{\varphi(z_k)})^{\frac{q+2}{p}}} & , \quad q+2 > p, \end{cases}$$

similar to the proof of Theorem 2, we obtain the following result.

Theorem 6. *Let $\alpha, p > 0$, $q > -2$ such that $q+2 \geq p$. Let K be a nonnegative nondecreasing function on $[0,\infty)$ such that (6) holds. Assume that φ is an analytic self-map of $\mathbb{D}$. Then the following statements are equivalent.*

(i) $C^0_{\varphi,g} : \mathcal{Q}_K(p,q) \to \mathcal{B}^\alpha$ is compact;

(ii) $C^0_{\varphi,g} : \mathcal{Q}_{K,0}(p,q) \to \mathcal{B}^\alpha$ is compact;

(iii) $C^0_{\varphi,g} : \mathcal{Q}_K(p,q) \to \mathcal{B}^\alpha$ is bounded and

$$\begin{cases} \lim_{|\varphi(z)|\to 1}(1-|z|^2)^\alpha |g(z)| \ln \frac{e}{1-|\varphi(z)|^2} = 0 & , \quad q+2=p; \\ \lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\alpha |g(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q-p}{p}}} = 0 & , \quad q+2>p. \end{cases}$$

Similar to the proofs of Theorems 3 and 4, we obtain Theorems 7 and 8 respectively. We omit the proofs.

Theorem 7. *Let $\alpha, p > 0$, $q > -2$ such that $q+2 \geq p$. Let K be a nonnegative nondecreasing function on $[0,\infty)$ such that (6) holds. Assume that φ is an analytic self-map of $\mathbb{D}$. Then $C^0_{\varphi,g} : \mathcal{Q}_{K,0}(p,q) \to \mathcal{B}^\alpha_0$ is bounded if and only if $C^0_{\varphi,g} : \mathcal{Q}_{K,0}(p,q) \to \mathcal{B}^\alpha$ is bounded and*

$$\lim_{|z|\to 1}(1-|z|^2)^\alpha |g(z)| = 0.$$

Theorem 8. *Let $\alpha, p > 0$, $q > -2$ such that $q+2 \geq p$. Let K be a nonnegative nondecreasing function on $[0,\infty)$ such that (6) holds. Assume that φ is an analytic self-map of $\mathbb{D}$. Then the following statements are equivalent.*

(i) $C^0_{\varphi,g} : \mathcal{Q}_K(p,q) \to \mathcal{B}^\alpha_0$ is compact;

(ii) $C^0_{\varphi,g} : \mathcal{Q}_{K,0}(p,q) \to \mathcal{B}^\alpha_0$ is compact;

(iii)

$$\begin{cases} \lim_{|z|\to 1}(1-|z|^2)^\alpha |g(z)| \ln \frac{e}{1-|\varphi(z)|^2} = 0 & , \quad q+2=p; \\ \lim_{|z|\to 1} \frac{(1-|z|^2)^\alpha |g(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q-p}{p}}} = 0 & , \quad q+2>p. \end{cases}$$

The proof of the following two theorems are similar to the proofs of Theorems 12-14 of [37]. We omit the details.

Theorem 9. *Let $\alpha, p > 0$, $q > -2$ such that $q+2 < p$. Let K be a nonnegative nondecreasing function on $[0,\infty)$ such that (6) holds. Assume that φ is an analytic self-map of $\mathbb{D}$. Then the following statements are equivalent.*

(i) $C^0_{\varphi,g} : \mathcal{Q}_K(p,q) \to \mathcal{B}^\alpha$ is bounded;

(ii) $C^0_{\varphi,g} : \mathcal{Q}_{K,0}(p,q) \to \mathcal{B}^\alpha$ is bounded;

(iii) $C^0_{\varphi,g} : \mathcal{Q}_K(p,q) \to \mathcal{B}^\alpha$ is compact;

(iv) $C^0_{\varphi,g} : \mathcal{Q}_{K,0}(p,q) \to \mathcal{B}^\alpha$ is compact;

