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ON 1-GAPS IN 3D DIGITAL OBJECTS

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Abstract

In Digital Geometry, a gap is a location of a digital object through which a discrete ray can penetrate with no intersection. More specifically, for a 3D digital object we distinguish between 0- and 1-gaps depending on the relative position of such a ray. Although in some applications it is important to know how many gaps has a set of voxels, it is quite complicated to find an efficient algorithm to directly count them. In this paper, we provide a formula that states the number of 1-gaps of a generic 3D object using the notion of free cell of dimension 1 and 2.

1 Introduction

Several definitions of gap are available in the literature (see, e.g., [2, 3, 9]). Roughly speaking, a *gap* is a location of a digital object that can be locally penetrated by some discrete path (usually called *ray*). In particular, the definition of a gap in a 3D digital space finds several useful applications in fields like computer aided design (CAD) and computer graphics, where it is relevant to know whether an apparently "solid" surface can have some "unreal" (or "immaterial") holes, and it is useful to understand the type of gap we are dealing with.

It is worth remembering that, at the moment, there is no efficient algorithm that can directly compute the number of gaps. For such a reason, it is necessary to reconduct the computation by means of some parameters which are relatively easier to count. Following this approach, in some recent works [5, 7] two equivalent formulas which gives the total number of gaps of a generic 2D digital object D by means of some basic parameters of the object were given.

In this paper, we extend to 3-dimensional case some results obtained in [7]. More precisely, we provide a formula that states the number of 1-gaps of a generic 3D object using the notion of free cell.

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In the next section, we introduce some fundamental notions and notations of Digital Geometry that we use throughout the paper. In Section 3, some basic definitions and propositions dealing with the notion of 1-gap and free cells are presented. In Section 4, we prove our main result, consisting in a formula which counts the number of 1-gap of a generic 3D digital object.

2 Preliminaries

Digital Geometry studies the geometric and topological properties of digital objects, i.e. collections of pixels (if we consider digital plane) or voxels (if we consider digital space). In this paper, we choose to represent digital objects by means of the *grid cell model* $\mathbb{C}_3$ (also referred as *cellular model*), originally introduced by Alexandroff and Hopf [1], adopting the notation and the terminology used in [10] and [12].

In the Euclidean space $\mathbb{R}^3$, we can consider some families of sets: cubes, faces, edges, and vertices which are generically called *cells*. More precisely, the cube centered at a point $p = (p_i) \in \mathbb{Z}^3$ is defined by $\prod_{i=1}^{3} \left[p_i - \frac{1}{2}, p_i + \frac{1}{2}\right]$. Such a set is called a 3-cell of $\mathbb{R}^3$ or, simply, a voxel. The set of all 3-cells of $\mathbb{R}^3$ is denoted by $\mathbb{C}_3^{(3)}$. The faces of every cube, i.e. the squares bounding the 3-cells, are called 2-cells of $\mathbb{R}^3$. The set of all 2-cells of $\mathbb{R}^3$ is denoted by $\mathbb{C}_3^{(2)}$. Finally, edges and vertices which are sides and points of the 3-cells are called 1- and 0-cells of $\mathbb{R}^3$, respectively. The set of all 1-cells (resp. 0-cells) of $\mathbb{R}^3$ is denoted by $\mathbb{C}_3^{(1)}$ (resp. $\mathbb{C}_3^{(0)}$). The digital 3-dimensional space considered as a cellular model is denoted by $\mathbb{C}_3$ and it is the union of all k-cells of $\mathbb{R}^3$ (with $k \in \{0, 1, 2, 3\}$), that is we set $\mathbb{C}_3 = \bigcup_{i=0}^{3} \mathbb{C}_3^{(i)}$. A *digital object* D is a finite subset of $\mathbb{C}_3^{(3)}$. We denote by $c_k(D)$ ($k = 0, 1, 2, 3$) (or simply by c_k if no confusion arises) the number of k-cells of D.

