

ON SOME VECTOR VALUED GENERALIZED DIFFERENCE MODULAR SEQUENCE SPACES

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Abstract

In this paper we generalize the modular sequence space $\ell\{M_k\}$ by introducing the sequence space $\ell\{M_k, p, q, s, \Delta_{(vm)}^n\}$. We give various properties relevant to algebraic and topological structures of this space and derived some other spaces .

1 Introduction

By $w(X)$, we shall denote the space of all X -valued sequences spaces, where (X, q) is a seminormed space, seminormed by q . For $X = C$, the space of complex numbers, these represent the corresponding scalar valued sequence spaces. The zero sequence is denoted by $\bar{\theta} = (\theta, \theta, \theta, \dots)$ where θ is the zero element of X .

The notion of difference sequence space was introduced by Kizmaz [5], who studied the difference sequence spaces $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Colak [3] by introducing the spaces $\ell_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_0(\Delta^n)$.

Let r, s be non-negative integers and $v = v_k$ be a sequence of non-zero scalars Also let $Z = \{\ell_\infty, c, c_0\}$. Dutta [2] define the following sequence spaces

$$Z\left(\Delta_{(vr)}^s\right) = \left\{x = (x_k) \in w : \left(\Delta_{(vr)}^s x_k\right) \in Z\right\},$$

where $\left(\Delta_{(vr)}^s x_k\right) = \left(\Delta_{(vr)}^{s-1} x_k - \Delta_{(vr)}^{s-1} x_{k-r}\right)$ and $\Delta_{(vr)}^0 x_k = v_k x_k$ for all $k \in N$, which is equivalent to the following binomial representation :

$$\Delta_{(vr)}^s = \sum_{i=0}^s (-1)^i \binom{s}{i} v_{k-ri} x_{k-ri}.$$

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In this expansion we take $v_k = 0$ and $x_k = 0$ for non-positive values of $k \in N$. Dutta [2] shown that these spaces can be made BK -spaces under the norm

$$\|x\| = \sup_k \left| \Delta_{(vr)}^s x_k \right|.$$

An Orlicz function is a function $M : [0, \infty) \rightarrow [0, \infty)$, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \rightarrow \infty$, as $x \rightarrow \infty$. If convexity of Orlicz function M is replaced by

$$M(x+y) \leq M(x) + M(y),$$

then this function is called a modulus function introduced by Nakano [9].

Lindenstrauss and Tzafriri [4] used the idea of Orlicz function to construct sequence space

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}.$$

The space ℓ_M becomes a Banach space, with the norm

$$\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}$$

which is called an Orlicz space. The space ℓ_M is closely related to the space ℓ_p which is an Orlicz sequence space with $M(x) = x^p$ for $1 \leq p < \infty$. Another generalization of Orlicz sequence spaces due to Woo [13]. Let $\{M_k\}$ be a sequence of Orlicz functions. Define the vector space $\ell\{M_k\}$ by

$$\ell\{M_k\} = \left\{ x \in w : \sum_{k=1}^{\infty} M_k\left(\frac{|x_k|}{\rho}\right) \leq \infty, \text{ for some } \rho > 0 \right\}$$

and this space has a norm defined by

$$\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M_k\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}$$

Then $\ell\{M_k\}$ becomes a Banach space and is called a modular sequence space. The space $\ell\{M_k\}$ also generalizes the concept of modular sequence space introduced by Nakano [10], who considered the space $\ell\{M_k\}$ when $M_k(x) = x^{\alpha_k}$, where $1 \leq \alpha_k < \infty$ for $k \geq 1$.

An Orlicz function M is said to satisfy the Δ_2 -condition for all values of u , if there exists a constant $K > 0$ such that $M(2u) \leq kM(u)$ ($u \geq 0$). The Δ_2 -condition is equivalent to the satisfaction of inequality $M(lu) \leq klM(u)$ for all values of u and for $l > 1$ (see; [7]). The Δ_2 -condition implies $M(lu) \leq Kl^{\log_2 K} M(u)$ for all values $u > 0$, $l > 1$.

Karakaya [6], Bektaş and Altin [1], Parasar and Choudhary [11], Mursaleen , Khan and Qamaruddin [8], Tripathy and Dutta [12] and many others have studied sequence spaces using Orlicz functions.

In [14], it is shown that a BK -spaces is a Banach space of complex sequences $x = (x_k)$ in which the co-ordinate maps are continuous , that is, $|x_k^n - x_k| \rightarrow 0$, whenever $\|x^n - x\| \rightarrow 0$ as $n \rightarrow \infty$, where $x^n = (x_k^n)$ for all $n \in N$ and $x = (x_k)$.

Let A denotes the set of all complex sequences which have only a finite number of non-zero coordinates, λ denotes a BK -space of sequences $x = (x_k)$ which contains A . An element $x = (x_k)$ of λ will be called sectionally convergent if

$$x^n = \sum_{k=1}^n x_k e_k \rightarrow x \text{ as } n \rightarrow \infty ,$$

where $e_k = (\delta_{ki})$, where $\delta_{kk} = 1, \delta_{ki} = 0$, for $k \neq i$.

