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ON THE SUM-CONNECTIVITY INDEX

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Abstract

The sum-connectivity index of a simple graph G is defined in mathematical chemistry as

$$R^+(G) = \sum_{uv\in E(G)} (d_u + d_v)^{-1/2},$$

where $E(G)$ is the edge set of G and d_u is the degree of vertex u in G. We give a best possible lower bound for the sum-connectivity index of a graph (a triangle-free graph, respectively) with n vertices and minimum degree at least two and characterize the extremal graphs, where $n \geq 11$.

1 Introduction

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$ [1]. For $u \in V(G)$, $d_u(G)$ or d_u denotes the degree of u in G. Let $N(u)$ be the set of neighbors of vertex u in G. Then $d_u = |N(u)|$.

The Randić connectivity index of a graph G, proposed by Randić in 1975, is defined as [2]

$$R(G) = \sum_{uv\in E(G)} (d_u d_v)^{-1/2}.$$

It is one of the most successful molecular descriptors in structure-property and structure-activity relationships studies [3–8]. Its mathematical properties [9, 10] and generalizations/variants [11–13] have also been studied extensively. We also call it the product-connectivity index.

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Motivated by Randić's definition of the product-connectivity index, the sum-connectivity index of a graph G was proposed in [14], which is defined as

$$R^+(G) = \sum_{uv \in E(G)} (d_u + d_v)^{-1/2}.$$

The applications of the sum-connectivity index have been investigated in [15, 16]. Some basic mathematical properties of the sum-connectivity index have been established in [14, 17–19].

Bollobás and Erdös [20] showed that for a graph G with n vertices and without isolated vertices, $R(G) \geq \sqrt{n-1}$ with equality if and only if G is the star. Then Delorme *et al.* [21] gave a best possible lower bound for the product-connectivity index of a graph with $n \geq 6$ vertices and minimum degree at least two. Later Liu *et al.* [22] found a best possible lower bound for the product-connectivity index of a triangle-free graph with $n \geq 6$ vertices and minimum degree at least two.

In [14], it was shown that for a graph G with $n \geq 5$ vertices and without isolated vertices, $R^+(G) \geq \frac{n-1}{\sqrt{n}}$ with equality if and only if G is the star. For $n = 4$, this is not true since for the union of two copies of path on two vertices, its sum-connectivity index is $\sqrt{2}$, less than $\frac{3}{2}$. In this paper, we establish a best possible lower bound for the sum-connectivity index of a graph (triangle-free graph, respectively) with $n \geq 11$ vertices and minimum degree at least two and characterize the extremal graphs.

2 Preliminaries

For a graph G with $u \in V(G)$ ($e \in E(G)$, respectively), $G - u$ ($G - e$, respectively) means the graph obtained feom G by deleting u and its incident edges (e, respectively).

For an edge $e = uv$ of a graph G, its weight is defined to be $(d_u + d_v)^{-1/2}$. The sum-connectivity index of G is the sum of weights over all its edges.

Lemma 2.1. *If e is an edge of maximal weight in G, then $R^+(G - e) < R^+(G)$.*

Proof. Let $e = uv$. Since uv is an edge of maximal weight in G, we have $d_w \geq d_v$ for $w \in N(u)$ and $d_w \geq d_u$ for $w \in N(v)$. Obviously, for positive a, $\frac{1}{\sqrt{x+a}} - \frac{1}{\sqrt{x+a-1}}$ and $\frac{x-1}{\sqrt{x}}$ are both increasing for $x \geq 1$. Then

$$\begin{aligned}
& R^+(G) - R^+(G - e) \\
= \ & \frac{1}{\sqrt{d_u + d_v}} + \sum_{w \in N(u) \setminus \{v\}} \left(\frac{1}{\sqrt{d_u + d_w}} - \frac{1}{\sqrt{d_u + d_w - 1}} \right) \\
& + \sum_{w \in N(v) \setminus \{u\}} \left(\frac{1}{\sqrt{d_v + d_w}} - \frac{1}{\sqrt{d_v + d_w - 1}} \right) \\
\geq \ & \frac{1}{\sqrt{d_u + d_v}} + (d_u - 1) \left(\frac{1}{\sqrt{d_u + d_v}} - \frac{1}{\sqrt{d_u + d_v - 1}} \right)
\end{aligned}$$

$$\begin{aligned}
&+(d_v-1)\left(\frac{1}{\sqrt{d_v+d_u}}-\frac{1}{\sqrt{d_v+d_u-1}}\right)\\
=\ &\frac{d_u+d_v-1}{\sqrt{d_u+d_v}}-\frac{d_u+d_v-2}{\sqrt{d_u+d_v-1}}\\
>\ &0.
\end{aligned}$$

The result follows. □

For $x \geq 3$, let $r(x) = 2\sqrt{x+1} + \frac{1}{\sqrt{2x-2}} - \frac{6}{\sqrt{x+1}}$.