Two voxels v, v' are *k-adjacent*, $k = 0, 1, 2$ iff $v \neq v'$ and $v \cap v' \subset \mathbb{C}_3^{(k)}$, that is iff they are distinct, and they share at least a k-cell. The symmetric, irreflexive relation of k-adjacency is denoted by A_k. Given a voxel v, the set of all voxels which are k-adjacent to v is denoted by $A_k(v)$, and it is called the *k-adjacent neighborhood* of v. If two voxels v, v' are k-adjacent for some $k = 0, 1, 2$, we simply say that they are adjacent.

Two voxels v and v' are *strictly k-adjacent* (with $k = 0, 1$) iff v and v' are k-adjacent but not j-adjacent, for any $j > k$. This is equivalent to say that v, v' share exactly one k-cell, i.e. $v \cap v' \in \mathbb{C}_3^{(k)}$. Finally, we say that two cells e and e' are *incident* each other, and we write eIe', iff either $e \subseteq e'$ or $e' \subseteq e$.

3 Basic Propositions

In this section, some basic definitions and propositions dealing with the notion of 1-gap and free cells are presented. These notions are the 3D generalization of the ones given in [5] and [7]. Let us start by introducing some particular configurations of voxel.

Definition 1. Let e be a k-cell (with $0 \leq k \leq 2$) of $\mathbb{C}_3$. A k-block centered in e is the union of all voxels containing e, i.e. $B_k(e) = \bigcup\{v \in \mathbb{C}_3^{(3)} : e \subset v\}$.

Let us note that, for any k-cell e, $B_k(e)$ is the union of exactly 2^{3-k} voxels and $e \in B_k(e)$.

Definition 2. An L-block is, up to symmetry, a 1-block from which we take away a voxel.

Definition 3. Let D be a digital object, v_1, v_2 be two voxels of D, and e be a k-cell, with $k = 0, 1, 2$. We say that $\{v_1, v_2\}$ forms a k-tandem of D over e if $D \cap B_k(e) = \{v_1, v_2\}$, v_1 and v_2 are strictly k-adjacent and $v_1 \cap v_2 = e$.

Definition 4. Let D be a digital object of $\mathbb{C}_3$ and e be a k-cell. We say that D has a k-gap ($k = 0, 1$) over e if there is a k-block $B_k(e)$ such that $B_k(e) \setminus D$ is a k-tandem over e. The cell e is called the k-hub of the gap. Finally, we denote by $g_k(D)$ (or simply by g_k if no confusion arises) the number of k-gaps of D.

Proposition 1. *A digital object D has a 1-gap over a 1-cell e iff there exist two voxels v_1 and v_2 of D such that:*

1) $e \subset v_1$ *and* $e \subset v_2$,

2) $v_1 \in A_1(v_2) \setminus A_2(v_2)$,

3) $A_2(v_1) \cap A_2(v_2) \cap D = \emptyset$.

Proof. Let us suppose that D has a 1-gap over a 1-cell e. So there exists a 1-block $B_1(e)$ such that $B_1(e) \setminus D$ is a 1-tandem over e. Hence $B_1(e) \setminus D$ is composed by two strictly 1-adjacent voxels v_1, v_2 and $v_1 \cap v_2 = e$. Since v_1 and v_2 are 1-adjacent, we have that $e \subset v_1$ and $e \subset v_2$.
Let suppose, by contradiction, that $v_1 \notin A_1(v_2) \setminus A_2(v_2)$. Then it should be $v_1 \notin A_1(v_2)$ or $v_1 \in A_2(v_2)$. Both expressions lead to a contradiction, since v_1 and v_2 are strictly 1-adjacent.
Finally, let us suppose that $A_2(v_1) \cap A_2(v_2) \cap D \neq \emptyset$. Then it should exist a voxel $v_3 \in D$ such that $v_3 \in A_2(v_1)$ and $v_3 \in A_2(v_2)$. So $\{v_1, v_2, v_3\}$ forms an L-block, and this contradicts the fact that $\{v_1, v_2\}$ is a 1-tandem of D.

