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SOME SYMMETRIC SEMI-CLASSICAL POLYNOMIAL SETS

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Abstract

We show that if v is a regular semi-classical form (linear functional), then the symmetric form u defined by the relation $x\sigma u = -\lambda v$ where σu is the even part of u, is also regular and semi-classical form for every complex λ except for a discrete set of numbers depending on v. We give explicitly the recurrence coefficients, integral representation and the structure relation coefficients of the orthogonal polynomials sequence associated with u and the class of the form u knowing that of v. We conclude with some illustrative examples.

1 Introduction

In many recent papers, different construction processes of semi-classical orthogonal polynomials (O.P) can be done from well known ones, particularly the classical ones. For instance, we can mention the adjunction of a finite number of Dirac's masses and their derivatives to semi-classical forms [2, 7-9], the product and the division of a form by a polynomial [1, 3, 6, 10, 13, 15].

The whole idea of the following work is to build a new construction process of semi-classical form, which has not yet been treated in the literature on semi-classical polynomials. The problem we tackle is as follows.
We study the form u, fulfilling $x\sigma u = -\lambda v, \quad \lambda \neq 0,\ (u)_{2n+1} = 0$, where σu is the even part of u and v is a given semi-classical form.

This paper is organized in sections : The first one is focused on the preliminary results and notations used in the sequel. We will also give the regularity condition and the coefficients of the three-term recurrence relation satisfied by the new family of O.P.. In the second , we compute the exact class of the semi-classical form obtained by the above modification and the structure relation of the O.P. sequence relatively to the form u will follow. In the final section, we apply our results to some examples. The regular forms found in the examples are semi-classical of class

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$\tilde{s} \in \{1,2,3\}$ and we present their integral representations.

Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and let $\mathcal{P}'$ be its dual. We denote by $\langle v, f\rangle$ the action of $v \in \mathcal{P}'$ on $f \in \mathcal{P}$. In particular, we denote by $(v)_n := \langle v, x^n\rangle, n \geq 0$, the moments of v. For any form v and any polynomial h let $Dv = v'$, hv, δ_0, and $(x-c)^{-1}v$ be the forms defined by: $\langle v', f\rangle := -\langle v, f'\rangle$, $\langle hv, f\rangle := \langle v, hf\rangle$, $\langle \delta_c, f\rangle := f(c)$,
and $\big\langle (x-c)^{-1}v, f\big\rangle := \langle v, \theta_c f\rangle$ where $\big(\theta_c f\big)(x) = \dfrac{f(x)-f(c)}{x-c}$, $c \in \mathbb{C}$, $f \in \mathcal{P}$.

Then, it is straightforward to prove that for $f \in \mathcal{P}$ and $v \in \mathcal{P}'$, we have

$$x^{-1}(xv) = v - (v)_0\delta_0 \ , \tag{1}$$

$$(fv)' = f'v + fv' \ . \tag{2}$$

Let us define the operator $\sigma : \mathcal{P} \longrightarrow \mathcal{P}$ by $(\sigma f)(x) := f(x^2)$. Then, we define the even part σv of v by $\langle \sigma v, f\rangle := \langle v, \sigma f\rangle$. Therefore, we have [5, 11]

$$f(x)(\sigma v) = \sigma(f(x^2)v) \ , \tag{3}$$

$$(\sigma v)_n = (v)_{2n} \ , \qquad n \geq 0 \ . \tag{4}$$

The form v will be called regular if there exists a sequence of polynomials $\{S_n\}_{n\geq 0}$ $\big(\deg(S_n) \leq n\ \big)$ such that $\langle v, S_nS_m\rangle = r_n\delta_{n,m}$, $n, m \geq 0$, $r_n \neq 0$, $n \geq 0$.

Then $\deg(S_n) = n$, $n \geq 0$, and we can always suppose each S_n is monic (i.e. $S_n(x) = x^n + \cdots$). The sequence $\{S_n\}_{n\geq 0}$ is said to be orthogonal with respect to v. It is a very well known fact that the sequence $\{S_n\}_{n\geq 0}$ satisfies the recurrence relation $\big($see, for instance, the monograph by Chihara [5]$\big)$

$$\begin{aligned} &S_{n+2}(x) = (x-\xi_{n+1})S_{n+1}(x) - \rho_{n+1}S_n(x) \ , \quad n \geq 0 \ , \\ &S_1(x) = x - \xi_0 \ , \qquad S_0(x) = 1 \ , \end{aligned} \tag{5}$$

with $\big(\xi_n, \rho_{n+1}\big) \in \mathbb{C} \times \mathbb{C} - \{0\}$, $n \geq 0$, by convention we set $\rho_0 = (v)_0 = 1$.

In this case, let $\{S_n^{(1)}\}_{n\geq 0}$ be the associated sequence of first kind for the sequence $\{S_n\}_{n\geq 0}$ satisfying the three-term recurrence relation

$$\begin{aligned} &S_{n+2}^{(1)}(x) = (x-\xi_{n+2})S_{n+1}^{(1)}(x) - \rho_{n+2}S_n^{(1)}(x) \ , \quad n \geq 0, \\ &S_1^{(1)}(x) = x - \xi_1, \qquad S_0^{(1)}(x) = 1 \ , \quad \big(S_{-1}^{(1)}(x) = 0\big) \ , \end{aligned} \tag{6}$$

Also, let $\{S_n(., \mu)\}_{n\geq 0}$ be the co-recursive polynomials for the sequence $\{S_n\}_{n\geq 0}$ satisfying [5]

$$S_n(x, \mu) = S_n(x) - \mu S_{n-1}^{(1)}(x), \quad n \geq 0. \tag{7}$$

A form v is called symmetric if $(v)_{2n+1} = 0, n \geq 0$. The conditions $(v)_{2n+1} = 0, n \geq 0$ are equivalent to the fact that the corresponding monic orthogonal polynomials sequence(MOPS) $\{S_n\}_{n\geq 0}$ satisfies the recurrence relation (5) with $\xi_n = 0, n \geq 0$ [5].