The space λ will be called AK -space if and only if each of its elements is sectionally convergent .Let $\mathbf{M} = (M_k)$ be a sequence of Orlicz functions, X be a semi-normed space with seminorm $q, p = (p_k)$ be a sequence of positive real numbers and $v = (v_k)$ be a fixed sequence of non-zero scalars. Then for non-negative real numbers s, m and n , we define

$$\ell \left\{ M_k, p, q, s, \Delta_{(vm)}^n \right\} = \left\{ x \in w(X) : \sum_{k=1}^{\infty} k^{-s} \left[M_k \left(q \left(\frac{\Delta_{(vm)}^n}{\rho} \right) \right) \right]^{p_k} \right\} < \infty \text{ for some } \rho > 0.$$

Considering $X = C$, $q(x) = |x|, p_k = l, v_k = 1$ for all $k \in N, s = 0$ and $n = 0$, we get the modular space $\ell \{M_k\}$ introduced and studied by Woo [13].

2 Main Results

In this section, we give the theorems that characterize the structure of the class of sequences $\ell \left\{ M_k, p, q, s, \Delta_{(vm)}^n \right\}$ and some other spaces which can be derived from this space.

Theorem 1. *Let $p = (p_k)$ be bounded sequence of positive reals, then $\ell \left\{ M_k, p, q, s, \Delta_{(vm)}^n \right\}$ is a linear space over the field C .*

Proof. Let $x, y \in \ell \left\{ M_k, p, q, s, \Delta_{(vm)}^n \right\}$ and $\alpha, \beta \in C$. Then there exist some $\rho_1 > 0$ and $\rho_2 > 0$ such that

$$\sum_{k=1}^{\infty} k^{-s} \left[M_k \left(q \left(\frac{\Delta_{(vm)}^n x_k}{\rho_1} \right) \right) \right]^{p_k} < \infty$$

and

$$\sum_{k=1}^{\infty} k^{-s} \left[M_k \left(q \left(\frac{\Delta_{(vm)}^n x_k}{\rho_2} \right) \right) \right]^{p_k} < \infty .$$

We consider $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since each M_k is non-decreasing and convex, and since q is a seminorm ,

$$\begin{aligned}
& \sum_{k=1}^{\infty} k^{-s} \left[M_k \left(q \left(\frac{\Delta_{(vm)}^n(\alpha x_k + \beta y_k)}{\rho_3} \right) \right) \right]^{p_k} \\
& \leq \sum_{k=1}^{\infty} k^{-s} \left[M_k \left(q \left(\frac{\Delta_{(vm)}^n(\alpha x_k)}{\rho_3} \right) + q \left(\frac{\Delta_{(vm)}^n(\beta y_k)}{\rho_3} \right) \right) \right]^{p_k} \\
& \leq \sum_{k=1}^{\infty} k^{-s} \left[M_k \left(q \left(\frac{\Delta_{(vm)}^n x_k}{\rho_1} \right) + q \left(\frac{\Delta_{(vm)}^n y_k}{\rho_2} \right) \right) \right]^{p_k} \\
& \leq D \sum_{k=1}^{\infty} k^{-s} \left[M_k \left(q \left(\frac{\Delta_{(vm)}^n x_k}{\rho_1} \right) \right) \right]^{p_k} \\
& \quad + D \sum_{k=1}^{\infty} k^{-s} \left[M_k \left(q \left(\frac{\Delta_{(vm)}^n y_k}{\rho_2} \right) \right) \right]^{p_k} \\
& < \infty
\end{aligned}$$

where, $D = \max\{1, 2^{H-1}\}$ and $H = \sup_k p_k$. Hence this completes the proof. ■

Theorem 2. $\ell \left\{ M_k, p, q, s, \Delta_{(vm)}^n \right\}$ is a paranormed space (need not total paranorm) space with paranorm g , defined as follows .

$$g(x) = \inf \left\{ \rho^{\frac{pn}{H}} : \sum_{k=1}^{\infty} k^{-s} M_k \left(q \left(\frac{\Delta_{(vm)}^n x_k}{\rho} \right) \right) \leq 1, n = 1, 2, \dots \right\},$$

where $H = \sup_k p_k$.

