

## STATISTICAL LIMIT SUPERIOR AND LIMIT INFERIOR IN PROBABILISTIC NORMED SPACES

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### Abstract

In this paper we study the concept of statistical limit superior and statistical limit inferior in probabilistic normed spaces. Our results are analogous to the results of Fridy and Orhan [Proc. Amer. Math. Soc. 125(1997), 3625-3631] but proofs are somewhat different and interesting. We also demonstrate through an example how to compute these points in PN-spaces.

## 1 Introduction

In [1] Menger introduced the notion of statistical metric space, now called probabilistic metric space, which is an interesting and important generalization of the notion of a metric space. Later on this notion was developed by many authors, for example [2], [3] and [4]. The notion of probabilistic metric space gives rise to the concept of probabilistic normed space [2] which is an important and useful generalization of the concept of normed space. These two concepts of PM and PN-spaces help us to deal with the fuzzy like situations. The concept of statistical convergence was first introduced by Fast [5] and then studied by many authors. In particular, active researches on this topic were started after the paper of Fridy [6]. This idea was extended for double sequences by Mursaleen and Edely [7]. The idea of statistical convergence in probabilistic normed space has been studied by Karakus [8]. Many of the results in the theory of ordinary convergence have been extended to the theory of statistical convergence. For instance, Fridy [9] introduced the concept of statistical limit points and Fridy and Orhan [10] introduced the statistical analogues of limit superior and limit inferior of a sequence of real numbers. Recently,

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statistical convergence and some of its related concepts for fuzzy numbers have been studied in [11], [12], [13], [14] and [15].

In this paper, we study the concept of statistical limit superior and statistical limit inferior in probabilistic normed space. An example is demonstrated to calculate these points in PN-space. We observe that our results are analogous to the results of Fridy and Orhan but proofs are somewhat different when we deal with these concepts in PN-spaces.

## 2 Preliminaries

Throughout  $\mathbb{N}$  and  $\mathbb{R}$  will denote the sets of positive integers and real numbers respectively. If  $K \subseteq \mathbb{N}$  then  $K_n := \{k \leq n : k \in K\}$ , and  $|K_n|$  denotes the cardinality of  $K_n$ .

**Definition 2.1.** A function  $f : \mathbb{R} \rightarrow \mathbb{R}_0^+$  is called a *distribution function* if it is non-decreasing and left-continuous with  $\inf_{t \in \mathbb{R}} f(t) = 0$  and  $\sup_{t \in \mathbb{R}} f(t) = 1$ . We will denote the set of all distribution functions by  $D$ .

**Definition 2.2.** A binary operation  $* : [0, 1] \times [0, 1] \rightarrow [0, 1]$  is said to be a continuous *t-norm* if it satisfies the following conditions:

- (a)  $*$  is associative and commutative,
- (b)  $*$  is continuous,
- (c)  $a * 1 = a$  for all  $a \in [0, 1]$ ,
- (d)  $a * b \leq c * d$  whenever  $a \leq c$  and  $b \leq d$  for each  $a, b, c, d \in [0, 1]$ .

For example,  $a * b = \max\{a + b - 1, 0\}$ ,  $a * b = ab$  and  $a * b = \min\{a, b\}$  on  $[0, 1]$  are t-norms.

**Definition 2.3 [8].** A triplet  $(X, N, *)$  is called a probabilistic normed space (in short *PN-space*) if  $X$  is a real vector space,  $N : X \rightarrow D$  (for  $x \in X$ , the distribution function  $N(x)$  is denoted by  $N_x$ , and  $N_x(t)$  is the value of  $N_x$  at  $t \in \mathbb{R}$ ) and  $*$  a continuous t-norm satisfying the following conditions:

- (i)  $N_x(0) = 0$ ,
- (ii)  $N_x(t) = 1$  for all  $t > 0$  if and only if  $x = 0$ ,
- (iii)  $N_{\alpha x}(t) = N_x(\frac{t}{|\alpha|})$  for all  $\alpha \in \mathbb{R} - \{0\}$ ,
- (iv)  $N_{x+y}(s+t) \geq N_x(s) * N_y(t)$  for all  $x, y \in X$  and  $s, t \in \mathbb{R}_0^+$ .