(v)

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|g(z)|<\infty.$$

Theorem 10. *Let $\alpha, p > 0$, $q > -2$ such that $q+2 < p$. Let K be a nonnegative nondecreasing function on $[0,\infty)$ such that (6) holds. Assume that φ is an analytic self-map of $\mathbb{D}$. Then the following statements are equivalent.*

(i) $C^0_{\varphi,g} : \mathcal{Q}_K(p,q) \to \mathcal{B}^\alpha_0$ is bounded;

(ii) $C^0_{\varphi,g} : \mathcal{Q}_{K,0}(p,q) \to \mathcal{B}^\alpha_0$ is bounded;

(iii) $C^0_{\varphi,g} : \mathcal{Q}_K(p,q) \to \mathcal{B}^\alpha_0$ is compact;

(iv) $C^0_{\varphi,g} : \mathcal{Q}_{K,0}(p,q) \to \mathcal{B}^\alpha_0$ is compact;

(v)

$$\lim_{|z|\to 1}(1-|z|^2)^\alpha|g(z)|=0.$$

References

[1] C. C. Cowen and B. D. MacCluer, *Composition Operators on Spaces of Analytic Functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, 1995.

[2] M. Kotilainen, On composition operators in $\mathcal{Q}_K$ type spaces, *J. Funct. Space Appl.* **5** (2007), 103-122.

[3] S. Li, Volterra composition operator between weighted Bergman space and Bloch type space, *J. Korea Math. Soc.* **45**(2008), 229-248

[4] S. Li and S. Stević, Generalized composition operators on Zygmund spaces and Bloch type spaces, *J. Math. Anal. Appl.* **338** (2008), 1282-1295.

[5] S. Li and S. Stević, Products of Volterra type operator and composition operator from H^∞ and Bloch spaces to the Zygmund space, *J. Math. Anal. Appl.* **345** (2008), 40-52.

[6] S. Li and S. Stević, Products of integral-type operators and composition operators between Bloch-type spaces, *J. Math. Anal. Appl.* **349** (2009), 596-610.

[7] S. Li and S. Stević, On an integral-type operator from iterated logarithmic Bloch spaces into Bloch-type spaces, *Appl. Math. Comput.* **215**(2009), 3106-3115.

[8] K. Madigan and A. Matheson, Compact composition operators on the Bloch space, *Trans. Amer. Math. Soc.* **347** (1995), 2679-2687.

[9] X. Meng, Some sufficient conditions for analytic functions to belong to $\mathcal{Q}_{K,0}(p,q)$ space, *Abstr. Appl. Anal.* Volume **2008** (2008), Article ID 404636, 9 pages.

[10] H. J. Schwartz, Composition operators on H^p, Thesis, University of Toledo, 1969.

[11] J. Shapiro, *Composition Operators and Classical Function Theory*, Springer-Verlag, New York, 1993.

[12] S. Stević, On an integral operator on the unit ball in $\mathbb{C}^n$, *J. Inequal. Appl.* **1** (2005), 81-88.

[13] S. Stević, Generalized composition operators between mixed norm space and some weighted spaces, *Numer. Funct. Anal. Opt.* **29** (7-8) (2008), 959-978.

[14] S. Stević, Generalized composition operators from logarithmic Bloch spaces to mixed-norm spaces, *Util. Math.* **77** (2008), 167-172.

[15] S. Stević, On a new operator from the logarithmic Bloch space to the Bloch-type space on the unit ball, *Appl. Math. Comput.* **206** (2008), 313-320.

[16] S. Stević, On a new operator from H^∞ to the Bloch-type space on the unit ball, *Util. Math.* **77** (2008), 257-263.

[17] S. Stević, On a new integral-type operator from the weighted Bergman space to the Bloch-type space on the unit ball, *Discrete Dyn. Nat. Soc.* Vol. 2008, Article ID 154263, (2008), 14 pages.