Proof. Clearly $g(x) = g(-x)$. Since $M_k(0) = 0$, for all $k \in N$, we get $\inf \left\{ \rho^{\frac{pn}{H}} \right\} = 0$ for $x = \theta$. Now let $x, y \in \ell \left\{ M_k, p, q, s, \Delta_{(vm)}^n \right\}$ and let us choose $\rho_1 > 0$ and $\rho_2 > 0$ such that

$$\sum_{k=1}^{\infty} k^{-s} M_k \left(q \left(\frac{\Delta_{(vm)}^n x_k}{\rho_1} \right) \right) \leq 1$$

and

$$\sum_{k=1}^{\infty} k^{-s} M_k \left(q \left(\frac{\Delta_{(vm)}^n y_k}{\rho_2} \right) \right) \leq 1$$

Let $\rho = \rho_1 + \rho_2$. Then we have

$$\begin{aligned}
\sum_{k=1}^{\infty} k^{-s} M_k \left(q \left(\frac{\Delta_{(vm)}^n (x_k + y_k)}{\rho} \right) \right) & \leq \left(\frac{\rho_1}{\rho_1 + \rho_2} \right) \sum_{k=1}^{\infty} k^{-s} M_k \left(q \left(\frac{\Delta_{(vm)}^n x_k}{\rho_1} \right) \right) \\
& \quad + \left(\frac{\rho_2}{\rho_1 + \rho_2} \right) \sum_{k=1}^{\infty} k^{-s} M_k \left(q \left(\frac{\Delta_{(vm)}^n y_k}{\rho_2} \right) \right) \\
& \leq 1
\end{aligned}$$

Hence $g(x + y) \leq g(x) + g(y)$.

Finally, let λ be a given non-zero scalar, then the continuity of the scalar multiplication follows from the following equality

$$\begin{aligned} g(\lambda x) &= \inf \left\{ \rho^{\frac{pn}{H}} : \sum_{k=1}^{\infty} k^{-s} M_k \left(q \left(\frac{\Delta_{(vm)}^n(\lambda x_k)}{\rho} \right) \right) \leq 1, n = 1, 2, \dots \right\} \\ &= \inf \left\{ (|\lambda|s)^{\frac{pn}{H}} : \sum_{k=1}^{\infty} k^{-s} M_k \left(q \left(\frac{\Delta_{(vm)}^n(\lambda x_k)}{s} \right) \right) \leq 1, n = 1, 2, \dots \right\}, \end{aligned}$$

where $s = \frac{\rho}{|\lambda|}$. This completes the proof. ■

The proof of the following Theorem is easy, so it is omitted.

Theorem 3. Let $\mathbf{M} = (M_k)$ and $\mathbf{T} = (T_k)$ be sequences of Orlicz functions. For any two sequences $p = (p_k)$ and $t = (t_k)$ of bounded positive real numbers and for any two seminorms q_1 and q_2 we have

- (i) If q_1 is stronger than q_2 , then $\ell \left\{ M_k, p, q_1, s, \Delta_{(vm)}^n \right\} \subset \ell \left\{ M_k, p, q_2, s, \Delta_{(vm)}^n \right\}$,
- (ii) $\ell \left\{ M_k, p, q_1, s, \Delta_{(vm)}^n \right\} \cap \ell \left\{ M_k, p, q_2, s, \Delta_{(vm)}^n \right\} \subset \ell \left\{ M_k, p, q_1 + q_2, s, \Delta_{(vm)}^n \right\}$,
- (iii) $\ell \left\{ M_k, p, q, s, \Delta_{(vm)}^n \right\} \cap \ell \left\{ T_k, p, q, s, \Delta_{(vm)}^n \right\} \subset \ell \left\{ M_k + T_k, p, q, s, \Delta_{(vm)}^n \right\}$,
- (iv) $\ell \left\{ M_k, p, q_1, s, \Delta_{(vm)}^n \right\} \cap \ell \left\{ M_k, t, q_2, s, \Delta_{(vm)}^n \right\} \neq \phi$,
- (v) If $s_1 \leq s_2$, then $\ell \left\{ M_k, p, q, s_1, \Delta_{(vm)}^n \right\} \subset \ell \left\{ M_k, p, q, s_2, \Delta_{(vm)}^n \right\}$,
- (vi) The inclusions $\ell \left\{ M_k, p, q, s, \Delta_{(vm)}^{n-1} \right\} \subset \ell \left\{ M_k, p, q, s, \Delta_{(vm)}^n \right\}$ are strict.

In general, $\ell \left\{ M_k, p, q, s, \Delta_{(vm)}^n \right\} \subset \ell \left\{ M_k, p, q, s, \Delta_{(vm)}^i \right\}$ for $i = 1, 2, 3, \dots, n - 1$ and the inclusion is strict.

Theorem 4. Let $\mathbf{M} = (M_k)$ and $\mathbf{T} = (T_k)$ be sequences of Orlicz functions which satisfy Δ_2 -condition and $s > 1$, then

$$\ell \left\{ M_k, p, q, s, \Delta_{(vm)}^n \right\} \subseteq \ell \left\{ T_k \circ M_k, p, q, s, \Delta_{(vm)}^n \right\}.$$

Proof. Let $x \in \ell \left\{ M_k, p, q, s, \Delta_{(vm)}^n \right\}$ and $\epsilon > 0$. We choose $0 < \delta < 1$ such that $M(u) < \epsilon$ for $0 \leq u \leq \delta$. We write $y_k = M_k \left(q \left(\frac{\Delta_{(vm)}^n x_k}{\rho} \right) \right)$ and consider

$$\sum_{k=1}^{\infty} k^{-s} [T_k(y_k)]^{p_k} = \sum_1 k^{-s} [T_k(y_k)]^{p_k} + \sum_2 k^{-s} [T_k(y_k)]^{p_k}$$

where the first summation is over $y_k \leq \delta$ and the second over $y_k > \delta$. Since $s > 1$, we have

$$\sum_1 k^{-s} [T_k(y_k)]^{p_k} < \max(1, \epsilon^H) \sum_{k=1}^{\infty} k^{-s} < \infty.$$