Proposition 1. [5,11] *If the form v is symmetric, then v is regular if and only if σv and $x\sigma v$ are both regular.*

Let v be a regular, normalized form (i.e. $(v)_0 = 1$) and $\{S_n\}_{n\geq 0}$ be its corresponding sequence of monic orthogonal polynomials. For a $\lambda \in \mathbb{C} - \{0\}$, we can define a new symmetric form u as follows

$$x\sigma u = -\lambda v \quad (u)_{2n+1} = 0 \ , \quad (u)_0 = 1 \ , \qquad n \geq 0. \tag{8}$$

From (1), we have

$$\sigma u = -\lambda x^{-1} v + \delta_0 \ . \tag{9}$$

Proposition 2. *The form u is regular if and only if $\lambda \neq \lambda_n, n \geq 0$ where* $\lambda_n = \dfrac{S_n(0)}{S^{(1)}_{n-1}(0)}$.

Proof. Since u is a symmetric form then, according to Proposition 1 u is regular if and only if $x\sigma u$ and σu are regular. But $x\sigma u = -\lambda v$ is regular. So u is regular if and only if $\sigma u = -\lambda x^{-1}\sigma v + \delta_0$ is regular. Or,it was shown in [13] that the form $-\lambda x^{-1} v + \delta_0$ is regular if and only if $\lambda \neq 0$, and $S_n(0,\lambda) \neq 0$, $n \geq 0$. Then, we deduce the desired result. $\square$

Remark. If w is the symmetrized form associated with the form v $\big($ i.e. $(w)_{2n} = (v)_n$ and $(w)_{2n+1} = 0, n \geq 0$ $\big)$, then (8) is equivalent to $x^2 u = -\lambda w$. Notice that w is not necessarily a regular form in the problem under study. In [1, 3], the authors have solved it only when w is regular.

When u is regular let $\{Z_n\}_{n\geq 0}$ be its MOPS satisfying the recurrence relation

$$\begin{aligned} &Z_{n+2}(x) = xZ_{n+1}(x) - \gamma_{n+1} Z_n(x) \ , \quad n \geq 0 \ , \\ &Z_1(x) = x \ , \quad Z_0(x) = 1 \ . \end{aligned} \tag{10}$$

Since $\{Z_n\}_{n\geq 0}$ is symmetric, let us consider its quadratic decomposition [11]:

$$Z_{2n}(x) = P_n(x^2) \ , \quad Z_{2n+1}(x) = xR_n(x^2) \ . \tag{11}$$

$$Z^{(1)}_{2n}(x) = R_n\left(x^2, -\gamma_1\right) \ , \quad Z^{(1)}_{2n+1}(x) = xP^{(1)}_n(x^2) \ . \tag{12}$$

The sequences $\{P_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ are respectively orthogonal with respect to σu and $x\sigma u$.

From (8), we have

$$R_n(x) = S_n(x) \ , \quad n \geq 0 \ . \tag{13}$$

Proposition 3. *We may write*

$$\gamma_1 = -\lambda \ , \quad \gamma_{2n+2} = a_n \ , \quad \gamma_{2n+3} = \frac{\rho_{n+1}}{a_n} \ , \quad n \geq 0 \tag{14}$$

where

$$a_n = -\frac{S_{n+1}(0,\lambda)}{S_n(0,\lambda)} \ , \quad n \geq 0 \ . \tag{15}$$

Proof. Using (8) and the condition $\langle u, Z_2 \rangle = 0$, we obtain $\gamma_1 = -\lambda$.
From (6) and (10) where $n \longrightarrow 2n$ and taking (12)-(13) into account, we get

$$S_{n+1}\left(x^2, -\gamma_1\right) = xZ^{(1)}_{2n+1}(x) - \gamma_{2n+2}S_n\left(x^2, -\gamma_1\right)$$

Substituting x by 0 in the above equation, we obtain $\gamma_{2n+2} = a_n$.
From (10), we have

$$\gamma_{2n+2}\gamma_{2n+3} = \frac{\left\langle u, Z^2_{2n+2}\right\rangle}{\left\langle u, Z^2_{2n+1}\right\rangle}\frac{\left\langle u, Z^2_{2n+3}\right\rangle}{\left\langle u, Z^2_{2n+2}\right\rangle} = \frac{\left\langle u, Z^2_{2n+3}\right\rangle}{\left\langle u, Z^2_{2n+1}\right\rangle} . \tag{16}$$

Using (11) , (8) and (5), equation (16) becomes

$$\gamma_{2n+2}\gamma_{2n+3} = \rho_{n+1}, \tag{17}$$

then, we deduce $\gamma_{2n+3} = \dfrac{\rho_{n+1}}{a_n}$. □

Corollary 1. When the form v is symmetric, then u is regular for every $\lambda \neq 0$. Moreover,

$$\begin{cases} \gamma_1 = -\gamma_2 = -\lambda \\ \gamma_{4n+3} = -\gamma_{4n+4} = -\dfrac{1}{\lambda}\displaystyle\prod_{k=0}^{n}\frac{\rho_{2k+1}}{\rho_{2k}}, \\ \gamma_{4n+5} = -\gamma_{4n+6} = \lambda\rho_{2n+2}\displaystyle\prod_{k=0}^{n}\frac{\rho_{2k}}{\rho_{2k+1}}, n \geq 0 . \end{cases} \tag{18}$$

Proof. Taking into account (5) and (6), with $\xi_n = 0$, we get $S_{n+2}(0) = -\rho_{n+1}S_n(0)$ and $S^{(1)}_{n+2}(0) = -\rho_{n+2}S^{(1)}_n(0)$. Then,

$$S_{2n+1}(0) = 0 , \quad S_{2n+2}(0) = (-1)^{n+1}\prod_{v=0}^{n}\rho_{2v+1} , \quad n \geq 0 , \tag{19}$$

$$S^{(1)}_{2n+1}(0) = 0, \quad S^{(1)}_{2n}(0) = (-1)^n\prod_{v=0}^{n}\rho_{2v} , \quad n \geq 0 . \tag{20}$$

Therefore, $S_{2n+1}(0, \lambda) = -\lambda S^{(1)}_{2n}(0) \neq 0$ and $S_{2n+2}(0, \lambda) = S_{2n+2}(0) \neq 0$. Hence u is regular for every $\lambda \neq 0$ according to proposition 2.
By virtue of (19)-(20), (14) becomes (18). □

We suppose that the form v has the following integral representation:

$$\langle v, f\rangle = \int_{-\infty}^{+\infty} V(x)f(x)dx , \quad f \in \mathcal{P}, \quad \text{with } (v)_0 = \int_{-\infty}^{+\infty} V(x)dx = 1$$

where V is a locally integrable function with rapid decay and continuous at the origin.
It is obvious that $f(x) = f^e(x^2) + xf^o(x^2), f \in \mathcal{P}$.