Conversely, let $v_1, v_2 \in D$ such that conditions 1), 2) and 3) are verified, and let us suppose by contradiction that, for any 1-cell e belonging to D, the set $E = B_k(e) \setminus D$ is not a 1-tandem over e. Then we have one of the following cases: E is either a k-block (with $k = 0, 1, 2$) or an L-block, but each of these cases contradicts our hypothesis. □

Definition 5. We say that a k-cell e (with $k = 0, 1, 2$) is free if $B_k(e) \nsubseteq D$.

The number of free k-cells (with $k = 0, 1, 2$) of a digital object D is denoted by $c_k^*(D)$ (or simply by c_k^* if no confusion arises). Moreover, we denote by $c'_k(D)$ (or simply by c'_k) the number of non-free k-cells.

Definition 6. Let D be a digital object of $\mathbb{C}_3$. We define the border of D, and we denote it by $\mathrm{bd}(D)$, the set of all cells e of D such that the block $B_k(e)$ (with $k = 0, 1, 2$) meets both D and $\mathbb{C}_3 \setminus D$.

Proposition 2. *A k-cell e of a digital object D is free iff $e \in \mathrm{bd}(D)$.*

Proof. The cell e is free iff $B_k(e) \not\subseteq D$, which is equivalent to $B_k(e) \cap (\mathbb{C}_3 \setminus D) \neq \emptyset$. Since it is also $e \in B_k(e) \cap D \neq \emptyset$, we have that $e \in \mathrm{bd}(D)$. □

Thanks to Proposition 2, we have that the border $bd(D)$ of a digital object D is the set of all free-cells of D. Moreover, c'_k (k=0,1,2) coincides with the number of k-blocks $B_k(e)$ such that $B_k(e) \subseteq D$.

4 Main Results

In this section, we present our main result, that is a formula which establishes a relation between the number of 1-gaps of a generic digital 3-object and the number of cells of its border. To this purpose, we point out the following facts.

Definition 7. Let D be a non-empty digital object in $\mathbb{C}_3$. We call *armor* of D, and we denote it by $\mathcal{A}(D)$, the graph (V, E) whose set of vertices is $V = \mathbb{C}_3^{(2)} \cap \mathrm{bd}(D)$ and the set of edges is $E = \{(e, e') \in V \times V : e \neq e' \text{ and } e \cap e' \in \mathbb{C}_3^{(1)}\}$, that is the set of all pairs of distinct free 2-cells that share a 1-cell.

Lemma 1. *Let D be a digital object and e be a 1-hub. Then the free 2-cells of D sharing e are exactly 4. More precisely, each of the two voxels forming the gap have 2 of such free 2-cells.*

Proof. The number of free 2-cells of D sharing e is 4. Moreover, since the pair of voxels composing the set $B_1(e) \setminus D$ is symmetric respect to e, it follows that each voxel contributes with the half of such free 2-cells, i.e. 2. □

Lemma 2. *Let $\mathcal{A}(D)$ be the armor of a digital object D and e be one of its vertices (i.e. a free 2-cell of D). Then:*

- $\deg(e) = 4$ *(respect to the armor $\mathcal{A}(D)$) iff the free 2-cell e does not incide any 1-hub.*
- $\deg(e) = 6$ *iff the free 2-cell e incides some 1-hub.*

Proof. By Definition 7, $\deg(e)$ (where e is considered as a vertex in the armor $\mathcal{A}(D)$) coincides with the number of all free 2-cells, distinct from e, which share a 1-cell with e.
Let us suppose that e does not incide a 1-hub. Then $\deg(e)$ is given by the number of 1-cells of a 2-cell, i.e. $\deg(e) = 4$.
Let us suppose, instead, that e incides a 1-hub e. Then, by Proposition 1, there exist two voxels v_1, v_2 that form the 1-gap. Without loss of generality, let us suppose that e is a 1-cell of v_1. Then $\deg(e)$ is given by the sum of the 4 free 2-cells of v_1, and, by Lemma 1, of the 2 free 2-cells of v_2, i.e $\deg(e) = 6$. □