**Example.** Suppose that  $(X, \|\cdot\|)$  is a normed space. Let  $\mu \in D$  with  $\mu(0) = 0$  and  $\mu \neq h$ , where

$$h(t) = \begin{cases} 0, & t \leq 0, \\ 1, & t > 0. \end{cases}$$

Define

$$N_x(t) = \begin{cases} h(t), & x = 0, \\ \mu\left(\frac{t}{\|x\|}\right), & x \neq 0, \end{cases}$$

where  $x \in X$ ,  $t \in \mathbb{R}$ . Then  $(X, N, *)$  is a PN-space. For example, if we define functions  $\mu$  and  $\mu'$  on  $\mathbb{R}$  by

$$\mu(x) = \begin{cases} 0, & x \leq 0, \\ \frac{x}{1+x}, & x > 0, \end{cases}$$

and

$$\mu'(x) = \begin{cases} 0, & x \leq 0, \\ \exp\left(\frac{-1}{x}\right), & x > 0, \end{cases}$$

then we obtain the following well known probabilistic norms

$$N_x(t) = \begin{cases} h(t), & x = 0, \\ \frac{t}{t+\|x\|}, & x \neq 0, \end{cases}$$

and

$$N'_x(t) = \begin{cases} h(t), & x = 0, \\ \exp\left(\frac{-\|x\|}{t}\right), & x \neq 0. \end{cases}$$

**Definition 2.4 [8].** Let  $(X, N, *)$  be a PN-space. Then a sequence  $x = (x_n)$  is said to be convergent to  $L$  with respect to the probabilistic norm  $N$  if for every  $\epsilon > 0$  and  $\lambda \in (0, 1)$ , there exists a positive integer  $k_\circ$  such that  $N_{x_n-L}(\epsilon) > 1 - \lambda$  whenever  $n \geq k_\circ$ . It is denoted by  $N\text{-}\lim x = L$  or  $x_n \xrightarrow{N} L$  as  $n \rightarrow \infty$ .

**Remark 1 [16].** Let  $(X, \|\cdot\|)$  be a real normed space, and  $N_x(t) = \frac{t}{t+\|x\|}$ , where  $x \in X$  and  $t \geq 0$ . Then  $x_n \xrightarrow{N} x$  if and only if  $x_n \xrightarrow{\|\cdot\|} x$ .

**Definition 2.5 [8].** Let  $(X, N, *)$  be a PN-space. Then a sequence  $x = (x_n)$  is said to be a Cauchy sequence with respect to the probabilistic norm  $N$  if for every  $\epsilon > 0$  and  $\lambda \in (0, 1)$  there exists a positive integer  $k_\circ$  such that  $N_{x_n-x_m}(\epsilon) > 1 - \lambda$  for all  $n, m \geq k_\circ$ .

**Definition 2.6 [6].** If  $K$  is a subset of  $\mathbb{N}$ , then the natural density of  $K$  denoted by  $\delta(K)$ , is defined by

$$\delta(K) := \lim_n \frac{1}{n} |\{k \leq n : k \in K\}|$$

whenever the limit exists. The natural density may not exist for each set  $K$ . But the *upper density*  $\bar{\delta}$  always exists for each set  $K$  identified as follows:

$$\bar{\delta}(K) := \limsup_n \frac{1}{n} |\{k \leq n : k \in K\}|.$$

**Definition 2.7 [6].** A sequence  $x = (x_n)$  of numbers is said to be statistically convergent to  $L$  if

$$\delta(\{k \in \mathbb{N} : |x_k - L| \geq \epsilon\}) = 0$$

for every  $\epsilon > 0$ . In this case we write  $st - \lim x = L$ .

**Definition 2.8 [10].** A sequence  $x = (x_n)$  of numbers is said to be statistically bounded if there is a number  $B$  such that

$$\delta(\{k \in \mathbb{N} : |x_k| > B\}) = 0.$$

**Definition 2.9 [8].** The real number sequence  $x$  is said to be statistically bounded with respect to the probabilistic norm  $N$  if there exists some  $t_0 \in \mathbb{R}$  and  $b \in (0, 1)$  such that  $\delta(\{k : N_{x_k}(t_0) \leq 1 - b\}) = 0$ .

**Definition 2.10 [8].** Let  $(X, N, *)$  be a PN-space. We say that a sequence  $x = (x_k)$  is statistically convergent to  $L \in X$  with respect to the probabilistic norm  $N$  provided that for every  $\epsilon > 0$  and  $b \in (0, 1)$

$$\delta(\{k \in \mathbb{N} : N_{x_k - L}(\epsilon) \leq 1 - b\}) = 0,$$

In this case we write  $st_N - \lim x = L$ , where  $L = st_N - \lim x$ .