For $y_k > \delta$, we use the fact that

$$y_k < \frac{y_k}{\delta} \leq 1 + \left(\frac{y_k}{\delta}\right).$$

Since each T_k is non-decreasing and convex, it follows that, for each $k \in N$,

$$T_k(y_k) < T_k\left(1 + \frac{y_k}{\delta}\right) < \frac{1}{2}T_k(2) + \frac{1}{2}T_k\left(2\frac{y_k}{\delta}\right).$$

Since each T_k is satisfy Δ_2 -condition, we have

$$T_k(y_k) < \frac{1}{2}K\frac{y_k}{\delta}T_k(2) + \frac{1}{2}K\frac{y_k}{\delta}T_k(2) = Ky_k\delta^{-1}T_k(2).$$

Hence

$$\sum_2 k^{-s} [T_k(y_k)]^{p_k} \leq \max\left(1, (K\delta^{-1}M(2))^H\right) \sum_{k=1}^{\infty} k^{-s} (y_k)^{p_k} < \infty.$$

Thus

$$\begin{aligned} \sum_{k=1}^{\infty} k^{-s} [T_k(y_k)]^{p_k} &= \sum_1 k^{-s} [T_k(y_k)]^{p_k} + \sum_2 k^{-s} [T_k(y_k)]^{p_k} \\ &\leq \max(1, \epsilon^H) \sum_{k=1}^{\infty} k^{-s} + \max\left(1, (K\delta^{-1}M(2))^H\right) \sum_{k=1}^{\infty} k^{-s} (y_k)^{p_k} \\ &< \infty. \end{aligned}$$

Hence $x \in \ell \left\{ T_k \circ M_k, p, q, s, \Delta_{(vm)}^n \right\}$. This completes the proof. ■

Taking $M_k(x) = x$, for all k in N , in the Theorem 4, we get the next Corollary.

Corollary 5. Let $\mathbf{M} = (M_k)$ be any sequence of Orlicz functions which satisfy Δ_2 -condition and $s > 1$, then

$$\ell \left\{ p, q, s, \Delta_{(vm)}^n \right\} \subseteq \ell \left\{ M_k, p, q, s, \Delta_{(vm)}^n \right\}.$$

We will write $f \approx g$ for non-negative functions f and g whenever $C_1 f \leq g \leq C_2 f$ for some $C_j > 0$, $j = 1, 2$.

Theorem 6. Let $\mathbf{M} = (M_k)$ and $\mathbf{T} = (T_k)$ be a sequence of Orlicz functions. If $M_k \approx T_k$ for each $k \in N$, then $\ell \left\{ M_k, p, q, s, \Delta_{(vm)}^n \right\} = \ell \left\{ T_k, p, q, s, \Delta_{(vm)}^n \right\}$.

Proof. Proof is obvious. ■

Theorem 7. Let $\mathbf{M} = (M_k)$ be a sequence of Orlicz functions. If $\lim_{t \rightarrow 0} \frac{M_k(t)}{t} > 0$ and $\lim_{t \rightarrow 0} \frac{M_k(t)}{t} < \infty$ for each $k \in N$, then

$$\ell \left\{ M_k, p, q, s, \Delta_{(vm)}^n \right\} = \ell \left\{ p, q, s, \Delta_{(vm)}^n \right\}$$

Proof. If the given conditions are satisfied, we have $M_k(t) \approx t$ for each k and the proof follows from Theorem 5. ■

If we take $s = 0$, the sequence space $\ell \left\{ M_k, p, q, s, \Delta_{(vm)}^n \right\}$ reduce to the following sequence space:

$$\ell \left\{ M_k, p, q, \Delta_{(vm)}^n \right\} = \left\{ x \in w(X) : \sum_{k=1}^{\infty} \left[M_k \left(q \left(\frac{\Delta_{(vm)}^n x_k}{\rho} \right) \right) \right]^{p_k} < \infty, \text{ for some } \rho > 0 \right\}.$$

Theorem 8. Let $p = (p_k)$ be bounded sequence of positive reals and (X, q) be a complete seminormed space, then $\ell \left\{ M_k, p, q, \Delta_{(vm)}^n \right\}$ is a complete paranormed space paranormed by h , defined by

$$h(x) = \inf \left\{ \rho^{\frac{p_n}{H}} : \sum_{k=1}^{\infty} M_k \left(q \left(\frac{\Delta_{(vm)}^n x_k}{\rho} \right) \right) \leq 1, n = 1, 2, \dots \right\},$$

where $H = \sup_k p_k$.