Lemma 2.2. *For $n \geq 11$, $2\sqrt{n} - \frac{4}{\sqrt{n}} - r(n) > 0$.*

Proof. For $11 \leq n \leq 15$, the result can be checked by direct calculation. Suppose that $n \geq 16$. For $a, b > 0$, it is easily seen that $\frac{1}{a} + \frac{1}{b} \geq \frac{4}{a+b}$ with equality if and only if $a = b$. This implies that

$$-\frac{2}{\sqrt{n}+\sqrt{n+1}} > -\frac{1}{2}\left(\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{n+1}}\right).$$

Then

$$\begin{aligned}
&2\sqrt{n}-\frac{4}{\sqrt{n}}-r(n)\\
=\ &2\sqrt{n}-\frac{4}{\sqrt{n}}-2\sqrt{n+1}-\frac{1}{\sqrt{2\,(n-1)}}+\frac{6}{\sqrt{n+1}}\\
=\ &-\frac{2}{\sqrt{n}+\sqrt{n+1}}-\frac{4}{\sqrt{n}}-\frac{1}{\sqrt{2\,(n-1)}}+\frac{6}{\sqrt{n+1}}\\
>\ &-\frac{1}{2}\left(\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{n+1}}\right)-\frac{4}{\sqrt{n}}-\frac{1}{\sqrt{2\,(n-1)}}+\frac{6}{\sqrt{n+1}}\\
=\ &-\frac{9}{2\sqrt{n}}+\frac{11}{2\sqrt{n+1}}-\frac{1}{\sqrt{2(n-1)}}\\
=\ &\left(-\frac{9}{2\sqrt{n}}+\frac{9}{2\sqrt{n+1}}\right)+\left(\frac{1}{\sqrt{2(n+1)}}-\frac{1}{\sqrt{2(n-1)}}\right)\\
&+\frac{\sqrt{2}-1}{\sqrt{2(n+1)}}\\
=\ &\frac{\sqrt{2}-1}{2\sqrt{2(n+1)}}+\left(-\frac{9}{2\sqrt{n}}+\frac{9}{2\sqrt{n+1}}\right)\\
&+\frac{\sqrt{2}-1}{2\sqrt{2(n+1)}}+\left(\frac{1}{\sqrt{2(n+1)}}-\frac{1}{\sqrt{2(n-1)}}\right)\\
=\ &\frac{1}{2\sqrt{2\,(n+1)}}\left(\sqrt{2}-1-\frac{9\sqrt{2}}{\sqrt{n}(\sqrt{n+1}+\sqrt{n})}\right)
\end{aligned}$$

$$
\begin{aligned}
& +\frac{1}{2\sqrt{2\,(n+1)}}\left(\sqrt{2}-1-\frac{4}{\sqrt{n-1}(\sqrt{n+1}+\sqrt{n-1})}\right) \\
\geq\ & \frac{1}{2\sqrt{2\,(n+1)}}\left(\sqrt{2}-1-\frac{9\sqrt{2}}{\sqrt{16}(\sqrt{17}+\sqrt{16})}\right) \\
& +\frac{1}{2\sqrt{2\,(n+1)}}\left(\sqrt{2}-1-\frac{4}{\sqrt{15}(\sqrt{17}+\sqrt{15})}\right) \\
>\ & 0.
\end{aligned}
$$

The result follows. □

Lemma 2.3. *For $x \geq 3+i$, $r(x)-r(x-i)$ is decreasing in x, where $i=2,3$.*

Proof. For $x \geq 3$, it is easily seen that $6\sqrt{2}\left(\frac{x-1}{x+1}\right)^{5/2} \geq 6\sqrt{2}\left(\frac{3-1}{3+1}\right)^{5/2} > 1$, implying that $\frac{3}{4\sqrt{2}}\,(x-1)^{-5/2} < \frac{9}{2}\,(x+1)^{-5/2}$. Then

$$
\begin{aligned}
r''(x) &= -\frac{1}{2}\,(x+1)^{-3/2}+\frac{3}{4\sqrt{2}}\,(x-1)^{-5/2}-\frac{9}{2}\,(x+1)^{-5/2} \\
&< -\frac{1}{2}\,(x+1)^{-3/2} \\
&< 0.
\end{aligned}
$$

By the Lagrange mean-value theorem, $r'(x)-r'(x-i)<0$ for $x \geq 3+i$, and thus the result follows. □

Let $f(x,y)=\frac{1}{\sqrt{x+y}}-\frac{1}{\sqrt{x+y-2}}+\frac{x-1}{\sqrt{x+2}}-\frac{x-2}{\sqrt{x+1}}+\frac{y-1}{\sqrt{y+2}}-\frac{y-2}{\sqrt{y+1}}$, where $x,y \geq 2$.