**Definition 2.11 [8].** Let  $(X, N, *)$  be a PN-space.  $l \in X$  is called a limit point of the sequence  $x = (x_k)$  with respect to the probabilistic norm  $N$  provided that there is a subsequence of  $x$  that converges to  $l$  with respect to the probabilistic norm  $N$ . Let  $L_N(x)$  denote the set of all limit points of the sequence  $x$  with respect to the probabilistic norm  $N$ .

**Definition 2.12 [8].** If  $\{x_{k(j)}\}$  is a subsequence of  $x = (x_k)$  and  $K := \{k(j) : j \in \mathbb{N}\}$ , then we abbreviate  $\{x_{k(j)}\}$  by  $\{x\}_K$ . If  $\delta(K) = 0$  then  $\{x\}_K$  is called a subsequence of density zero or a thin subsequence. On the other hand,  $\{x\}_K$  is a nonthin subsequence of  $x$  if  $K$  does not have density zero.

**Definition 2.13 [8].** Let  $(X, N, *)$  be a PN-space. Then  $\xi \in X$  is called a statistical limit point of the sequence  $x = (x_k)$  with respect to the probabilistic norm  $N$  provided that there is a nonthin subsequence of  $x$  that converges to  $\xi$  with respect to the probabilistic norm  $N$ . In this case we say  $\xi$  is an  $st_N$ -limit point of sequence  $x = (x_k)$ . Let  $\Lambda_N(x)$  denote the set of all  $st_N$ -limit points of the sequence  $x$ .

**Definition 2.14 [8].** Let  $(X, N, *)$  be a PN-space. Then  $\eta \in X$  is called a statistical cluster point of the sequence  $x = (x_k)$  with respect to the probabilistic norm  $N$  provided that for every  $\epsilon > 0$  and  $a \in (0, 1)$ ,

$$\bar{\delta}(\{k \in \mathbb{N} : N_{x_k - \eta}(\epsilon) > 1 - a\}) > 0.$$

In this case we say  $\eta$  is an  $st_N$ -cluster point of the sequence  $x$ . Let  $\Gamma_N(x)$  denote the set of all  $st_N$ -cluster points of the sequence  $x$ .

### 3 Statistical Limit Superior and Inferior

In this section we define the concept of statistical limit superior and statistical limit inferior in probabilistic normed spaces and demonstrate through an example how to compute these points in a PN-space.

**Definition 3.1.** The real number sequence  $x$  is said to be bounded with respect to the probabilistic norm  $N$  if there exists some  $t_o \in \mathbb{R}$  and for every  $b \in (0, 1)$  such that  $N_{x_k}(t_o) > 1 - b$  for all  $k$ .

For a real sequence  $x$  let us define the sets  $B_x^N$  and  $A_x^N$  by

$$B_x^N := \{b \in (0, 1) : \delta(\{k : N_{x_k}(\epsilon) < 1 - b\}) \neq 0\}$$

$$A_x^N := \{a \in (0, 1) : \delta(\{k : N_{x_k}(\epsilon) > 1 - a\}) \neq 0\}$$

Note that throughout this paper the statement  $\delta(\{K\}) \neq 0$  means that either  $\delta(\{K\}) > 0$  or  $K$  does not have natural density.

**Definition 3.2.** If  $x$  is a real number sequence then the statistical limit superior of  $x$  with respect to the probabilistic norm  $N$  is defined by

$$st_N - \lim \sup x := \begin{cases} \sup B_x^N & \text{if } B_x^N \neq \emptyset, \\ 0 & \text{if } B_x^N = \emptyset. \end{cases}$$

Also, the statistical limit inferior of  $x$  with respect to the probabilistic norm  $N$  is defined by

$$st_N - \lim \inf x := \begin{cases} \inf A_x^N & \text{if } A_x^N \neq \emptyset, \\ 1 & \text{if } A_x^N = \emptyset. \end{cases}$$

**Example.** A simple example will help to illustrate the concepts just defined. Let  $X = \mathbb{R}$  and  $N_x(t) = \frac{t}{t+|x|}$ . Define the sequence  $x = (x_k)$  by

$$x_k := \begin{cases} 2k, & \text{if } k \text{ is an odd square,} \\ -1, & \text{if } k \text{ is an even square,} \\ 1/2, & \text{if } k \text{ is an odd nonsquare,} \\ 0, & \text{if } k \text{ is an even nonsquare.} \end{cases}$$