Proof. Let (x^i) be a Cauchy sequence in $\ell \left\{ M_k, p, q, \Delta_{(vm)}^n \right\}$. Let $\delta > 0$ be fixed and $r > 0$ be such that for a given $0 < \epsilon < 1$, $\frac{\epsilon}{r\delta} > 0$, and $r\delta \geq 1$. Then there exists a positive integer n_0 such that

$$h(x^i - x^j) < \frac{\epsilon}{r\delta}$$

for all $i, j \geq n_0$

$$h(x^i - x^j) = \inf \left\{ \rho^{\frac{p_n}{H}} : \sum_{k=1}^{\infty} M_k \left(q \left(\frac{\Delta_{(vm)}^n x_k^i - \Delta_{(vm)}^n x_k^j}{\rho} \right) \right) \leq 1 \right\} < \frac{\epsilon}{r\delta}$$

for all $i, j \geq n_0$. Hence we have

$$\sum_{k=1}^{\infty} M_k \left(q \left(\frac{\Delta_{(vm)}^n x_k^i - \Delta_{(vm)}^n x_k^j}{h(x^i - x^j)} \right) \right) \leq 1$$

for all $i, j \geq n_0$. It follows that

$$M_k \left(q \left(\frac{\Delta_{(vm)}^n x_k^i - \Delta_{(vm)}^n x_k^j}{h(x^i - x^j)} \right) \right) \leq 1$$

for all $i, j \geq n_0$ and $k \in N$. For $r > 0$ with $M_k \left(\frac{r\delta}{2} \right) \geq 1$, we have

$$M_k \left(q \left(\frac{\Delta_{(vm)}^n x_k^i - \Delta_{(vm)}^n x_k^j}{h(x^i - x^j)} \right) \right) \leq M_k \left(\frac{r\delta}{2} \right),$$

for all $i, j \geq n_0$ and $k \in N$. Since M_k is non-decreasing for each $k \in N$, we have

$$q \left(\Delta_{(vm)}^n x_k^i - \Delta_{(vm)}^n x_k^j \right) \leq \frac{r\delta}{2} \cdot \frac{\epsilon}{r\delta} = \frac{\epsilon}{2}.$$

Hence $\left(\Delta_{(vm)}^n x_k^i \right)$ is a Cauchy sequence in (X, q) for each $k \in N$. But (X, q) is complete and so $\left(\Delta_{(vm)}^n x_k^i \right)$ is convergent in (X, q) for each $k \in N$.

Let $\lim_{i \rightarrow \infty} \Delta_{(vm)}^n x_k^i = y_k$ for all $k \geq 1$. Let $k = 1$, then we have

$$\lim_{i \rightarrow \infty} \Delta_{(vm)}^n x_1^i = \lim_{i \rightarrow \infty} \sum_{v=0}^n (-1)^v \binom{n}{v} v_{1-mv} x_{1-mv}^i = \lim_{i \rightarrow \infty} v_1 x_1^i = y_1 \quad (1)$$

Similary we have ,

$$\lim_{i \rightarrow \infty} \Delta_{(vm)}^n x_k^i = \lim_{i \rightarrow \infty} v_k x_k^i = y_k \quad , \text{ for } k = 1, \dots, nm \quad (2)$$

Thus from (2.1) and (2.2),we have $\lim_{i \rightarrow \infty} x_{1+nm}^i$ exists. Let $\lim_{i \rightarrow \infty} x_{1+nm}^i = x_{1+nm}$. Proceeding in this way inductively, we have $\lim_{i \rightarrow \infty} x_k^i = x_k$ for each $k \in N$. Now we have for all $i, j \geq n_0$,

$$\inf \left\{ \rho^{\frac{pn}{H}} : \sum_{k=1}^{\infty} M_k \left(q \left(\frac{\Delta_{(vm)}^n x_k^i - \Delta_{(vm)}^n x_k^j}{\rho} \right) \right) \leq 1 \right\} < \epsilon.$$

Then we have

$$\lim_{j \rightarrow \infty} \left\{ \inf \left\{ \rho^{\frac{pn}{H}} : \sum_{k=1}^{\infty} M_k \left(q \left(\frac{\Delta_{(vm)}^n x_k^i - \Delta_{(vm)}^n x_k^j}{\rho} \right) \right) \leq 1 \right\} \right\} < \epsilon$$

for all $i \geq n_0$. Using the continuity of Orlicz functions , we have

$$\inf \left\{ \rho^{\frac{pn}{H}} : \sum_{k=1}^{\infty} M_k \left(q \left(\frac{\Delta_{(vm)}^n x_k^i - \Delta_{(vm)}^n \lim_{j \rightarrow \infty} x_k^j}{\rho} \right) \right) \leq 1 \right\} < \epsilon$$

for all $i \geq n_0$. This implies

$$\inf \left\{ \rho^{\frac{pn}{H}} : \sum_{k=1}^{\infty} M_k \left(q \left(\frac{\Delta_{(vm)}^n x_k^i - \Delta_{(vm)}^n x_k}{\rho} \right) \right) \leq 1 \right\} < \epsilon$$

for all $i \geq n_0$. It follows that $(x^i - x) \in \ell \left\{ M_k, p, q, \Delta_{(vm)}^n \right\}$. Since $(x^i) \in \ell \left\{ M_k, p, q, \Delta_{(vm)}^n \right\}$ and $\ell \left\{ M_k, p, q, \Delta_{(vm)}^n \right\}$ is a linear space , so we have $x = x^i - (x^i - x) \in \ell \left\{ M_k, p, q, \Delta_{(vm)}^n \right\}$. This completes the proof ■