Lemma 2.4. *For $x,y \geq 3$, $f(x,y)$ is decreasing in x and y.*

Proof. Let $g(x)=(x+2)\,x^{-3/2}-(x+1)^{-3/2}$ for $x \geq 4$. Then

$$
g'(x)=-\left(\frac{1}{2}x+3\right)x^{-5/2}+\frac{3}{2}\,(x+1)^{-5/2}<0,
$$

i.e., $g(x)$ is decreasing in x. It is easily seen that

$$
\begin{aligned}
\frac{\partial f(x,y)}{\partial x} &= \frac{1}{2}(x+5)(x+2)^{-3/2}-\frac{1}{2}(x+4)(x+1)^{-3/2} \\
&\quad -\frac{1}{2}(x+y)^{-3/2}+\frac{1}{2}(x+y-2)^{-3/2},
\end{aligned}
$$

and thus

$$
\frac{\partial}{\partial y}\left(\frac{\partial f(x,y)}{\partial x}\right)=-\frac{3}{4}(x+y-2)^{-5/2}+\frac{3}{4}(x+y)^{-5/2}<0,
$$

implying that

$$
\frac{\partial f(x,y)}{\partial x} \leq \frac{\partial f(x,3)}{\partial x}
$$

$$
\begin{aligned}
&= \frac{1}{2}(x+5)(x+2)^{-3/2} - \frac{1}{2}(x+3)(x+1)^{-3/2} - \frac{1}{2}(x+3)^{-3/2} \\
&= \frac{1}{2}\left(g(x+2) - g(x+1)\right) \\
&< 0.
\end{aligned}
$$

Similarly, $\frac{\partial f(x,y)}{\partial y} < 0$. Now the result follows. □

Let $K_{a,b}$ be the complete bipartite graph with a and b vertices in its two partite sets, respectively. For $n \geq 4$, let $K^*_{2,n-2}$ be the graph obtained from $K_{2,n-2}$ by joining an edge between the two vertices of degree $n-2$. Obviously, $R^+(K^*_{2,n-2}) = r(n)$. Let $\delta(G)$ be the minimum degree of the graph G.

Lemma 2.5. *Let G be a graph with n vertices and $\delta(G) = 2$. Let u be a vertex of degree two with two adjacent neighbors, both of degree at least three. Then $R^+(G) - R^+(G-u) \geq f(n-1, n-1)$ with equality if and only if $G = K^*_{2,n-2}$.*

Proof. Let $N(u) = \{v, w\}$. Obviously, $\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x-1}}$ is increasing for $x > 1$. We have

$$
\begin{aligned}
& R^+(G) - R^+(G-u) \\
= \;& \frac{1}{\sqrt{d_v+2}} + \frac{1}{\sqrt{d_w+2}} + \frac{1}{\sqrt{d_v+d_w}} - \frac{1}{\sqrt{d_v+d_w-2}} \\
& + \sum_{z \in N(v)\setminus\{u,w\}} \left(\frac{1}{\sqrt{d_v+d_z}} - \frac{1}{\sqrt{d_v+d_z-1}} \right) \\
& + \sum_{z \in N(w)\setminus\{u,v\}} \left(\frac{1}{\sqrt{d_w+d_z}} - \frac{1}{\sqrt{d_w+d_z-1}} \right) \\
\geq \;& \frac{1}{\sqrt{d_v+2}} + \frac{1}{\sqrt{d_w+2}} + \frac{1}{\sqrt{d_v+d_w}} - \frac{1}{\sqrt{d_v+d_w-2}} \\
& + (d_v-2)\left(\frac{1}{\sqrt{d_v+2}} - \frac{1}{\sqrt{d_v+2-1}} \right) \\
& + (d_w-2)\left(\frac{1}{\sqrt{d_w+2}} - \frac{1}{\sqrt{d_w+2-1}} \right) \\
= \;& f(d_v, d_w),
\end{aligned}
$$

and thus $R^+(G) - R^+(G-u) \geq f(d_v, d_w)$ with equality if and only if $d_z = 2$ for $z \in N(v) \setminus \{u,w\}$ or $z \in N(w) \setminus \{u,v\}$. By Lemma 2.4, $R^+(G) - R^+(G-u) \geq f(n-1, n-1)$ with equality if and only if $d_v = d_w = n-1$ and $d_z = 2$ for $z \in N(v) \setminus \{u,w\}$ or $z \in N(w) \setminus \{u,v\}$, i.e., $G = K^*_{2,n-2}$. □

Lemma 2.6. *Let G be a triangle-free graph with n vertices and $\delta(G) = 2$. Let u be a vertex of degree two in G. Then $R^+(G) - R^+(G-u) \geq 2\left(\frac{n-2}{\sqrt{n}} - \frac{n-2-1}{\sqrt{n-1}}\right)$ with equality if and only if $G = K_{2,n-2}$.*