Lemma 3. *The number of* 2*-cells of a digital object that are incident to some* 1*-hub is* $4g_1$*.*

Proof. It directly follows from Lemma 1. □

Lemma 4. *Let* $\mathcal{H}$ *be the set of all* 1*-hubs of a digital object* D*. Then* $\mathcal{H}$ *generates* $\chi = 6g_1$ *edges of* $A(D)$*.*

Proof. First, let us note that $|\mathcal{H}| = g_1$. Let $e \in \mathcal{H}$. Since e is incident to four 2-cells of D, it generates a number of edges of $A(D)$ equal to the 2-combination of four elements, that is $\binom{4}{2} = 6$. Hence $\chi = 6|\mathcal{H}| = 6g_1$. □

Lemma 5. *Let* $\mathcal{N}$ *be the set of free* 1*-cells that are not* 1*-hubs of* D*. Then* $\mathcal{N}$ *generates* $\eta = c_1^* - g_1$ *edges of* $A(D)$*.*

Proof. Let $e \in \mathcal{N}$. Since only a couple of 2-cells sharing e exists, it follows that e generates $\binom{2}{2} = 1$ edge of $A(D)$. Moreover, $\mathcal{N}$ is the set of free 1-cells that do not belong to $\mathcal{H}$. Hence $\mathcal{N}$ generates $\eta = c_1^* - |\mathcal{H}| = c_1^* - g_1$ edges. □

We are now able to prove the main theorem which expresses the number of 1-gap of a digital object by means of the number of its free cells.

Theorem 1. *Let* D *be a digital object of* $\mathbb{C}_3$*. Then:*

$$g_1 = 2c_2^* - c_1^*.$$

Proof. Let $A(D) = (V, E)$ be the armor of D. Since $A(D)$ is a graph, it is $\sum \deg(v) = 2l$, where $l = |E|$ and $\deg(v)$ is the degree of the vertex $v \in V$. Let $\mathcal{F}$ be the set of free 2-cells that incides some 1-hub. By Lemma 3, it is $|\mathcal{F}| = 4g_1$. Moreover, by Lemma 2, we have

$$\sum \deg(v) = 4(f^* - |\mathcal{F}|) + 6|\mathcal{F}| = 4f^* + 2|\mathcal{F}| = 4f^* + 8g_1.$$

E is the disjoint union of the edges generated by $\mathcal{H}$ and the ones generated by $\mathcal{N}$. Hence $l = |E| = \chi + \eta$, and, thanks to Lemma 4 and 5, we get $l = c_2^* + 5g_1$.
Combining the previous expressions, we obtain $4c_2^* + 8g_1 = 2(c_1^* + 5g_1)$, that is $g_1 = 2c_2^* - c_1^*$. □

Proposition 3. *Let* D *be a 3D digital object with* c_3 *voxels and* c_2' *non-free* 2*-cells. Then, the number of* 2*-cells of* D *is given by* $c_2 = 6c_3 - c_2'$*.*

Proof. First, let us note that there are six 2-cells for each voxel of D. Furthermore, some of these 2-cells, more precisely the ones that are centers of 2-blocks contained in D, are repeated in different 3-cells, and their number coincides with c_2'. Hence, the number of 2-cells of D is given by $c_2 = 6c_3 - c_2'$. □

Corollary 1. *Let* D *be a digital object, and let* c_k $(k = 0, 1, 2, 3)$*, and* B_1 *the number of* k*-cells, and* 1*-blocks of* D*, respectively. Then* $g_1 = 4c_2 - c_1 - 12c_3 + B_1$*.*

Proof. By Proposition 3, we have $2f - 12p + 2f' = 0$. Hence $g_1 = 2c_2^* - c_1^* = 2c_2 - 2c_2' - c_1 + B_1 + 2c_2 - 12c_3 + 2c_2' = 4c_2 - 12c_3 - c_1 + B_1$. □