If we take $s = 0$ and $p_k = l$, the sequence space $\ell \left\{ M_k, p, q, s, \Delta_{(vm)}^n \right\}$ reduce to the following sequence space :

$$\ell \left\{ M_k, q, \Delta_{(vm)}^n \right\} = \left\{ x \in w(X) : \sum_{k=1}^{\infty} M_k \left(q \left(\frac{\Delta_{(vm)}^n x_k}{\rho} \right) \right) < \infty, \text{ for some } \rho > 0 \right\}.$$

Theorem 9. *Let (X, q) be a complete normed space, then $\ell \left\{ M_k, q, \Delta_{(vm)}^n \right\}$ is a Banach space normed by $\|\cdot\|$, defined by*

$$\|x\| = \inf \left\{ \rho : \sum_{k=1}^{\infty} M_k \left(q \left(\frac{\Delta_{(vm)}^n x_k}{\rho} \right) \right) \leq 1 \right\}.$$

Proof. We prove that $\|\cdot\|$ is a norm on $\ell \left\{ M_k, q, \Delta_{(vm)}^n \right\}$. The completeness part can be proved using similar arguments as applied to prove above Theorem .

If $x = \theta$, then it is obvious that $\|x\| = 0$. Conversely assume $\|x\| = 0$. Then using the definition of norm, we have

$$\inf \left\{ \rho : \sum_{k=1}^{\infty} M_k \left(q \left(\frac{\Delta_{(vm)}^n x_k}{\rho} \right) \right) \leq 1 \right\} = 0.$$

This implies that for a given $\epsilon > 0$, there exists some $\rho_\epsilon (0 < \rho_\epsilon < \epsilon)$ such that

$$\sum_{k=1}^{\infty} M_k \left(q \left(\frac{\Delta_{(vm)}^n x_k}{\rho_\epsilon} \right) \right) \leq 1.$$

Thus

$$M_k \left(q \left(\frac{\Delta_{(vm)}^n x_k}{\epsilon} \right) \right) \leq M_k \left(q \left(\frac{\Delta_{(vm)}^n}{\rho_\epsilon} \right) \right) \leq 1, \forall k \in N.$$

Suppose that $\Delta_{(vm)}^n x_{n_i} \neq 0$ for some i . Let $\epsilon \rightarrow 0$, then $\frac{|\Delta_{(vm)}^n x_{n_i}|}{\epsilon} \rightarrow \infty$. It follows that $M_k \left(\frac{|\Delta_{(vm)}^n x_{n_i}|}{\epsilon} \right) \rightarrow \infty$ as $\epsilon \rightarrow 0$ for some $n_i \in N$. This is a contradiction. Therefore $\Delta_{(vm)}^n x_k = 0$ for all $k \in N$. Let $k = 1$, then $\Delta_{(vm)}^n x_1 = \sum_{i=0}^n (-1)^i \binom{n}{v} v_{1-mi} x_{1-mi} = 0$ and so $v_1 x_1 = 0$, by putting $v_{1-mi} = 0$ and $x_{1-mi} = 0$ for $i = 1, 2, \dots, n$.

Hence $x_1 = 0$, since (λ_k) is a sequence of non-zero scalars. Similarly taking $k = 2, \dots, mn$, we have $x_2 = \dots = x_{mn} = 0$. Next let $k = mn + 1$, then $\Delta_{(vm)}^n x_{mn+1} = \sum_{i=0}^n (-1)^i \binom{n}{v} v_{1+mn-mi} x_{1+mn-mi} = 0$. Since $x_1 = x_2 = \dots = x_{mn} = 0$, we must have $v_{mn+1} x_{mn+1} = 0$ and thus $x_{mn+1} = 0$. Proceeding in this way we can conclude that $x_k = 0$ for all $k \geq 1$. Hence $x = \theta$. Again proof of the properties $\|x + y\| \leq \|x\| + \|y\|$ and for any scalar α , $\|\alpha x\| = |\alpha| \|x\|$ are similar to that Theorem 2. It is easy to see that $\|x^i\| \rightarrow 0$ implies that $x_k^i \rightarrow 0$ for each $i \geq 1$. ■

Proposition 10. $\ell \left\{ M_k, q, \Delta_{(vm)}^n \right\}$ is a BK-space.

Now we study the AK-characteristic of the space $\ell \left\{ M_k, q, s, \Delta_{(vm)}^n \right\}$. Before that we give a new definition and prove some results those will be required.