Therefore, $\langle u, f\rangle = \langle u, f^e(x^2)\rangle = \langle \sigma u, f^e(x)\rangle$ since u is symmetric.
Using (8) and taking into account that $f^e(0) = f(0)$, we obtain

$$\langle u, f\rangle = f(0)\left\{1 + \lambda P\int_{-\infty}^{+\infty} \frac{V(x)}{x}dx\right\} - \lambda P\int_{-\infty}^{+\infty} \frac{V(x)}{x}f^e(x)dx\ , \tag{21}$$

where

$$P\int_{-\infty}^{+\infty} \frac{V(x)}{x}f(x)dx = \lim_{\epsilon\longrightarrow 0}\left\{\int_{-\infty}^{-\epsilon} \frac{V(x)}{x}f(x)dx + \int_{\epsilon}^{+\infty} \frac{V(x)}{x}f(x)dx\right\} .$$

It is easy to see that

$$P\int_{-\infty}^{+\infty} \frac{V(x)}{x}f(x)dx = \lim_{\epsilon\longrightarrow 0}\left\{\int_{\epsilon}^{+\infty} \frac{V(x)}{x}f^e(x)dx - \int_{\epsilon}^{+\infty} \frac{V(-x)}{x}f^e(-x)dx\right\} .$$

Using the fact that $f^e(x) = \dfrac{f(\sqrt{x}) + f(-\sqrt{x})}{2}$ and $f^e(-x) = \dfrac{f(i\sqrt{x}) + f(-i\sqrt{x})}{2}$ for $x \geq 0$ and making the change of variables $t = \sqrt{x}$, we get

$$\begin{aligned} P\int_{-\infty}^{+\infty} \frac{V(x)}{x}f(x)dx = -\lim_{\epsilon\longrightarrow 0}\int_{\sqrt{\epsilon}}^{+\infty} \frac{V(-t^2)}{t}(f(it) + f(-it))dt + \\ + \lim_{\epsilon\longrightarrow 0}\int_{\sqrt{\epsilon}}^{+\infty} \frac{V(t^2)}{t}(f(t) + f(-t))dt\ . \end{aligned}$$

Inserting the last equation into (21), we get after a change variables in the obtained equation

$$\begin{aligned} \langle u, f\rangle = f(0)\left\{1 + \lambda P\int_{-\infty}^{+\infty} \frac{V(x)}{x}dx\right\} + \\ + \lambda P\int_{-\infty}^{+\infty} \frac{V(-x^2)}{|x|}f(ix)dx - \lambda P\int_{-\infty}^{+\infty} \frac{V(x^2)}{|x|}f(x)dx\ . \end{aligned} \tag{22}$$

Remark. When v is symmetric, (22) becomes

$$\langle u, f\rangle = f(0) - \lambda P\int_{-\infty}^{+\infty} \frac{V(x^2)}{|x|}(f(x) - f(ix))dx\ . \tag{23}$$

Our aim is to give examples of semi-classical forms (8) through data of semi-classical form v.

2 The semi-classical case

Let us recall that a form v is called semi-classical when it is regular and there exist two polynomials Φ and Ψ such that:

$$(\Phi v)' + \Psi v = 0\ ,\quad \deg(\Psi) \geq 1\ ,\quad \Phi\ \textit{monic.} \tag{24}$$

The class of the semi-classical form v is $s = \max\big(\deg \Psi - 1, \deg \Phi - 2\big)$ if and only if the following condition is satisfied

$$\prod_c \left(|\Phi'(c) + \Psi(c)| + \left|\left\langle u, \theta_c \Psi + \theta_c^2 \Phi \right\rangle\right|\,\right) > 0\ , \tag{25}$$

where c goes over the roots set of Φ [12].

The corresponding orthogonal sequence $\{S_n\}_{n\geq 0}$ is also called semi-classical of class s.

We can state characterizations of semi-classical orthogonal sequences. $\{S_n\}_{n\geq 0}$ is semi-classical of class s if and only if one of the following statements holds:

(a) The formal Stieltjes function of v, namely

$$S\big(v\big)(z) = -\sum_{n\geq 0} \frac{(v)_n}{z^{n+1}} \tag{26}$$

satisfies a linear non-homogeneous first order differential equation [4,12]

$$\Phi(z) S'\big(v\big)(z) = C_0(z) S\big(v\big)(z) + D_0(z), \tag{27}$$

where

$$C_0(x) = -\Phi'(x) - \Psi(x). \tag{28}$$

and

$$D_0(z) = -\big(v\theta_0 \Phi\big)'(x) - \big(v\theta_0 \Psi\big)(x). \tag{29}$$

with $\quad (v\theta_0 f)\,(x) = \left\langle v, \dfrac{f(x) - f(\zeta)}{x - \zeta} \right\rangle$, $f \in \mathcal{P}$. Φ and Ψ are the same polynomials as in (24).

(b) $\{S_n\}_{n\geq 0}$ fulfills the following structure recurrence relation (written in a compact form):

$$\Phi(x) S'_{n+1}(x) = \frac{C_{n+1}(x) - C_0(x)}{2} S_{n+1}(x) - \rho_{n+1} D_{n+1}(x) S_n(x)\ ,\ n \geq 0 \tag{30}$$

where

$$\begin{cases} C_{n+1}(x) = -C_n(x) + 2(x - \beta_n) D_n(x)\ , \quad n \geq 0\ , \\ \rho_{n+1} D_{n+1}(x) = -\Phi(x) + \rho_n \tilde{D}_{n-1}(x) - (x - \xi_n) C_n(x) + \\ \qquad\qquad (x - \xi_n)^2 D_n(x)\ ,\ n \geq 0\ , \end{cases} \tag{31}$$

Φ, Ψ, C_0 and D_0 are the same polynomials introduced in (a); ξ_n, ρ_n are the coefficients of the three term recurrence relation (5). Notice that $D_{-1}(x) = 0, \deg C_n \leq s + 1$ and $\deg D_n \leq s, n \geq 0$ [12].