Proof. Let $N(u) = \{v_1, v_2\}$. Since G is a triangle-free graph, $d_{v_1}, d_{v_2} \le n-2$ and $v_1v_2 \notin E(G)$. Note that $\frac{x}{\sqrt{\delta+x}} - \frac{x-1}{\sqrt{\delta+x-1}}$ is decreasing for $x \ge 1$. We have

$$\begin{aligned}
& R^+(G) - R^+(G-u) \\
= & \sum_{i=1}^{2}\left(\frac{1}{\sqrt{2+d_{v_i}}} + \sum_{z\in N(v_i)\setminus\{u\}}\left(\frac{1}{\sqrt{d_z+d_{v_i}}} - \frac{1}{\sqrt{d_z+d_{v_i}-1}}\right)\right) \\
\ge & \sum_{i=1}^{2}\left[\frac{1}{\sqrt{2+d_{v_i}}} + (d_{v_i}-1)\left(\frac{1}{\sqrt{2+d_{v_i}}} - \frac{1}{\sqrt{2+d_{v_i}-1}}\right)\right] \\
= & \sum_{i=1}^{2}\left(\frac{d_{v_i}}{\sqrt{2+d_{v_i}}} - \frac{d_{v_i}-1}{\sqrt{2+d_{v_i}-1}}\right) \\
\ge & 2\left(\frac{n-2}{\sqrt{2+(n-2)}} - \frac{(n-2)-1}{\sqrt{2+(n-2)-1}}\right) \\
= & 2\left(\frac{n-2}{\sqrt{n}} - \frac{n-3}{\sqrt{n-1}}\right)
\end{aligned}$$

with equalities if and only if $d_{v_1} = d_{v_2} = n-2$ and $d_z = 2$ for $z \in N(v_i) \setminus \{u\}$ with $i = 1, 2$, i.e., $G = K_{2,n-2}$. □

3 Result

Now we prove our main results.

Theorem 3.1. *Let G be a graph with $n \ge 11$ vertices and $\delta(G) \ge 2$. Then $R^+(G) \ge r(n)$ with equality if and only if $G = K^*_{2,n-2}$.*

Proof. Assume that G is a counterexample with minimal number of vertices for which $R^+(G)$ is minimal. If $\delta(G) \ge 3$, then by Lemma 2.1, the deletion of an edge of maximal weight yields a graph G' of minimal degree at least two such that $R^+(G') < R^+(G)$, which is a contradiction to the choice of G. Hence $\delta(G) = 2$.
Claim 1. The neighbors of every vertex of degree two are adjacent.

Suppose that the claim is false. Let u be a vertex of degree two with $N(u) = \{v, w\}$ and $vw \notin E(G)$. Then $G_1 = G - u + vw$ is not a counterexample, and thus $R^+(G_1) \ge r(n-1)$.

Let $t(x,y) = \frac{1}{\sqrt{2+x}} + \frac{1}{\sqrt{2+y}} - \frac{1}{\sqrt{x+y}}$, where $x, y \ge 2$. Then $\frac{\partial t(x,y)}{\partial x} = -\frac{1}{2}(2+x)^{-3/2} + \frac{1}{2}(x+y)^{-3/2}$, and thus $\frac{\partial}{\partial y}\left(\frac{\partial t(x,y)}{\partial x}\right) = -\frac{3}{4}(x+y)^{-5/2} < 0$, implying that $\frac{\partial t(x,y)}{\partial x} \le \frac{\partial t(x,2)}{\partial x} = 0$. Similarly, $\frac{\partial t(x,y)}{\partial y} \le 0$. Since $2 \le d_v, d_w \le n-2$, we have $t(d_v, d_w) \ge t(n-2, n-2)$. By Lemma 2.2, we have

$$R^+(G) = R^+(G_1) + \frac{1}{\sqrt{2+d_v}} + \frac{1}{\sqrt{2+d_w}} - \frac{1}{\sqrt{d_v+d_w}}$$

$$\begin{aligned}
&= R^+(G_1) + t\,(d_v, d_w) \\
&\geq r(n-1) + t\,(n-2, n-2) \\
&= 2\sqrt{n} - \frac{4}{\sqrt{n}} \\
&> r(n),
\end{aligned}$$

which is a contradiction. Claim 1 follows.

Claim 2. Every pair of adjacent vertices of degree two has no common neighbor.

Suppose that the claim is false. Let u_1 and u_2 be two adjacent vertices of degree two and u_3 a common neighbor of them. Obviously, $2 \leq d_{u_3} \leq n-1$.

Suppose that $d_{u_3} = 2$. Then $G_2 = G - u_1 - u_2 - u_3$ is not a counterexample, and thus $R^+(G_2) \geq r(n-3)$. By Lemma 2.3, $r(n) - r(n-3) \leq r(11) - r(8) = 1.1525 < \frac{3}{2}$, implying that

$$R^+(G) = R^+(G_2) + \frac{3}{2} \geq r(n-3) + \frac{3}{2} > r(n),$$

which is a contradiction.