[18] S. Stević, On a new integral-type operator from the Bloch space to Bloch-type spaces on the unit ball, *J. Math. Anal. Appl.* **354** (2009), 426-434.

[19] S. Stević, Products of integral-type operators and composition operators from the mixed norm space to Bloch-type spaces, *Siberian Math. J.* **50** (4) (2009), 726-736.

[20] S. Stević, On an integral-type operator from logarithmic Bloch-type and mixed-norm spaces to Bloch-type spaces, *Nonlinear Anal. TMA* **71**(2009), 6323-6342.

[21] S. Stević, Integral-type operators from a mixed norm space to a Bloch-type space on the unit ball, *Siberian Math. J.* **50** (6) (2009), 1098-1105.

[22] S. Stević, On an integral operator from the Zygmund space to the Bloch-type space on the unit ball, *Glasg. J. Math.* **51** (2009), 275-287.

[23] S. Stević, On operator P_g from the logarithmic Bloch-type space to the mixed-norm space on unit ball, *Appl. Math. Comput.* **215** (2010), 4248-4255.

[24] S. Stević, On an integral operator between Bloch-type spaces on the unit ball, Bull. Sci. Math. 134 (2010), 329-339.

[25] S. Stević, Norm and essential norm of an integral-type operator from the Dirichlet space to the Bloch-type space on the unit ball, *Abstr. Appl. Anal.* Vol. 2010, Article ID 134969, (2010), 9 pages.

[26] S. Stević, On an integral-type operator from Zygmund-type spaces to mixed-norm spaces on the unit ball, *Abstr. Appl. Anal.* Vol. 2010, Article ID 198608, (2010), 7 pages.

[27] S. Stević and S. I. Ueki, On an integral-type operator between weighted-type spaces and Bloch-type spaces on the unit ball, *Appl. Math. Comput.* **217** (2010), 3127-3136.

[28] S. Stević and S. I. Ueki, On an integral-type operator acting between Bloch-type spaces on the unit ball, *Abstr. Appl. Anal.* Vol. 2010, Article ID 214762, (2010), 14 pages.

[29] H. Wulan and J. Zhou, $\mathcal{Q}_K$ type spaces of analytic functions, *J. Funct. Spaces Appl.* **4** (2006), 73-84.

[30] W. Yang, Products of composition and differentiation operators from $\mathcal{Q}_K(p,q)$ spaces to Bloch-type spaces, *Abstr. Appl. Anal.* vol. 2009, no. 1, Article ID 741920, 14 pages, 2009.

[31] W. Yang and X. Meng, Generalized composition operators from $F(p,q,s)$ spaces to Bloch-type spaces, *Appl. Math. Comput.* **217**(2010), 2513-2519.

[32] Y. Yu and Y. Liu, On a Li-Stević integral-type operators between different weighted Bloch-type spaces, *J. Inequal. Appl.* vol. 2008, Article ID 780845, 14 pages, 2008.

[33] F. Zhang and Y. Liu, Generalized composition operators from Bloch type spaces to $\mathcal{Q}_K$ spaces , *J. Funct. Spaces Appl.* **8** (2010), 55-66.

[34] K. Zhu, Bloch type spaces of analytic functions, *Rocky Mountain J. Math.* **23** (1993), 1143-1177.

[35] X. Zhu, Generalized composition operators from generalized weighted Bergman spaces to Bloch type spaces, *J. Korea Math. Soc.* **46** (2009), 1219-1232.

[36] X. Zhu, Generalized composition operators and Volterra composition operators on Bloch spaces in the unit ball, *Complex Var. Elliptic Equ.*, **54**(2009), 95-102.

[37] X. Zhu, Volterra composition operators from generalized weighted Bergman spaces to μ-Bloch type spaces, *J. Funct. Space Appl.* **7** (2009), 225-240.

[38] X. Zhu, An integral-type operator from H^∞ to Zygmund-type spaces, *Bull. Malaysian Math. Sci. Soc.* to appear.

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