(c) Each polynomial of $\{S_n\}_{n\geq 0}$ satisfies a second order differential equation of Laguerre-Perron type, i.e.

$$\begin{aligned} &\Phi D_{n+1} S''_{n+1} + \{C_0 D_{n+1} - W(\Phi, D_{n+1})\} S'_{n+1} + \\ &+ \left\{ W\left(\frac{C_{n+1} - C_0}{2}, D_{n+1}\right) - D_{n+1} \sum_{k=0}^{n} D_k \right\} S_{n+1} = 0\ , \quad n \geq 0\ , \end{aligned} \tag{32}$$

where $W(f,g) = fg' - f'g$. Φ, D_n, C_n, $n \geq 0$ are the same parameters introduced in the previous characterizations [4,14].

Remark. The structure relation gives information about the multiplicity of the zeros of orthogonal polynomials.

In the sequel the form v will be supposed semi-classical of class s satisfying (24) − (25).

Proposition 4. *If v is a semi-classical form and satisfies* (24), *then for every $\lambda \in \mathbb{C} - \{0\}$ such that $S_n(0,\lambda) \neq 0, n \geq 0$, the form u defined by* (8) *is regular and semi-classical. It satisfies*

$$\left(\tilde{\Phi}u\right)' + \tilde{\Psi}u = 0 \tag{33}$$

with

$$\tilde{\Phi}(x) = x\Phi(x^2)\ , \quad \tilde{\Psi}(x) = 2x^2\Psi(x^2)\ . \tag{34}$$

and u is of class $\tilde{s}$ with $\tilde{s} \leq 2s+3$.

Proof. Assume that v fulfils (24). To prove that u satisfies (33)-(34), we will show that the forms $(\tilde{\Phi}u)'$ and $-\tilde{\Psi}u$ coincide on the basis $\{x^n\}_{n\geq 0}$ of $\mathcal{P}$.
Taking into account (34) and using the operator σ, we obtain

$$\left\langle (\tilde{\Phi}u)', x^{2n} \right\rangle = -2n \left\langle \Phi(x^2)u, x^{2n} \right\rangle = -2n \left\langle \Phi(x)\sigma u, x^n \right\rangle, n \geq 1.$$

By virtue of (8) and (24), we deduce

$$\left\langle (\tilde{\Phi}u)', x^{2n} \right\rangle = -2\lambda \left\langle (\Phi(x)v)', x^n \right\rangle = 2\lambda \left\langle \Psi(x)v, x^n \right\rangle .$$

Now, using (8) again and the definition of the operator σ, we get

$$\left\langle (\tilde{\Phi}u)', x^{2n} \right\rangle = -\left\langle \tilde{\Psi}u, x^{2n} \right\rangle .$$

Since u is symmetric, it is clear that $\left\langle (\tilde{\Phi}u)', x^{2n+1} \right\rangle = -\left\langle \tilde{\Psi}u, x^{2n+1} \right\rangle = 0$.

Thus, (33)-(34) is proved.

Finally, we have $s = \max\big(\deg\Psi - 1, \deg\Phi - 2\big)$, then $\deg(\tilde{\Phi}) \leq 2s+5$ and $\deg(\tilde{\Psi}) = \tilde{p} \leq 2s+4$. Thus $\tilde{s} \leq 2s+3$. □

Proposition 5. *The class of u depends only on the zero $x=0$.*

For the proof, we use the following lemma:

Lemma 1. For $c \in \mathbb{C}$ such that c^2 be a root of Φ, we have

$$\left\langle u, \theta_c\tilde{\Psi} + \theta_c^2\tilde{\Phi} \right\rangle = -2c\lambda \left\langle v, \theta_{c^2}\Psi + \theta_{c^2}^2\Phi \right\rangle + 2c\big(\Phi'(c^2) + \Psi(c^2)\big) \tag{35}$$

and

$$\tilde{\Psi}(c) + \tilde{\Phi}'(c) = 2c^2\big(\Phi'(c^2) + \Psi(c^2)\big). \tag{36}$$

Proof. Using the definition of the operator θ_c, it is easy to prove that, for two polynomials f and g, we have

$$\theta_c\big(fg\big)(x) = g(x)\big(\theta_c f\big)(x) + f(c)\big(\theta_c g\big)(x)\ , \tag{37}$$

$$\theta_c\big(f(\xi^2)\big)(x) = (x+c)\big(\theta_{c^2} f\big)(x^2)\ . \tag{38}$$

Let $c \in \mathbb{C}$ such that c^2 be a root of Φ.

Using successively (37) and (38), we obtain

$\left(\theta_c\tilde{\Phi}\right)(x) = x\left(\theta_c \Phi(\xi^2)\right)(x) = x(x+c)\left(\theta_{c^2}\Phi\right)(x^2),\quad$ since $\Phi(c^2) = 0$. Then,

$$\left(\theta_c^2\tilde{\Phi}\right)(x) = x(x+c)^2\left(\theta_{c^2}^2\Phi\right)(x^2) + (x+2c)\Phi'(c^2)\ , \tag{39}$$

because $\theta_c\big(\xi(\xi+c)\big)(x) = x + 2c$, $\theta_c\left((\theta_{c^2}\Phi)(\xi^2)\right)(x) = (x+c)\left(\theta_{c^2}^2\Phi\right)(x^2)$ and $\big(\theta_{c^2}\Phi\big)(c^2) = \Phi'(c^2)$.

Using the same procedure, we prove that

$$\theta_c\tilde{\Psi}(x) = x^2(x+c)\big(\theta_{c^2}\Psi\big)(x^2) + (x+c)\Psi(c^2)\ . \tag{40}$$

Therefore, with (39)-(40) and the fact u is symmetric, we obtain

$$\left\langle u, \theta_c\tilde{\Psi} + \theta_c^2\tilde{\Phi}\right\rangle = \left\langle x^2 u, 2\theta_{c^2}\Psi + \theta_{c^2}^2\Phi\right\rangle + 2c\big(\Phi'(c^2) + \Psi(c^2)\big)\ . \tag{41}$$

Now applying the operator σ for (41) and using (8), we get (35). Finally, from (34), we easily get (36). $\square$

Proof of Proposition 5. Let c be a root of $\tilde{\Phi}$ such that $c \neq 0$.
If $\Phi'(c^2) + \Psi(c^2) \neq 0$ then $\tilde{\Phi}'(c) + \tilde{\Psi}(c) \neq 0$, from (36).
If $\Phi'(c^2) + \Psi(c^2) = 0$, using (35), we have $\left\langle u, \theta_c\Psi + \theta_c^2\Phi\right\rangle \neq 0$, since v is semi-classical and so satisfies (25).
In any case, we cannot simplify by $x - c$. $\square$

Proposition 6. *Under the conditions of proposition* 4, *for the class of* u, *we have the four different cases*

1) $\tilde{s} = 2s + 3 \quad$ *if* $\quad \Phi(0) \neq 0$.