Now suppose that $d_{u_3} \geq 4$. Then $G_3 = G - u_1 - u_2$ is not a counterexample, and thus $R^+(G_3) \geq r(n-2)$. Then

$$\begin{aligned}
R^+(G) &= R^+(G_3) + \sum_{v \in N(u_3)\setminus\{u_1,u_2\}} \left(\frac{1}{\sqrt{d_v + d_{u_3}}} - \frac{1}{\sqrt{d_v + d_{u_3} - 2}} \right) \\
&\quad + \frac{2}{\sqrt{2 + d_{u_3}}} + \frac{1}{2} \\
&\geq r(n-2) + (d_{u_3} - 2) \left(\frac{1}{\sqrt{2 + d_{u_3}}} - \frac{1}{\sqrt{2 + d_{u_3} - 2}} \right) \\
&\quad + \frac{2}{\sqrt{2 + d_{u_3}}} + \frac{1}{2} \\
&= r(n-2) + \frac{d_{u_3}}{\sqrt{2 + d_{u_3}}} - \frac{d_{u_3} - 2}{\sqrt{d_{u_3}}} + \frac{1}{2}.
\end{aligned}$$

It is easily seen that $\frac{a}{\sqrt{2+a}} - \frac{a-2}{\sqrt{a}}$ is decreasing for $a \geq 2$. If $11 \leq n \leq 20$, then $d_{u_3} \leq n-1$, and by Lemma 2.3 and direct calculation, we have

$$\begin{aligned}
R^+(G) - r(n) &\geq (r(n-2) - r(n)) + \left(\frac{d_{u_3}}{\sqrt{2 + d_{u_3}}} - \frac{d_{u_3} - 2}{\sqrt{d_{u_3}}} \right) + \frac{1}{2} \\
&\geq (r(11-2) - r(11)) + \left(\frac{19}{\sqrt{2 + 19}} - \frac{19 - 2}{\sqrt{19}} \right) + \frac{1}{2} \\
&> 0.
\end{aligned}$$

It is easily seen that $\frac{a-2}{\sqrt{a}}$ is increasing for $a \geq 2$. If $n \geq 21$, then by Lemma 2.3 and direct calculation, we have

$$R^+(G) - r(n) \geq (r(n-2) - r(n)) + \left(\frac{d_{u_3}}{\sqrt{2 + d_{u_3}}} - \frac{d_{u_3} - 2}{\sqrt{d_{u_3}}} \right) + \frac{1}{2}$$

$$\begin{aligned} &\geq (r(21-2)-r(21))+\frac{1}{2} \\ &> 0. \end{aligned}$$

Thus $R^+(G) \geq r(n)$, which is a contradiction.

Suppose that $d_{u_3} = 3$. Denote by u_4 the neighbor of u_3 in G different from u_1 and u_2, where $2 \leq d_{u_4} \leq n-3$. First suppose that $d_{u_4} = 2$. Denote by u_5 the neighbor of u_4 in G different from u_3. By Claim 1, $u_3u_5 \in E(G)$. Since $d_{u_3} = 3$, the neighbors of u_3 are u_1, u_2, u_4, which is a contradiction. Then $d_{u_4} \neq 2$. Next suppose that $3 \leq d_{u_4} \leq n-3$. Then $G_4 = G-u_1-u_2-u_3$ is not a counterexample, and thus $R^+(G_4) \geq r(n-3)$. By Lemma 2.3, $r(n)-r(n-3) \leq r(11)-r(8) < \frac{1}{2}+\frac{1}{\sqrt{6}}+\frac{1}{\sqrt{5}}$. Then

$$\begin{aligned} R^+(G) &= R^+(G_4) + \sum_{v\in N(u_4)\setminus\{u_3\}} \left(\frac{1}{\sqrt{d_v+d_{u_4}}} - \frac{1}{\sqrt{d_v+d_{u_4}-1}}\right) \\ &\quad + \frac{1}{\sqrt{3+d_{u_4}}} + \frac{1}{2} + \frac{2}{\sqrt{5}} \\ &\geq r(n-3) + (d_{u_4}-1)\left(\frac{1}{\sqrt{2+d_{u_4}}} - \frac{1}{\sqrt{2+d_{u_4}-1}}\right) \\ &\quad + \frac{1}{\sqrt{3+d_{u_4}}} + \frac{1}{2} + \frac{2}{\sqrt{5}} \\ &= r(n-3) + \frac{1}{\sqrt{3+d_{u_4}}} - \frac{1}{\sqrt{2+d_{u_4}}} + \frac{1}{2} + \frac{2}{\sqrt{5}} \\ &\quad + \left(\frac{d_{u_4}}{\sqrt{2+d_{u_4}}} - \frac{d_{u_4}-1}{\sqrt{1+d_{u_4}}}\right) \\ &> r(n-3) + \frac{1}{\sqrt{3+d_{u_4}}} - \frac{1}{\sqrt{2+d_{u_4}}} + \frac{1}{2} + \frac{2}{\sqrt{5}} \\ &\geq r(n-3) + \frac{1}{\sqrt{3+3}} - \frac{1}{\sqrt{2+3}} + \frac{1}{2} + \frac{2}{\sqrt{5}} \\ &= r(n-3) + \frac{1}{\sqrt{6}} + \frac{1}{\sqrt{5}} + \frac{1}{2} \\ &> r(n), \end{aligned}$$

which is a contradiction.