This is clearly unbounded sequence with respect to  $N$ . On contrary, let  $x$  be bounded. Then

$$N_{x_k}(t_o) > 1 - b$$

for all  $k$  and for some  $b \in (0, 1)$ ,  $t_o \in \mathbb{R}$ . Then we have  $\frac{t_o}{t_o + x_k} > 1 - b$  for all  $k$  and for some  $b \in (0, 1)$  and  $t_o \in \mathbb{R}$ . This implies that  $x_k < \frac{bt_o}{1-b}$  for all  $k$  and for some  $b \in (0, 1)$  and  $t_o \in \mathbb{R}$ . But  $x_k = 2k$  if  $k$  is an odd square. Therefore  $\sqrt{2k} < \frac{bt_o}{1-b}$  for all  $k = m^2$  where  $m$  is odd, which is not possible. Hence  $x$  must be unbounded with respect to  $N$ .

On the other hand it is statistically bounded with respect to  $N$ . For this

$$\begin{aligned} \delta(\{k \leq n : N_{x_k}(t_o) \leq 1 - b\}) &= \delta(\{k \leq n : \frac{t_o}{t_o + |x_k|} \leq 1 - b\}), \\ &= \delta(\{k \leq n : |x_k| \geq \frac{bt_o}{1-b}\}). \end{aligned}$$

Since  $0 < b < 1$ ,  $\frac{1}{b} - 1 > 0$ . Choose  $t_o = \frac{1-b}{3b}$ . Then  $t_o > 0$  and

$$\begin{aligned} \delta(\{k \leq n : N_{x_k}(t_o) \leq 1 - b\}) &= \delta(\{k \leq n : |x_k| \geq \frac{b}{1-b} \times \frac{1-b}{3b} = \frac{1}{3}\}) \\ &= \delta(\{k \leq n : |x_k| \geq \frac{1}{3}\}) = \lim_{n \rightarrow \infty} \frac{1}{n} \times \sqrt{n} = 0 \end{aligned}$$

Hence it is statistically bounded with respect to  $N$ .

To find  $B_x^N$ , we have to find those  $b \in (0, 1)$  such that

$$\delta(\{k : N_{x_k}(\epsilon) \leq 1 - b\}) \neq 0.$$

Now,

$$\begin{aligned} \delta(\{k : N_{x_k}(\epsilon) \leq 1 - b\}) &= \delta(\{k \leq n : \frac{\epsilon}{\epsilon + |x_k|} \leq 1 - b\}), \\ &= \delta(\{k \leq n : |x_k| \geq \frac{b\epsilon}{1-b}\}). \end{aligned}$$

We can easily choose any  $\epsilon > 0$  as  $\epsilon < \frac{1}{3}(\frac{1}{b} - 1)$  for  $0 < b < 1$ , so that

$$0 < \frac{b\epsilon}{1-b} < \frac{b}{1-b} \times \frac{1-b}{3b} = \frac{1}{3}.$$

Therefore

$$\delta(\{k \leq n : |x_k| \geq \frac{b\epsilon}{1-b}\}) = \delta(\{k \leq n : |x_k| \geq r = \frac{b\epsilon}{1-b}\}),$$

and by the above condition  $r \in (0, 1)$ . Now the number of members of the sequence which satisfy the above condition is always greater than  $n - \frac{n}{2}$  or  $n - \frac{n-1}{2}$  for the case  $n$  is even or odd respectively. Therefore

$$\delta(\{k \leq n : |x_k| \geq r = \frac{b\epsilon}{1-b}\}) > \lim_{n \rightarrow \infty} \frac{1}{n} \times \frac{n}{2} = \frac{1}{2} \text{ or } \lim_{n \rightarrow \infty} \frac{1}{n} \times \frac{n+1}{2} = \frac{1}{2}.$$

Thus

$$\delta(\{k \leq n : |x_k| \geq r = \frac{b\epsilon}{1-b}\}) \neq 0 \text{ for all } b \in (0, 1).$$