2) $\tilde{s} = 2s + 2 \quad$ *if* $\quad \Phi(0) = 0$ and $X_1 = -2\lambda\left\langle v, \theta_0\Psi + \theta_0^2\Phi\right\rangle + 2\big(\Phi'(0) + \Psi(0)\big) \neq 0$.

3) $\tilde{s} = 2s + 1 \quad$ *if* $\quad \Phi(0) = 0$, $X_1 = 0$ and $X_2 = 3\Phi'(0) + 2\Psi(0) \neq 0$.

4) $\tilde{s} = 2s \quad$ *if* $\quad \Phi(0) = 0$, $X_1 = 0$ and $X_2 = 0$.

Proof. 1) From (34), we have $\tilde{\Phi}'(0) + \tilde{\Psi}(0) = \Phi(0)$ and $\left\langle u, \theta_0\tilde{\Psi} + \theta_0^2\tilde{\Phi}\right\rangle = \left\langle u, 2x\Psi(x^2) + x(\theta_0\Phi)(x^2)\right\rangle = 0$, since u is symmetric. Therefore, if $\Phi(0) \neq 0$ it is not possible to simplify (33)-(34), which means that the class of u is $\tilde{s} = 2s + 3$.

2) If $\Phi(0)=0$, then it is possible to simplify by x. Then, u fulfils (33) with

$$\tilde{\Phi}(x)=\Phi(x^2)\ ,\quad \tilde{\Psi}(x)=x\big((\theta_0\Phi)(x^2)+2\Psi(x^2)\big)\ . \tag{42}$$

Here, we have $\tilde{\Phi}'(0)+\tilde{\Psi}(0)=0$ and $\left\langle u,\theta_0\tilde{\Psi}+\theta_0^2\tilde{\Phi}\right\rangle=\left\langle u,2\Psi(x^2)+2(\theta_0\Phi)(x^2)\right\rangle$.
Applying the operator σ for the second equation and using (9), we obtain $\left\langle u,\theta_0\tilde{\Psi}+\theta_0^2\tilde{\Phi}\right\rangle=-2\lambda\left\langle v,\theta_0\Psi+\theta_0^2\Phi\right\rangle+2\big(\Phi'(0)+\Psi(0)\big)=X_1$.
Therefore, if $X_1\neq 0$ it is not possible to simplify, which means that the class of u is $\tilde{s}=2s+2$.
3) If $\Phi(0)=0$ and $X_1=0$, then it is possible to simplify (33)-(34) by x^2. Then, u fulfils (33) with

$$\tilde{\Phi}(x)=x(\theta_0\Phi)(x^2)\ ,\quad \tilde{\Psi}(x)=2\big((\theta_0\Phi)(x^2)+\Psi(x^2)\big)\ . \tag{43}$$

Here, we have $\tilde{\Phi}'(0)+\tilde{\Psi}(0)=3\Phi'(0)+2\Psi(0)=X_2$ and $\left\langle u,\theta_0\tilde{\Psi}+\theta_0^2\tilde{\Phi}\right\rangle=\left\langle u,x\left(2\left(\theta_0\Psi\right)(x^2)+\left(\theta_0^2\Phi\right)(x^2)\right)\right\rangle=0$, since u is symmetric. Therefore, if $X_2\neq 0$ it is not possible to simplify, which means that the class of u is $\tilde{s}=2s+1$.
4) If $\Phi(0)=0$, $X_1=0$ and $X_2=0$, then it is possible to simplify (33)-(34) by x^3. Then, u fulfils (33) with

$$\tilde{\Phi}(x)=(\theta_0\Phi)(x^2)\ ,\quad \tilde{\Psi}(x)=x\big(3(\theta_0^2\Phi)(x^2)+2(\theta_0\Psi)(x^2)\big)\ . \tag{44}$$

Under these conditions $x=0$ can't be a root of $(\theta_0\Phi)(x^2)$. Assuming the contrary, that $(\theta_0\Phi)(0)=\Phi'(0)=0$, then from the conditions $\Phi(0)=0$, $X_1=0$ and $X_2=0$ we obtain $\left\langle v,\theta_0\Psi+\theta_0^2\Phi\right\rangle=0$ and $\Phi'(0)+\Psi(0)=0$ which is a contradiction with (25). Then it is not possible to simplify, which means that the class of u is $\tilde{s}=2s$.
□

Proposition 7. *If v is a semi-classical form and satisfies* (27)*, then for every $\lambda\in\mathbb{C}-\{0\}$ such that $S_n(0,\lambda)\neq 0, n\geq 0$, the form u defined by* (8) *is regular and semi-classical. It satisfies*

$$\tilde{\Phi}(z)S'\big(u\big)(z)=\tilde{C}_0(z)S\big(u\big)(z)+\tilde{D}_0(z), \tag{45}$$

where

$$\begin{cases}\tilde{\Phi}(z)=z\Phi(z^2),\\ \tilde{C}_0(z)=-\Phi(z^2)+2z^2C_0(z^2),\\ \tilde{D}_0(z)=-2z\lambda D_0(z^2)+2zC_0(z^2).\end{cases} \tag{46}$$

Proof. From (26), we have

$$S\big(v\big)(z^2)=-\sum_{n\geq 0}\frac{(v)_n}{z^{2n+2}}.$$

Using (8), we get

$$-\lambda S\big(v\big)(z^2)=zS\big(u\big)(z)+1, \tag{47}$$

Deriving (47), we obtain

$$-2z\lambda S'\big(v\big)(z^2) = zS'\big(u\big)(z) + S\big(u\big)(z). \tag{48}$$

Make a change of variable $z \longrightarrow z^2$ in (27) and multiply by $-2\lambda z$, we obtain (45)-(46) by taking into account (47)-(48). □

We are going to establish the expression of structure relation coefficients $\tilde{C}_n$ and $\tilde{D}_n$, $n \geq 0$ of $\{Z_n\}_{n\geq 0}$ in terms of those of the sequence $\{S_n\}_{n\geq 0}$.