Now Claim 2 follows.

Let $v \in V(G)$ be a vertex of degree two with neighbors v_1 and v_2. By Claim 1, v_1 and v_2 are adjacent. By Claim 2, $d_{v_1}, d_{v_2} \geq 3$. By Lemma 2.5,

$$\begin{aligned} R^+(G) &\geq R^+(G-v) + f(n-1,n-1) \\ &\geq r(n-1) + \frac{1}{\sqrt{2(n-1)}} - \frac{1}{\sqrt{2(n-2)}} + \frac{2(n-2)}{\sqrt{n+1}} - \frac{2(n-3)}{\sqrt{n}} \\ &= r(n) \end{aligned}$$

with equality if and only if $G = K^*_{2,n-2}$, which is a contradiction. $\square$

It is easily checked that $R^+(K_{2,8}) = \frac{16}{\sqrt{2+8}} = 5.05964 < 5.05988 = 2\sqrt{10+1} + \frac{1}{\sqrt{20-2}} - \frac{6}{\sqrt{10+1}}$. Thus the condition $n \geq 11$ in Theorem 1 is necessary.

Theorem 3.2. *Let G be a triangle-free graph of order $n \geq 11$ with $\delta(G) \geq 2$. Then $R^+(G) \geq \frac{2(n-2)}{\sqrt{n}}$ with equality if and only if $G = K_{2,n-2}$.*

Proof. Assume that G is a counterexample with minimal number of vertices for which $R^+(G)$ is minimal. By Lemma 2.1, we have $\delta(G) = 2$. Let V_2 be the set of vertices of degree two in G. Suppose that there exists a vertex $z \in V_2$ with $N(z) \cap V_2 = \emptyset$. Let $N(z) = \{z_1, z_2\}$. Then $z_i \notin V_2$ for $i = 1, 2$. Note that $2 \leq \delta(G - z) \leq \frac{n-1}{2}$ as $G - z$ is triangle-free. By the assumption of G, we have $R^+(G) \geq \frac{2(n-1-2)}{\sqrt{n-1}}$. By Lemma 2.6, we have

$$\begin{aligned} R^+(G) &\geq R^+(G - z) + 2\left(\frac{n-2}{\sqrt{n}} - \frac{n-2-1}{\sqrt{n-1}}\right) \\ &\geq \frac{2(n-2-1)}{\sqrt{n-1}} + 2\left(\frac{n-2}{\sqrt{n}} - \frac{n-2-1}{\sqrt{n-1}}\right) \\ &= \frac{2(n-2)}{\sqrt{n}} = R^+(K_{2,n-2}) \end{aligned}$$

with equalities if and only if $G = K_{2,n-2}$, which is a contradiction to the choice of G. Thus $N(z) \cap V_2 \neq \emptyset$ for any $z \in V_2$.

Choose a vertex $u \in V_2$ such that $|N(u) \cap V_2|$ is as small as possible. Let $N(u) = \{u_1, u_2\}$ with $u_1 \in V_2$ and $d_{u_2} \geq 2$.

Claim 1. $N(u_1) \cap N(u_2) \setminus \{u\} \neq \emptyset$.

Suppose that the claim is false. Then $G_1 = G - u + u_1u_2$ is not a counterexample, i.e., $R^+(G_1) \geq \frac{2(n-3)}{\sqrt{n-1}}$. It is easily seen that $\frac{2(n-3)}{\sqrt{n-1}} - \frac{2(n-2)}{\sqrt{n}}$ is increasing for $n \geq 11$, implying that $\frac{2(n-3)}{\sqrt{n-1}} - \frac{2(n-2)}{\sqrt{n}} \geq \frac{2(11-3)}{\sqrt{11-1}} - \frac{2(11-2)}{\sqrt{11}} > -\frac{1}{2}$. Thus

$$\begin{aligned} R^+(G) &= R^+(G_1) + \frac{1}{\sqrt{2+d_{u_2}}} + \frac{1}{2} - \frac{1}{\sqrt{2+d_{u_2}}} \\ &\geq \frac{2(n-3)}{\sqrt{n-1}} + \frac{1}{2} \\ &> \frac{2(n-2)}{\sqrt{n}}, \end{aligned}$$

which is a contradiction. Claim 1 follows.

Let v be the neighbor of u_1, u_2 different from u. For $u_1 \in V_2$, by Claim 1, we have $N(u_1) \cap N(u_2) = \{u, v\}$. Since G is a triangle-free graph, we have $d_v + d_{u_2} \leq n$.

Claim 2. $v \in V_2$.