Hence  $B_x^N = (0, 1)$ , and  $st_N - \limsup x = 1$ . The above sequence has two subsequences

$$x = (x_{n_i}) \text{ where } x_{n_i} = 1 \text{ for each } n_i \in \{3, 5, 7, 11, 13, \dots\},$$

and

$$x = (x_{n_j}) \text{ where } x_{n_j} = 0 \text{ for each } n_j \in \{2, 6, 8, 10, 12, \dots\},$$

$i, j \in \mathbb{N}$ ; which are of positive density and clearly convergent to 1 and 0 respectively. Therefore  $x$  is not statistically convergent. Similarly we have  $A_x^N = (0, 1)$ . Hence  $st_N - \liminf x = 0$ . Now by applying the definition, we get the set of statistical cluster points of  $x$  as  $\{0, 1\}$ , where  $st_N - \liminf x = \text{least element}$  and  $st_N - \limsup x = \text{greatest element}$  of the above set.

This observation suggests the main idea of our first theorem of the next section.

## 4 Main Results

The following results are analogues of the results due to Fridy and Orhan [10], while the proofs are different which show the technique to work with PN-spaces.

**Theorem 4.1.** *If  $b = st_N - \limsup x$  is finite, then for every positive numbers  $\epsilon$  and  $\gamma$*

$$\delta(\{k : N_{x_k}(\epsilon) < 1 - b + \gamma\}) \neq 0 \text{ and } \delta(\{k : N_{x_k}(\epsilon) < 1 - b - \gamma\}) = 0. \quad (1)$$

*Conversely, if (1) holds for every positive  $\epsilon$  and  $\gamma$  then  $b = st_N - \limsup x$ .*

*Proof.* Let  $b = st_N - \limsup x$  where  $b$  be finite. Then

$$\delta(\{k : N_{x_k}(\epsilon) < 1 - b\}) \neq 0. \quad (2)$$

Since  $N_{x_k}(\epsilon) < 1 - b + \gamma$  for every  $k$  and for any  $\epsilon, \gamma > 0$ ,

$$\delta(\{N_{x_k}(\epsilon) < 1 - b + \gamma\}) \neq 0.$$

Now applying the definition of  $st_N - \limsup x$  we have  $1 - b$  as the least value satisfying (2). Now if possible,

$$N_{x_k}(\epsilon) < 1 - b - \gamma \text{ for some } \gamma > 0.$$

Then  $1 - b - \gamma$  is another value with  $1 - b - \gamma < 1 - b$  which satisfies (2). This observation contradicts the fact that  $1 - b$  is least value which satisfies the above condition. Hence,

$$\delta(\{N_{x_k}(\epsilon) < 1 - b - \gamma\}) = 0 \text{ for every } \gamma > 0.$$

Conversely, if (1) holds for every positive  $\epsilon$  and  $\gamma$ , then

$$\delta(\{k : N_{x_k}(\epsilon) < 1 - b + \gamma\}) \neq 0 \quad \text{and} \quad \delta(\{k : N_{x_k}(\epsilon) < 1 - b - \gamma\}) = 0.$$

Therefore

$$\delta(\{k : N_{x_k}(\epsilon) \leq 1 - b\}) \neq 0 \quad \text{and} \quad \delta(\{k : N_{x_k}(\epsilon) = 1 - b\}) = 0.$$

That is

$$\delta(\{k : N_{x_k}(\epsilon) < 1 - b\}) \neq 0 \quad \text{for every } \epsilon > 0.$$

Hence  $b = st_N - \limsup x$ .

This completes the proof of the theorem.

The dual statement for  $st_N - \liminf x$  can also be proved similarly.

**Theorem 4.1'.** *If  $a = st_N - \liminf x$  is finite, then for every positive number  $\epsilon$  and  $\gamma$*

$$\delta(\{k : N_{x_k}(\epsilon) > 1 - a - \gamma\}) \neq 0 \quad \text{and} \quad \delta(\{k : N_{x_k}(\epsilon) > 1 - a + \gamma\}) = 0. \quad (1')$$

*Conversely, if (1') holds for every positive  $\epsilon$  and  $\gamma$  then  $a = st_N - \liminf x$ .*

**Remark.** From the definition of statistical cluster points in [9] we see that Theorems 4.1 and 4.1' can be interpreted as saying that  $st_N - \limsup x$  and  $st_N - \liminf x$  are the greatest and the least statistical cluster points of  $x$ , respectively.