Proposition 8. *The sequence $\{Z_n\}_{n\geq 0}$ fulfills*

$$\tilde{\Phi}(x)Z'_{n+1}(x) = \frac{\tilde{C}_{n+1}(x) - \tilde{C}_0(x)}{2} Z_{n+1}(x) - \gamma_{n+1}\tilde{D}_{n+1}(x)Z_n(x)\ ,\ n \geq 0 \tag{49}$$

with

$$\begin{cases} \tilde{C}_{2n+1}(x) = 2x^2C_n(x^2) + \Phi(x^2) + 4\gamma_{2n+1}x^2D_n(x^2),\ n \geq 0\ , \\ \tilde{D}_{2n+1}(x) = 2x^3D_n(x^2),\ n \geq 0\ , \end{cases} \tag{50}$$

$$\begin{cases} \tilde{C}_{2n+2}(x) = 2x^2C_{n+1}(x^2) - \Phi(x^2) + 4\gamma_{2n+2}x^2D_n(x^2),\ n \geq 0\ , \\ \tilde{D}_{2n+2}(x) = x\left(C_{n+1}(x^2) - C_n(x^2)\right) + 2x\big(\gamma_{2n+3}D_{n+1}(x^2) - \\ \qquad -\gamma_{2n+1}D_n(x^2)\big) + 2x^3D_n(x^2),\ n \geq 0\ , \end{cases} \tag{51}$$

$\tilde{C}_0(x)$ and $\tilde{D}_0(x)$ are given by (46) and γ_{n+1} by (14)-(15).

Proof. Change $x \longrightarrow x^2$ in (29) and multiply by $2x^3$ we obtain by taking (11) and (13) into account,

$$x\Phi(x^2)Z'_{2n+3}(x) = \left(x^2\left(C_{n+1}(x^2) - C_0(x^2)\right) + \Phi(x^2)\right)Z_{2n+3}(x) - \\ -2x^2D_{n+1}(x^2)\rho_{n+1}Z_{2n+1}(x).$$

Using (16) and (10) where $n \longrightarrow 2n$, the last equation becomes

$$\tilde{\Phi}(x)Z'_{2n+3}(x) = \left(x^2\left(C_{n+1}(x^2) - C_0(x^2)\right) + \Phi(x^2) + 2x^2\gamma_{2n+3}D_{n+1}(x^2)\right)Z_{2n+3}(x) - \\ -2\gamma_{2n+3}x^2D_{n+1}(x^2)Z_{2n+2}(x).$$

From (49) and the above equation, we have

$$\left\{\frac{\tilde{C}_{2n+3}(x) - \tilde{C}_0(x)}{2} - \left(x^2\left(C_{n+1}(x^2) - C_0(x^2)\right) + \Phi(x^2) + 2x^2\gamma_{2n+3}D_{n+1}(x^2)\right)\right\} \times \\ \times Z_{2n+3}(x) = \gamma_{2n+3}\left\{\tilde{D}_{2n+3}(x) - 2x^2D_{n+1}(x^2)\right\}Z_{2n+2}(x).$$

Z_{2n+3} and Z_{2n+2} have no common roots, then Z_{2n+3} divides $\tilde{D}_{2n+3}(x) - 2x^2D_{n+1}(x^2)$, which is a polynomial of degree at most equal to $2s+3$. Then we have necessarily $\tilde{D}_{2n+3}(x) = 2x^2D_{n+1}(x^2)$ for $n > s$, and also $\dfrac{\tilde{C}_{2n+3}(x) - \tilde{C}_0(x)}{2} = x^2\left(C_{n+1}(x^2) - C_0(x^2)\right) +$ $\Phi(x^2) + 2x^2\gamma_{2n+3}D_{n+1}(x^2), n > s$.

Then, by (45), we get (49) for $n > s$.

By virtue of the recurrence relation (30) and (46), we can easily prove by induction

that the system (50) is valid for $0 \le n \le s$. Hence (50) is valid for $n \ge 0$.
After a derivation of (10) where $n \longrightarrow 2n+1$ multiplying by $x\Phi(x^2)$ and using (49), we obtain

$$x^2\Phi(x^2)Z'_{2n+2}(x) = \frac{\tilde{C}_{2n+3}(x) - \tilde{C}_0(x)}{2} Z_{2n+3}(x) - \gamma_{2n+3}\tilde{D}_{2n+3}(x)Z_{2n+2}(x) - x\Phi(x^2)Z_{2n+2}(x) + \gamma_{2n+2}\left\{\frac{\tilde{C}_{2n+1}(x) - \tilde{C}_0(x)}{2} Z_{2n+1}(x) - \gamma_{2n+1}\tilde{D}_{2n+1}(x)Z_{2n}(x)\right\}.$$

Applying the recurrence relation (10), we get

$$x^2\Phi(x^2)Z'_{2n+2}(x) = \left\{x\frac{\tilde{C}_{2n+3}(x) - \tilde{C}_0(x)}{2} - \gamma_{2n+3}\tilde{D}_{2n+3}(x) - x\Phi(x^2) + \gamma_{2n+2}\tilde{D}_{2n+1}(x)\right\} \times \times Z_{2n+2}(x) - \gamma_{2n+2}\left\{\frac{\tilde{C}_{2n+3}(x) - \tilde{C}_{2n+1}(x)}{2} + x\tilde{D}_{2n+1}(x)\right\} Z_{2n+1}.$$

Now, using (49) and taking into account the fact that $Z_{2n+2}(x)$ and $Z_{2n+1}(x)$ are coprime, we get from the last equation after simplification by x (51) for $n > s$. Finally, by virtue of the recurrence relation (30) and (50) with $n = 0$, we can easily prove by induction that the system (51) is valid for $0 \le n \le s$. Hence (51) is also proved for $n \ge 0$. □

Using (32), Proposition 8. and simplifying, we get the following result:

Corollary 2. Each polynomial of $\{Z_n\}_{n\ge 0}$ satisfies a second order differential equation of Laguerre-type, (or holonomic second order differential equation)

$$J(x,n)Z''_{n+1}(x) + K(x,n)Z'_{n+1}(x) + L(x,n)Z_{n+1}(x) = 0, \quad n \ge 1,$$

with

$$\begin{cases} J(x,2n+1) = \Phi(x^2)\big\{x\left(C_{n+1}(x^2) - C_n(x^2)\right) + 2x\big(\gamma_{2n+3}D_{n+1}(x^2) - \gamma_{2n+1}D_n(x^2)\big) + 2x^3D_n(x^2)\big\} \\ K(x,2n+1) = 2x^2\left(\Phi'(x^2) + C_0(x^2)\right)\big\{C_{n+1}(x^2) - C_n(x^2) + 2\big(\gamma_{2n+3}D_{n+1}(x^2) - \gamma_{2n+1}D_n(x^2)\big) + \\ \qquad +2x^2D_n(x^2)\big\} + \Phi(x^2)\big\{C_{n+1}(x^2) - C_n(x^2) + 2x^2\left(C'_{n+1}(x^2) - C'_n(x^2)\right) + \\ \qquad +2\big(\gamma_{2n+3}D_{n+1}(x^2) - \gamma_{2n+1}D_n(x^2)\big) + 4x^2 2\big(\gamma_{2n+3}D'_{n+1}(x^2) - \gamma_{2n+1}D'_n(x^2)\big) + \\ \qquad +6x^2D_n(x^2) + 4x^4D'_n(x^2)\big\}, \\ L(x,2n+1) = x\big\{C_{n+1}(x^2) + 2\gamma_{2n+2}D_n(x^2) - C_0(x^2)\big\}\big\{C_n(x^2) - C_{n+1}(x^2) - 2\big(\gamma_{2n+3}D_{n+1}(x^2) - \\ \qquad -\gamma_{2n+1}D_n(x^2)\big) + 2x^2D_n(x^2) + 2x^2\left(C'_{n+1}(x^2) - C'_n(x^2)\right) + 4x^2\big(\gamma_{2n+3}D'_{n+1}(x^2) - \\ \qquad -\gamma_{2n+1}D'_n(x^2)\big) + 4x^4D'_n(x^2)\big\} - x\big\{C_{n+1}(x^2) - C_n(x^2) + 2\gamma_{2n+3}D_{n+1}(x^2) - \\ \qquad -2\gamma_{2n+1}D_n(x^2) + 2x^2D_n(x^2)\big\}\big\{2x^2C'_{n+1}(x^2) + 4\gamma_{2n+2}x^2D'_n(x^2) - 2x^2C'_0(x^2) + \\ \qquad +C_n(x^2) + C_0(x^2) + 2\gamma_{2n+1}D_n(x^2) - 2\lambda D_0(x^2) + 4x^2\sum_{k=0}^{n-1}D_k(x^2) + 2x^2D_n(x^2)\big\}. \end{cases}$$

and

$$\begin{cases} J(x,2n) = 2x^3\Phi(x^2)D_n(x^2), \\ K(x,2n) = 2x^2D_n(x^2)\big(2x^2\Phi'(x^2) + 2x^2C_0(x^2) - 3\Phi(x)\big) - 4x^4\Phi(x^2)D'_n(x^2), \\ L(x,2n) = 2xD_n(x^2)\big(3\Phi(x^2) - 2x^2C_0(x^2) - 2x^4C'_n(x^2) - 2x^2\Phi'(x^2) + 2x^4C'_0(x^2) + \\ \qquad +2\lambda x^2D_0(x^2) - 4x^4\sum_{k=0}^{n-1}D_k(x^2)\big) + 4x^3D'_n(x^2)\big(x^2C_n(x^2) - x^2C_0(x^2) + \Phi(x^2)\big). \end{cases}$$

3 Illustrative examples

(1) We study the problem (8), with $v = \mathcal{L}(\alpha)$ where $\mathcal{L}(\alpha)$ is the Laguerre form. In this case, the form v is not symmetric. This form is classical (semi-classical of class

$s=0$). We have [12]

$$\xi_n = 2n+\alpha+1, \quad \rho_{n+1} = (n+1)(n+\alpha+1), \quad n \geq 0, \tag{52}$$

the regularity condition is $\quad \alpha \neq -n,\ n \geq 1$

$$\Phi(x) = x, \quad \Psi(x) = x-\alpha-1, \tag{53}$$

$$C_n(x) = -x+(2n+\alpha), \quad D_n(x) = -1, \quad n \geq 0. \tag{54}$$

Using (5) and (52), we get

$$S_n(0) = (-1)^n \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)}\ , \quad n \geq 0\ . \tag{55}$$

From (6) and (52), we obtain by induction for $n \geq 0$

$$S_n^{(1)}(0) = \begin{cases} \dfrac{(-1)^{n+1}}{\alpha}\left(\Gamma(n+2) - \dfrac{\Gamma(n+\alpha+2)}{\Gamma(\alpha+1)}\right)\ , & \alpha \neq 0, \\ (-1)^n \Gamma(n+2) \displaystyle\sum_{k=0}^{n} \frac{1}{k+1}\ , & \alpha = 0\ . \end{cases} \tag{56}$$

By virtue of (7) and (55)-(56), we deduce

$$S_n(0,\lambda) = \frac{(-1)^n \Gamma(n+\alpha+1) d_{\alpha,n}}{\alpha \Gamma(\alpha+1)}\ , \quad n \geq 0 \tag{57}$$

where

$$d_{\alpha,n} = \begin{cases} (\alpha+\lambda) - \dfrac{\lambda \Gamma(\alpha+1)\Gamma(n+1)}{\Gamma(n+\alpha+1)}\ , & \alpha \neq 0\ , \quad n \geq 0\ , \\ 1 + \lambda \displaystyle\sum_{k=0}^{n-1} \frac{1}{k+1}\ , \quad \alpha = 0\ , \quad n \geq 0\ . & \left(\displaystyle\sum_{0}^{-1} = 0\right) \end{cases} \tag{58}$$

Then, u is regular for every $\lambda \neq 0$ such that

$$\lambda \neq \begin{cases} -\alpha + \dfrac{\lambda \Gamma(\alpha+1)\Gamma(n+1)}{\Gamma(n+\alpha+1)}\ , & \alpha \neq 0\ , \quad n \geq 0\ , \\ -\left(\displaystyle\sum_{k=0}^{n-1} \frac{1}{k+1}\right)^{-1}\ , & \alpha = 0\ , \quad n \geq 1\ . \end{cases} \tag{59}$$

(15) and (57) give

$$a_n = \frac{(n+\alpha+1) d_{\alpha,n+1}}{d_{\alpha,n}}\ , \quad n \geq 0\ . \tag{60}$$