Suppose that the claim is false. Suppose that $u_2 \notin V_2$. We have $3 \leq d_v, d_{u_2} \leq n - 3$ as G is triangle-free. Since $G_2 = G - u - u_1$ is not a counterexample, we have $R^+(G_2) \geq \frac{2(n-4)}{\sqrt{n-2}}$. Let $g(x) = \frac{2}{\sqrt{x-2}} + \frac{2(x-3)}{\sqrt{x-1}}$ with $x \geq 11$. Then $g''(x) = \frac{3}{2}(x-2)^{-5/2} - \left(\frac{1}{2}x + \frac{5}{2}\right)(x-1)^{-5/2} < 0$, implying that $g(x) - g(x+1)$ is increasing for $x \geq 11$.

By Lemma 2.4, we have $f(d_v, d_{u_2}) \geq f(n-3, n-3)$, and thus

$$\begin{aligned}
R^+(G) &= R^+(G_2) + \frac{1}{2} + \frac{1}{\sqrt{2+d_v}} + \frac{1}{\sqrt{2+d_{u_2}}} + \frac{1}{\sqrt{d_v + d_{u_2}}} \\
&\quad - \frac{1}{\sqrt{d_v + d_{u_2} - 2}} \\
&\quad + \sum_{w \in N(v)\setminus\{u_1,u_2\}} \left(\frac{1}{\sqrt{d_w + d_v}} - \frac{1}{\sqrt{d_w + d_v - 1}} \right) \\
&\quad + \sum_{w \in N(u_2)\setminus\{u,v\}} \left(\frac{1}{\sqrt{d_w + d_{u_2}}} - \frac{1}{\sqrt{d_w + d_{u_2} - 1}} \right) \\
&\geq \frac{2(n-4)}{\sqrt{n-2}} + \frac{1}{2} + \frac{1}{\sqrt{2+d_v}} + \frac{1}{\sqrt{2+d_{u_2}}} + \frac{1}{\sqrt{d_v + d_{u_2}}} \\
&\quad - \frac{1}{\sqrt{d_v + d_{u_2} - 2}} \\
&\quad + (d_v - 2)\left(\frac{1}{\sqrt{2+d_v}} - \frac{1}{\sqrt{1+d_v}} \right) \\
&\quad + (d_{u_2} - 2)\left(\frac{1}{\sqrt{2+d_{u_2}}} - \frac{1}{\sqrt{1+d_{u_2}}} \right) \\
&= \frac{2(n-4)}{\sqrt{n-2}} + \frac{1}{2} + f(d_v, d_{u_2}) \\
&\geq \frac{2(n-4)}{\sqrt{n-2}} + \frac{1}{2} + f(n-3, n-3) \\
&= \frac{2(n-2)}{\sqrt{n}} + \frac{1}{2} + \left(\frac{2}{\sqrt{n-2}} - \frac{2(n-2)}{\sqrt{n}} + \frac{2(n-4)}{\sqrt{n-1}} \right) \\
&\quad + \left(\frac{1}{\sqrt{2(n-3)}} - \frac{1}{\sqrt{2(n-4)}} \right) \\
&= \frac{2(n-2)}{\sqrt{n}} + \frac{1}{2} + (g(n) - g(n+1)) \\
&\quad + \left(\frac{1}{\sqrt{2(n-3)}} - \frac{1}{\sqrt{2(n-4)}} \right) \\
&\geq \frac{2(n-2)}{\sqrt{n}} + \frac{1}{2} + (g(11) - g(11+1)) \\
&\quad + \left(\frac{1}{\sqrt{2\cdot(11-3)}} - \frac{1}{\sqrt{2\cdot(11-4)}} \right)
\end{aligned}$$

$$> \frac{2(n-2)}{\sqrt{n}},$$

which is a contradiction.

Now suppose that $u_2 \in V_2$. Then $3 \le d_v \le n-2$ and $G_3 = G - u - u_1 - u_2$ is not a counterexample, and thus $R^+(G_3) \ge \frac{2(n-5)}{\sqrt{n-3}}$. Let $h(x) = \frac{x-2}{\sqrt{x}}$ with $x \ge 2$. Then $h''(x) = -\frac{3}{2}\left(\frac{1}{6}x+1\right)x^{-5/2} < 0$, implying that $h(x-3) - h(x) = \frac{x-5}{\sqrt{x-3}} - \frac{x-2}{\sqrt{x}}$ and $h(x-3) - h(x-2) = \frac{x-5}{\sqrt{x-3}} - \frac{x-4}{\sqrt{x-2}}$ are both increasing in x. Then