**Theorem 4.2.** *For any sequence  $x$ ,  $st_N - \liminf x \leq st_N - \limsup x$ .*

*Proof.* First consider the case in which  $st_N - \limsup x = 0$ , which implies that

$$B_x^N = \emptyset.$$

Then for every  $b \in (0, 1)$ ,

$$B_x^N = \delta(\{k : N_{x_k}(\epsilon) < 1 - b\}) = 0,$$

that is

$$\delta(\{k : N_{x_k}(\epsilon) \geq 1 - b\}) = 1.$$

Also for every  $a \in (0, 1)$ , we have

$$\delta(\{k : N_{x_k}(\epsilon) > 1 - a\}) \neq 0.$$

Hence,  $st_N - \liminf x = 0$ .

The case in which  $st_N - \limsup x = 1$ , is trivial.



Suppose that  $b = st_N - \limsup x$ , and  $a = st_N - \liminf x$ ; where  $a$  and  $b$  are finite. Now for given any  $\gamma$ , we show that  $1 - b - \gamma \in A_x^N$ . By Theorem 4.1, we have

$$\delta(\{k : N_{x_k}(\epsilon) < 1 - b - \frac{\gamma}{2}\}) = 0, \quad \text{where } 1 - b = \text{least upper bound of } B_x^N.$$

Therefore

$$\delta(\{k : N_{x_k}(\epsilon) \geq 1 - b - \frac{\gamma}{2}\}) = 1,$$

which in turn gives

$$\delta(\{k : N_{x_k}(\epsilon) > 1 - b - \gamma\}) = 1.$$

Hence,  $1 - b - \gamma \in A_x^N$ .

By definition

$$a = \inf A_x^N,$$

so we conclude that

$$1 - b - \gamma \leq 1 - a.$$

Since  $\gamma$  is arbitrary, we have

$$1 - b \leq 1 - a,$$

that is

$$a \leq b.$$

This completes the proof of the theorem.

**Theorem 4.3.** *In PN-space  $(X, N, *)$  the statistically bounded sequence  $x$  is statistically convergent if and only if*

$$st_N - \liminf x = st_N - \limsup x.$$

*Proof.* Let  $\alpha, \beta$  be  $st_N - \liminf x$  and  $st_N - \limsup x$  respectively. Now we assume that  $st_N - \lim x = L$ . Then for every  $\epsilon > 0$  and  $b \in (0, 1)$ ,

$$\delta(\{k : N_{x_k - L}(\epsilon) \leq 1 - b\}) = 0,$$

so that

$$\delta(\{k : N_{x_k}(\frac{\epsilon}{2}) * N_L(\frac{\epsilon}{2}) \leq 1 - b\}) = 0.$$

Let for every  $\epsilon > 0$ ,

$$\sup_{\epsilon} N_{x_k}(\frac{\epsilon}{2}) = 1 - b_1 \quad \text{and} \quad \sup_{\epsilon} N_L(\frac{\epsilon}{2}) = 1 - b_2$$

such that

$$(1 - b_1) * (1 - b_2) \leq 1 - b. \tag{1}$$

Then

$$\delta(\{k : N_{x_k}(\frac{\epsilon}{2}) \leq 1 - b_1\}) = 0, \tag{2}$$

and therefore

$$\delta(\{k : N_{x_k}(\frac{\epsilon}{2}) < 1 - b_1 - \gamma\}) = 0 \text{ for every } \gamma > 0. \quad (3)$$

Now applying Theorem 4.1 and the definition of  $st_N - \lim \sup x$ , we get

$$\delta(\{k : N_{x_k}(\frac{\epsilon}{2}) < 1 - \beta - \gamma\}) = 0 \text{ for every } \gamma > 0. \quad (4)$$

From (3) and (4) and by the definition of  $st_N - \lim \sup x$ , we get

$$1 - b_1 - \gamma \leq 1 - \beta - \gamma,$$

that is,

$$\beta \leq b_1. \quad (5)$$

Now we find those  $k$  such that

$$N_{x_k}(\frac{\epsilon}{2}) > 1 - b_1 + \gamma.$$

We can easily observe that no such  $k$  exists which satisfy (1) together with the above condition. Therefore this implies that

$$\delta(\{k : N_{x_k}(\frac{\epsilon}{2}) > 1 - b_1 + \gamma\}) = 0.$$

Since  $\alpha = st_N - \lim \inf x$ , by Theorem 4.1', we get

$$\delta(\{k : N_{x_k}(\frac{\epsilon}{2}) > 1 - \alpha + \gamma\}) = 0.$$

By the definition of  $st_N - \lim \inf x$ , we have

$$1 - \alpha + \gamma \leq 1 - b_1 + \gamma,$$

that is,

$$b_1 \leq \alpha. \quad (6)$$

From (4) and (5), we get  $\beta \leq \alpha$ . Now combining Theorem 4.2 and the above inequality, we conclude  $\alpha = \beta$ .