Then, with (14), we get

$$\begin{cases} \gamma_1 \quad = -\lambda, \\ \gamma_{2n+3} = \dfrac{(n+1) d_{\alpha,n}}{d_{\alpha,n+1}}\ , \quad n \geq 0\ , \\ \gamma_{2n+2} = \dfrac{(n+\alpha+1) d_{\alpha,n+1}}{d_{\alpha,n}}\ , \quad n \geq 0\ . \end{cases} \tag{61}$$

Taking into account that the form v is semi-classical and by virtue of Proposition 4., the form u is also semi-classical. It satisfies (33) and (45)with

$$\begin{array}{l}\tilde{\Phi}(x)=x^2\ ,\quad \tilde{\Psi}(x)=2x^3-(2\alpha+1)x,\\ \tilde{C}_0(x)=-2x^3+(2\alpha-1)x\ ,\quad \tilde{D}_0(x)=-2x^2+2(\alpha+\lambda).\end{array} \tag{62}$$

From (53), we have

$\Phi(0)=0, X_1=-2(\alpha+\lambda)$ and $X_2=1-2\alpha$ (we take $\lambda=-\alpha$ in calculation of X_2). Now, it is enough to use Proposition 6. in order to obtain the following results:

⋆ If $\lambda\neq-\alpha$ and verifies (59), then the class of u is $\tilde{s}=2$.

⋆ If $\lambda=-\alpha$ and $2\alpha\neq1$, then the class of u is $\tilde{s}=1$.

⋆ If $\lambda=-\alpha$ and $2\alpha=1$, then the class of u is $\tilde{s}=0$.

Now, we are going to give the elements of the structure relation of the sequence $\{Z_n\}_{n\geq0}$.
Using (53), (54) and Proposition 8., we obtain after simplifying by x

$$\begin{cases} \tilde{C}_0(x) \quad =-2x^3+(2\alpha-1)x\ ,\quad C_1(x)=-2x^3+(2\alpha+4\lambda+1)x\ ,\\ \tilde{C}_{2n+2}(x)=-2x^3-X_n\ ,\quad \tilde{C}_{2n+3}(x)=-2x^3+X_{n+1}\ ,\\ \tilde{D}_0(x) \quad =-2x^2+2(\alpha+\lambda)\ ,\quad \tilde{D}_{2n+1}(x)=-2x^2\ ,\\ \tilde{D}_{2n+2}(x)=-2x^2-\dfrac{2\left(\alpha^2+\delta_{0,\alpha}\right)(\alpha+\lambda)\lambda\Gamma(\alpha+1)\Gamma(n+1)}{\Gamma(n+\alpha+2)d_{\alpha,n}d_{\alpha,n+1}}\ ,n\geq0.\end{cases} \tag{63}$$

where $X_n=\left(2\alpha+1+\dfrac{4\lambda\left(\alpha+\delta_{0,\alpha}\right)\Gamma(\alpha+1)\Gamma(n+1)}{\Gamma(n+\alpha+1)d_{\alpha,n}}\right)x$.

The form v has the following integral representation[5]

$$\langle v,f\rangle=\frac{1}{\Gamma(\alpha+1)}\int_0^{+\infty}x^\alpha e^{-x}f(x)dx,\ \ \Re(\alpha)>-1,\ \ f\in\mathcal{P}. \tag{64}$$

Then, using (22), we obtain the following integral representation of u

$$\langle u,f\rangle=\left(1+\frac{\lambda}{\alpha}\right)f(0)-\frac{\lambda}{\Gamma(\alpha+1)}\int_{-\infty}^{+\infty}|x|^{2\alpha-1}e^{-x^2}f(x)dx\ ,\Re(\alpha)>0\ . \tag{65}$$

(2) Let us describe the case $v:=\mathcal{H}$ where $\mathcal{H}$ denotes the Hermite form . In this case, the form v is symmetric. This form is classical (semi-classical of class $s=0$). Here [12]

$$\xi_n=0,\quad \rho_{n+1}=\frac{1}{2}(n+1),\quad n\geq0, \tag{66}$$

$$\Phi(x)=1,\quad \Psi(x)=2x, \tag{67}$$

$$C_n(x)=-2x,\quad D_n(x)=-2,\quad n\geq0. \tag{68}$$

In accordance with Corollary 1. and (66), u is regular for every $\lambda \neq 0$ and we have

$$\begin{cases} \gamma_1 = -\gamma_2 = \lambda \\ \gamma_{4n+3} = -\gamma_{4n+4} = -\dfrac{1}{\lambda}\dfrac{\Gamma(2n+2)}{2^{2n+1}\Gamma^2(n+1)}, \\ \gamma_{4n+5} = -\gamma_{4n+6} = \lambda\dfrac{2^{2n+1}\Gamma(n+1)\Gamma(n+2)}{\Gamma(2n+2)}, n \geq 0 \end{cases} \tag{69}$$

By virtue of Proposition 6. and Proposition 7., the form u is semi-classical of class $\tilde{s} = 3$ for any $\lambda \neq 0$ and fulfils (33) and (45) with

$$\tilde{\Phi}(x) = x, \quad \tilde{\Psi}(x) = 4x^3, \quad \tilde{C}_0(x) = -4x^4 - 1, \quad \tilde{D}_0(x) = -4x^3 + 4\lambda x. \tag{70}$$

According to Proposition 8., (67)and (68), we have, for $n \geq 0$

$$\begin{cases} \tilde{C}_0(x) \quad = -4x^4 - 1 \,, \\ \tilde{C}_{2n+1}(x) = -4x^4 - 8\gamma_{2n+1}x^2 + 1 \,, \\ \tilde{C}_{2n+2}(x) = -4x^4 - 8\gamma_{2n+2}x^2 - 1 \,, \\ \tilde{D}_0(x) \quad = -4x^3 + 4\lambda x \,, \\ \tilde{D}_{2n+1}(x) = -4x^3 \,, \\ \tilde{D}_{2n+2}(x) = -4x^3 + 4(\gamma_{2n+1} - \gamma_{2n+3})x \,. \end{cases} \tag{71}$$

The form v has the following integral representation[5]

$$\langle v, f \rangle = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-x^2} f(x)dx \,, \quad f \in \mathcal{P} \,. \tag{72}$$

Therefore, for $\lambda \neq 0$ and $f \in \mathcal{P}$, (23) becomes

$$\langle u, f \rangle = f(0) - \frac{\lambda}{\sqrt{\pi}} P \int_{-\infty}^{+\infty} \frac{e^{-x^4}}{|x|}(f(x) - f(ix))dx \,. \tag{73}$$

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