$$\begin{aligned}
R^+(G) &= R^+(G_3) + 1 + \frac{2}{\sqrt{2+d_v}} \\
&\quad + \sum_{w \in N(v)\setminus\{u_1,u_2\}} \left(\frac{1}{\sqrt{d_w+d_v}} - \frac{1}{\sqrt{d_w+d_v-2}}\right) \\
&\ge \frac{2(n-5)}{\sqrt{n-3}} + 1 + \frac{2}{\sqrt{2+d_v}} + (d_v-2)\left(\frac{1}{\sqrt{2+d_v}} - \frac{1}{\sqrt{d_v}}\right) \\
&= \frac{2(n-5)}{\sqrt{n-3}} + 1 + \frac{d_v}{\sqrt{2+d_v}} - \frac{d_v-2}{\sqrt{d_v}} \\
&\ge \frac{2(n-5)}{\sqrt{n-3}} + 1 + \frac{n-2}{\sqrt{(n-2)+2}} - \frac{(n-2)-2}{\sqrt{n-2}} \\
&= \frac{2(n-2)}{\sqrt{n}} + \left(\frac{n-5}{\sqrt{n-3}} - \frac{n-2}{\sqrt{n}}\right) + \left(\frac{n-5}{\sqrt{n-3}} - \frac{n-4}{\sqrt{n-2}}\right) + 1 \\
&\ge \frac{2(n-2)}{\sqrt{n}} + \left(\frac{11-5}{\sqrt{11-3}} - \frac{11-2}{\sqrt{11}}\right) + \left(\frac{11-5}{\sqrt{11-3}} - \frac{11-4}{\sqrt{11-2}}\right) + 1 \\
&> \frac{2(n-2)}{\sqrt{n}},
\end{aligned}$$

which is a contradiction. Claim 2 follows.

Claim 3. $u_2 \notin V_2$.

Suppose that the claim is false. Then $G_4 = G - u - u_1 - u_2 - v$ is not a counterexample. It is easily seen that $\frac{2(n-6)}{\sqrt{n-4}} - \frac{2(n-2)}{\sqrt{n}} \ge \frac{2\cdot(11-6)}{\sqrt{11-4}} - \frac{2\cdot(11-2)}{\sqrt{11}} > -2$, and thus $R^+(G) = R^+(G_4) + 2 \ge \frac{2(n-6)}{\sqrt{n-4}} + 2 > \frac{2(n-2)}{\sqrt{n}}$, which is a contradiction.

By Claims 2 and 3, we have $v \in V_2$ and $3 \le d_{u_2} \le n-2$ as G is triangle-free. Now we will complete our proof by considering the following two cases.

Case 1. $d_{u_2} \ge 4$. Then $G - u - u_1 - v$ is not a counterexample. Replacing u_2 by v in the proof of the Claim 2 for the case $u_2 \in V_2$, we may derive a contradiction.

Case 2. $d_{u_2} = 3$. Let $N(u_2) \setminus \{u, v\} = \{x\}$ and $y \in N(x) \setminus \{u_2\}$. If $d_x = 2$, then $N(y) \cap N(u_2) \setminus \{x\} = \emptyset$ as $d_u = d_v = 2$. Thus we can derive a contradiction by the same argument as in the proof of Case 1 by setting $u_2 = y$. Hence $d_x \ge 3$. Note that $G_5 = G - u - u_1 - u_2 - v$ is not a counterexample. Then $R^+(G_5) \ge \frac{2(n-6)}{\sqrt{n-4}}$. Since $\frac{1}{\sqrt{3+d}} - \frac{1}{\sqrt{2+d}}$ is increasing for $3 \le d \le n-4$, noting the properties of $h(x)$

used above, we have

$$\begin{aligned}
R^+(G) &= R^+(G_5) + 1 + \frac{2}{\sqrt{5}} + \frac{1}{\sqrt{3+d_x}} \\
&\quad + \sum_{w \in N(x)\setminus\{u_2\}} \left(\frac{1}{\sqrt{d_x+d_w}} - \frac{1}{\sqrt{d_x+d_w-1}} \right) \\
&\geq \frac{2(n-6)}{\sqrt{n-4}} + 1 + \frac{2}{\sqrt{5}} + \frac{1}{\sqrt{3+d_x}} \\
&\quad + (d_x-1)\left(\frac{1}{\sqrt{d_x+2}} - \frac{1}{\sqrt{d_x+2-1}} \right) \\
&= \frac{2(n-6)}{\sqrt{n-4}} + 1 + \frac{2}{\sqrt{5}} + \frac{1}{\sqrt{3+d_x}} + \frac{d_x-1}{\sqrt{d_x+2}} - \frac{d_x-1}{\sqrt{d_x+1}} \\
&> \frac{2(n-6)}{\sqrt{n-4}} + 1 + \frac{2}{\sqrt{5}} + \frac{1}{\sqrt{3+d_x}} - \frac{1}{\sqrt{d_x+2}} \\
&\geq \frac{2(n-6)}{\sqrt{n-4}} + 1 + \frac{2}{\sqrt{5}} + \frac{1}{\sqrt{3+3}} - \frac{1}{\sqrt{3+2}} \\
&> \frac{2(n-2)}{\sqrt{n}},
\end{aligned}$$

which is a contradiction.

The proof of our theorem is completed. □

It is easily checked that for the cycle C_{10} (on 10 vertices), $R^+(C_{10}) = \frac{10}{2} < \frac{2(10-2)}{\sqrt{10}}$. Thus the condition $n \geq 11$ in Theorem 2 is necessary.

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