Definition 1. For any sequence of Orlicz functions $\mathbf{M} = (M_k)$,

$$h \left\{ M_k, q, \Delta_{(vm)}^n \right\} = \left\{ x \in w(X) : \sum_{k=1}^{\infty} M_k \left(q \left(\frac{\Delta_{(vm)}^n x_k}{\rho} \right) \right) < \infty, \text{ for every } \rho > 0 \right\}.$$

Clearly $h \left\{ M_k, q, \Delta_{(vm)}^n \right\}$ is a subspace of $\ell \left\{ M_k, q, \Delta_{(vm)}^n \right\}$. The topology of $h \left\{ M_k, q, \Delta_{(vm)}^n \right\}$ is the one it inherits from $\|\cdot\|$.

Proposition 11. Let $\mathbf{M} = (M_k)$ be a sequence of Orlicz functions which satisfy Δ_2 -condition. Then

$$\ell \left\{ M_k, q, \Delta_{(vm)}^n \right\} = h \left\{ M_k, q, \Delta_{(vm)}^n \right\}.$$

Proof. It is enough to prove that $\ell \left\{ M_k, q, \Delta_{(vm)}^n \right\} \subseteq h \left\{ M_k, q, \Delta_{(vm)}^n \right\}$. Let $x \in h \left\{ M_k, q, \Delta_{(vm)}^n \right\}$, then for some $\rho > 0$,

$$\sum_{k=1}^{\infty} M_k \left(q \left(\frac{\Delta_{(vm)}^n x_k}{\rho} \right) \right) < \infty$$

Therefore

$$M_k \left(q \left(\frac{\Delta_{(vm)}^n x_k}{\rho} \right) \right) < \infty \text{ for every } k \geq 1.$$

Choose an arbitrary $\eta > 0$. If $\rho \leq \eta$, then $M_k \left(q \left(\frac{\Delta_{(vm)}^n x_k}{\eta} \right) \right) < \infty$ for every $k \geq 1$ and so

$$\sum_{k=1}^{\infty} M_k \left(q \left(\frac{\Delta_{(vm)}^n x_k}{\eta} \right) \right) < \infty$$

Let now $\eta < \rho$ and put $l = \frac{\rho}{\eta} > 1$. Since M satisfied the Δ_2 -condition, there exists a constant K such that

$$M_k \left(q \left(\frac{\Delta_{(vm)}^n x_k}{\eta} \right) \right) \leq K \left(\frac{\rho}{\eta} \right)^{\log_2 K} M_k \left(q \left(\frac{\Delta_{(vm)}^n x_k}{\rho} \right) \right)$$

for every $k \geq 1$. Now we can find $U > 0$ with $s > 1$ such that

$$M_k \left(q \left(\frac{\Delta_{(vm)}^n x_k}{\rho} \right) \right) < U k^{-s}$$

for the fixed $\rho > 0$ and for every $k \geq 1$.Then it follows that for every $\eta > 0$, we have

$$\sum_{k=1}^{\infty} M_k \left(q \left(\frac{\Delta_{(vm)}^n x_k}{\eta} \right) \right) < K \left(\frac{\rho}{\eta} \right)^{\log_2 K} M \sum_{k=1}^{\infty} k^{-s} < \infty .$$

This completes the proof . ■

Proposition 12. *Let (X, q) be a complete normed space , then $h \left\{ M_k, q, \Delta_{(vm)}^n \right\}$ is an AK -space.*

Proof. Let $x \in h \left\{ M_k, q, \Delta_{(vm)}^n \right\}$. Then for each $\epsilon, 0 < \epsilon < 1$, we can find an s_0 such that

$$\sum_{k \geq s_0} M_k \left(q \left(\frac{\Delta_{(vm)}^n x_k}{\epsilon} \right) \right) \leq 1.$$

Hence for $s \geq s_0$,

$$\begin{aligned} \|x - x^{[s]}\| &= \inf \left\{ \rho > 0 : \sum_{k \geq s+1} M_k \left(q \left(\frac{\Delta_{(vm)}^n x_k}{\epsilon} \right) \right) \leq 1 \right\} \\ &\leq \inf \left\{ \rho > 0 : \sum_{k \geq s} M_k \left(q \left(\frac{\Delta_{(vm)}^n x_k}{\rho} \right) \right) \leq 1 \right\} < \epsilon. \end{aligned}$$

Thus we can conclude that $h \left\{ M_k, q, \Delta_{(vm)}^n \right\}$ is an AK space. ■

Combining Proposition 1 and Proposition 2, we have the following Theorem .