5 Conclusion, Perspective and Acknowledgements

Although in some applications it is important to know how many gaps has a given digital object, it is well-known that, at the moment, there exists no efficient algorithm for counting them. For such a reason, in the present paper, we have approached the problem and we solved it in an indirect way, finding a formula which expresses the number of 1-gaps of a digital object D by means of the number of free 1- and 2-cells only. Since it is relatively easy to automatically compute the number of free cells of a digital object, that formula simplifies the calculation of total number of 1-gaps in a $3D$-spaces. Such a task can be tackled in several ways, but it always need the use of an appropriate data structure that can save both informations about all the voxels of the digital object and the relative adjacency relation between them. The study of such a data structure and of the algorithm to compute efficiently the number of free cells of a 3D digital object will be the focus of some our forthcoming paper.

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References

[1] P. Alexandroff and H. Hopf, *Topologie, Erster Band: Grundbegriffe der mengentheoretischen Topologie · Topologie der Komplexe · Topologische Invarianzstze und anschließende Begriffsbildungen · Verschlingungen im mdimensionalen euklidischen Raum · stetige Abbildungen von Polyedern*, Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, Band XLV, Verlag von Julius Springer, Berlin, 1935.

[2] E. Andres, R. Acharya and C. Sibata, *Discrete analytical hyperplanes*, Graphical Models and Image Processing **59** (1997), 302-309.

[3] E. Andres, Ph. Nehlig and J. Françon, *Tunnel-free supercover 3D polygons and polyhedra*, in: D. Fellner and L. Szirmay-Kalos (Guest Eds.), EUROGRAPHICS'97, 1997, C3-C13.

[4] F.G. Arenas, *Alexandroff spaces*, Acta Mathematica, University of Comenianae, Vol. LXVIII-1 (1999) 17–25.

[5] V.E. Brimkov, A. Maimone, G. Nordo, R.P. Barneva and R. Klette, *The number of gaps in binary pictures*, in: Lecture Notes in Computer Science (Ed. G. Bebis, R. Boyle, D. Koracin and B. Parvin), Proceedings of the ISVC 2005, Lake Tahoe, NV, USA, Vol. **3804** (2005), 35 - 42.

[6] V.E. Brimkov, A. Maimone and G. Nordo, *An explicit formula for the number of tunnels in digital objects*, ARXIV (2005).

[7] V.E. Brimkov, A. Maimone and G. Nordo, *Counting gaps in binary pictures*, in: Lecture Notes in Computer Science (Ed. R. Reulke, U. Eckardt, B. Flach, U. Knauer, K. Polthier), Proceedings of the 11th International Workshop, IW-CIA 2006, Berlin, GERMANY, LNCS **4040** (2006), 16 - 24.

[8] V.E. Brimkov, *Formulas for the number of (n-2)-gaps of binary objects in arbitrary dimension*, Discrete Applied Mathematics, Elsevier, 1**57**(3) (2009) 452-463.

[9] D. Cohen-Or and A. Kaufman, *3D line voxelization and connectivity control*, IEEE Computer Graphics and Applications **17** (**6**) (1997) 80 - 87.

[10] R. Klette and A. Rosenfeld, *Digital Geometry - Geometric Methods for Digital Picture Analysis*, Morgan Kaufmann, San Francisco, 2004.

[11] V.A. Kovalevsky, *Algorithms and data structures for computer topology*, in: Digital and Image Geometry (Ed. G. Bertrand, A. Imiya, R. Klette), LNCS **2243** (2001), Springer, 37-58.

[12] V.A. Kovalevsky, *Algorithms in digital geometry based on cellular topology*, in: Combinatorial image analysis: 10th international workshop, IWCIA 2004 (ed. R. Klette. and J. Zunic), LNCS **3322**, Springer Verlag, (2004), 366-393.

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