Conversely, suppose that  $\alpha = \beta$  and  $\sup_{\epsilon} N_L(\epsilon) = 1 - \alpha$ . Then for any  $\gamma > 0$ , Theorems 4.1 and 4.1' together imply that

$$\delta(\{k : N_{x_k}(\frac{\epsilon}{2}) < 1 - \alpha - \frac{\gamma}{2}\}) = 0, \quad (7)$$

and

$$\delta(\{k : N_{x_k}(\frac{\epsilon}{2}) > 1 - \alpha + \frac{\gamma}{2}\}) = 0. \quad (8)$$

Now

$$1 - \alpha \geq N_L(\epsilon) = N_{x_k - (x_k - L)}(\epsilon) \geq N_{x_k}(\frac{\epsilon}{2}) * N_{x_k - L}(\frac{\epsilon}{2}).$$

Therefore

$$N_{x_k}(\frac{\epsilon}{2}) * N_{x_k-L}(\frac{\epsilon}{2}) \leq 1 - \alpha. \quad (9)$$

Let

$$\sup_{\epsilon} N_{x_k-L}(\frac{\epsilon}{2}) = 1 - a_1 \quad \text{where } a_1 \in (0, 1) \text{ and (7) and (9) hold.}$$

Then

$$\delta(\{k : N_{x_k-L}(\frac{\epsilon}{2}) < 1 - a_1 - \frac{\gamma}{2}\}) = 0,$$

which is true for all  $\gamma > 0$ . Hence

$$\delta(\{k : N_{x_k-L}(\frac{\epsilon}{2}) \leq 1 - a_1\}) = 0,$$

which is true for all  $a \leq a_1 \in (0, 1)$ , because  $1 - a_1$  is the least upper bound. Now repeat the process by taking (8) and (9) instead of (7) and (9). If (8) and (9) are satisfied, then  $\inf_{\epsilon} N_{x_k-L}(\frac{\epsilon}{2}) = 1 - a_1$ . On contrary suppose that  $1 - a_1 \neq \inf_{\epsilon} N_{x_k-L}(\frac{\epsilon}{2})$  while conditions (8) and (9) hold. This implies that there exists some  $t$  in  $\{N_{x_k-L}(\frac{\epsilon}{2}) : \epsilon > 0 \text{ is arbitrary}\}$  such that  $t > 1 - a_1$ . Let us suppose  $\inf_{\epsilon} N_{x_k-L}(\frac{\epsilon}{2}) = 1 - a_2$ . Then, we have

$$1 - a_2 > 1 - a_1, \quad (10)$$

and by (9), we get

$$N_{x_k}(\frac{\epsilon}{2}) * (1 - a_2) \leq 1 - \alpha.$$

Using (8) we get,

$$(1 - \alpha + \frac{\gamma}{2}) * (1 - a_2) \leq 1 - \alpha, \quad \text{for all } \gamma > 0.$$

Clearly,

$$(1 - \alpha - \frac{\gamma}{2}) * (1 - a_2) \leq 1 - \alpha. \quad (11)$$

Now

$$1 - a_1 = \sup_{\epsilon} N_{x_k-L}(\frac{\epsilon}{2}) \quad \text{where } a_1 \in (0, 1) \text{ and which satisfy (7) and (9).}$$

From (11) we conclude that  $1 - a_2$  is another value satisfying (7) and (9). Hence  $1 - a_2 < 1 - a_1$ . which contadicts (10). Hence  $1 - a_1 = \inf_{\epsilon} N_{x_k-L}(\frac{\epsilon}{2})$  satisfying conditions (8) and (9). Therefore the inequality becomes true for all  $a \geq a_1 \in (0, 1)$ , because  $1 - a_1$  is the greatest lower bound, and hence

$$\delta(\{k : N_{x_k-L}(\frac{\epsilon}{2}) \leq 1 - a\}) = 0,$$

for each  $\epsilon > 0$  and  $a \in (0, 1)$ . Therefore  $st_N - \lim x = L$ .

This completes the proof of the theorem.

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