Theorem 13. *Let $M = (M_k)$ be a sequence of Orlicz functions which satisfy Δ_2 -condition, then $\ell \left\{ M_k, q, \Delta_{(vm)}^n \right\}$ is an AK -space.*

Proposition 14. *$h \left\{ M_k, q, \Delta_{(vm)}^n \right\}$ is a closed subspace of $\ell \left\{ M_k, q, \Delta_{(vm)}^n \right\}$.*

Proof. Let $\{x^s\}$ be a sequence in $h \left\{ M_k, q, \Delta_{(vm)}^n \right\}$ such that $\|x^s - x\| \rightarrow 0$, where $x \in h \left\{ M_k, q, \Delta_{(vm)}^n \right\}$. To complete the proof we need to show that $x \in h \left\{ M_k, q, \Delta_{(vm)}^n \right\}$, i.e. ,

$$\sum_{k \geq 1} M_k \left(q \left(\frac{\Delta_{(vm)}^n x_k}{\rho} \right) \right) < \infty \text{ for every } \rho > 0.$$

For $\rho > 0$, there corresponds an l such that $\|x^l - x\| \leq \frac{\rho}{2}$. Then using convexity of

each M_k ,

$$\begin{aligned} \sum_{k \geq 1} M_k \left(q \left(\frac{\Delta_{(vm)}^n x_k}{\rho} \right) \right) &= \sum_{k \geq 1} M_k \left(q \left(\frac{2 \left| \Delta_{(vm)}^n x_k^l \right| - 2 \left(\left| \Delta_{(vm)}^n x_k^l \right| - \left| \Delta_{(vm)}^n x_k \right| \right)}{2\rho} \right) \right) \\ &\leq \frac{1}{2} \sum_{k \geq 1} M_k \left(q \left(\frac{2 \left| \Delta_{(vm)}^n x_k^l \right|}{\rho} \right) \right) \\ &\quad + \frac{1}{2} \sum_{k \geq 1} M_k \left(q \left(\frac{2 \left| \Delta_{(vm)}^n (x_k^l - x_k) \right|}{\rho} \right) \right) \\ &\leq \frac{1}{2} \sum_{k \geq 1} M_k \left(q \left(\frac{2 \left| \Delta_{(vm)}^n x_k^l \right|}{\rho} \right) \right) \\ &\quad + \frac{1}{2} \sum_{k \geq 1} M_k \left(q \left(\frac{2 \left| \Delta_{(vm)}^n (x_i^l - x_i) \right|}{\|x^l - x\|} \right) \right) \end{aligned}$$

Now from Theorem 8, using the definition of norm $\|\cdot\|$, we have

$$\sum_{k \geq 1} M_k \left(q \left(\frac{2 \left| \Delta_{(vm)}^n (x_i^l - x_i) \right|}{\|x^l - x\|} \right) \right) \leq 1$$

It follows that

$$\sum_{k \geq 1} M_k \left(q \left(\frac{\Delta_{(vm)}^n x_k}{\rho} \right) \right) < \infty \text{ for every } \rho > 0$$

Thus $x \in h \left\{ M_k, q, \Delta_{(vm)}^n \right\}$ ■

Hence we have the following Corollary

Corollary 15. $h \left\{ M_k, q, \Delta_{(vm)}^n \right\}$ is a BK- space .

References

- [1] A. Ç. Bektaş, Y. Altın, *The sequence space $\ell_M(p, q, s)$ on seminormed spaces*, Indian J. Pure Appl. Math., 34(2003), 4, 529-534.
- [2] H. Dutta, *Characterization of certain matrix classes involving generalized difference summability spaces*, Applied Sciences (APPS), (2009),11, 60-67.
- [3] M. Et, R. Çolak, *On generalized difference sequence spaces*, Soochow J. Math., 21(1995), 4,377-386.

- [4] J. Lindenstrauss, L. Tzafriri, *On Orlicz sequence spaces*, Israel J. Math., (1971),10, 379-390
- [5] H. Kizmaz, *On certain sequence spaces*, Canad. Math. Bull., (1981),24, 169-176.
- [6] V. Karakaya, *Some new sequence spaces defined by a sequence of Orlicz functions*,Taiwanese J.Math.9 (2005), 4, 617-627.
- [7] M. A. Krasnoselskii, Y. B. Rutitsky, *Convex Functions and Orlicz Spaces*, Groningen, Netherlands, 1961
- [8] Mursaleen, A. Khan, Qamaruddin, *Difference sequence spaces defined by Orlicz functions*, Demonstratio Math. 32 (1999), no. 1, 145–150.
- [9] H. Nakano, *Concave modulars*, J. Math. Soc. Japan, (1953),5, 29-49.
- [10] H. Nakano, *Modulared sequence spaces*, Proc. Japan Acad. 27, (1951). 508–512.
- [11] S. D. Parasar, B. Choudhary, *Sequence spaces defined by Orlicz functions*, Indian J. Pure Appl. Math, 25 (1994), 4, 419-428.
- [12] B. C. Tripathy, H. Dutta, *Some difference paranormed sequence spaces defined by Orlicz functions*,Fasc. Math. No. 42 (2009), 121–131.
- [13] J. Y. T. Woo, *On modular sequence spaces*, Studia Math., 48 (1973), 271-289
- [14] K. Zeller, *Theorie der Limitierungsverfahren*, Berlin, 1958.

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