Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Filomat **25:3** (2011), 105–110 DOI: 10.2298/FIL1103105U

UNIVALENCE CRITERION FOR TWO INTEGRAL OPERATORS

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Abstract. In this paper we extend some results obtained by Breaz et al. in [3], for a general integral operators $G_{n,\alpha}(z)$, $G_{\alpha_1,\ldots,\alpha_n,n}(z)$ and $J_{\alpha_1,\alpha_2,\ldots,\alpha_n,\gamma}$.

1 Introduction and preliminaries

Let $\mathcal{U} = \{z : |z| < 1\}$ the unit disc and $\mathcal{A}$ the class of all functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in $\mathcal{U}$. We denote by $\mathcal{S}$ the class of all functions in $\mathcal{A}$ which are univalent in $\mathcal{U}$.

Lemma 1.1. *(***Schwarz Lemma***)[5] Let the analytic function f be regular in the open unit disk $\mathcal{U}$ and let $f(0) = 0$. If $|f(z)| \le 1$, for all $z \in \mathcal{U}$, then*

$$|f(z)| \le |z| \qquad (z \in \mathcal{U}), \tag{2}$$

where the equality holds true only if

$$|f(z)| = Kz \qquad (z \in \mathcal{U}) \quad \textit{and} \quad K = 1. \tag{3}$$

Pescar has proved the following univalent condition:

Theorem 1.1. *[7] Let $\alpha \in \mathbb{C}$ $(\mathrm{Re}(\alpha) > 0)$ and $c \in \mathbb{C}$ $(|c| \le 1; c \ne -1)$. Suppose also that the function $f(z)$ given by (1) is analytic in $\mathcal{U}$. If*

$$\left| c|z|^{2\alpha} + (1 - |z|^{2\alpha}) \frac{zf''(z)}{\alpha f'(z)} \right| \le 1 \qquad (z \in \mathcal{U}),$$

2010 *Mathematics Subject Classifications.* 30C45.
Key words and Phrases. Analytic, univalent, unit disk, integral operators, univalent functions.
Received: January 14, 2011
Communicated by H. M. Srivastava

then the function $F_\alpha(z)$ *defined by*

$$F_\alpha(z) := \left(\alpha \int_0^z t^{\alpha-1} f'(t)dt\right)^{\frac{1}{\alpha}} = z + \dots$$

is analytic and univalent in $\mathcal{U}$.

The following theorem is another univalent condition which was proved by Ozaki and Nunokawa [6]:

Theorem 1.2. *[6] Let* $f \in \mathcal{A}$ *satisfy the following inequality:*

$$\left|\frac{z^2 f'(z)}{(f(z))^2} - 1\right| \leq 1 \qquad (z \in \mathcal{U}) \tag{4}$$

then f *is univalent in* $\mathcal{U}$.

D. Breaz et al. in [3] proved next theorem:

Theorem 1.3. *[3] Let* $M \geq 1$ *and suppose that each of the functions* $g_j \in \mathcal{A}, (j \in \{1, \dots, n\})$ *satisfies the inequality (4). Also let* $\alpha \in \mathbb{R}$, $\left(\alpha \in \left[1, \frac{(2M+1)n}{(2M+1)n-1}\right]\right)$ *and* $c \in \mathbb{C}$. *If*

$$|c| \leq 1 + \left(\frac{1-\alpha}{\alpha}\right)(2M+1)n$$

and

$$|g_j(z)| \leq M \ (z \in \mathcal{U}; j \in \{1, \dots, n\}),$$

then the function

$$G_{n,\alpha}(z) = \left((n(\alpha-1)+1) \int_0^z (g_1(t))^{\alpha-1} \dots (g_n(t))^{\alpha-1} dt\right)^{\frac{1}{n(\alpha-1)+1}} \tag{5}$$

is in the univalent function class $\mathcal{S}$.

2 Main results

Theorem 2.1. *Let* $g_i \in \mathcal{A}$ *for* $i \in \{1, \dots, n\}$ *all the functions which satisfies the inequality (4) and* $M_i \geq 1$.
We consider $\alpha_i \in \mathbb{R}$ $\left(\alpha_i \in \left[1, \frac{1}{n} \cdot \max\left\{\frac{(2M_i+1)n}{(2M_i+1)n-1} - 1\right\}_{i=\overline{1,n}} + 1\right]\right)$ *and* $c \in \mathbb{C}$.
If

$$|c| \leq 1 + \frac{n}{n(\sum\limits_{i=1}^{n} \alpha_i - 1) + 1} \cdot \max_{1 \leq i \leq n} (\alpha_i - 1)(2M_i + 1) \tag{6}$$

and

$$|g_i(z)| \le M_i \qquad (z \in \mathcal{U}, i \in \{1, \dots, n\}), \tag{7}$$

then the function

$$G_{\alpha_1,\dots,\alpha_n,n}(z) = \left(\left(n \left(\sum_{i=1}^{n} \alpha_i - 1 \right) + 1 \right) \int_0^z (g_1(t))^{\alpha_1 - 1} \dots (g_n(t))^{\alpha_n - 1} dt \right)^{\frac{1}{n(\sum\limits_{i=1}^{n} \alpha_i - 1) + 1}} \tag{8}$$

belongs to the univalent function class $\mathcal{S}$.

Proof. We consider the function

$$f(z) = \int_0^z \prod_{i=1}^{n} \left(\frac{g_i(t)}{t} \right)^{\alpha_i - 1} dt.$$

From here we have that

$$\frac{zf''(z)}{f'(z)} = \sum_{i=1}^{n} (\alpha_i - 1) \left(\frac{zg_i'(z) - g_i(z)}{g_i(z)} - 1 \right)$$

So

$$\left| c|z|^{2(n(\sum\limits_{i=1}^{n} \alpha_i - 1) + 1)} + (1 - |z|^{2(n(\sum\limits_{i=1}^{n} \alpha_i - 1) + 1)}) \frac{zf''(z)}{(n(\sum\limits_{i=1}^{n} \alpha_i - 1) + 1) f'(z)} \right|$$

$$= \left| c|z|^{2(n(\sum\limits_{i=1}^{n} \alpha_i - 1) + 1)} + (1 - |z|^{2(n(\sum\limits_{i=1}^{n} \alpha_i - 1) + 1)}) \frac{1}{(n(\sum\limits_{i=1}^{n} \alpha_i - 1) + 1)} \sum_{i=1}^{n} (\alpha_i - 1) \left(\frac{zg_i'(z)}{g_i(z)} - 1 \right) \right|$$

$$\le |c| + \left(\frac{1}{n(\sum\limits_{i=1}^{n} \alpha_i - 1) + 1} \right) \cdot \sum_{i=1}^{n} \left(\left| \frac{z^2 g_i'(z)}{(g_i(z))^2} \right| \cdot \frac{|g_i(z)|}{|z|} + 1 \right).$$

From (7), because $|g_i(z)| \le M_i$ for $i \in \{1, \dots, n\}$, and (4) we obtain that

$$\left| c|z|^{2(n(\sum\limits_{i=1}^{n} \alpha_i - 1) + 1)} + (1 - |z|^{2(n(\sum\limits_{i=1}^{n} \alpha_i - 1) + 1)}) \frac{zf''(z)}{(n(\sum\limits_{i=1}^{n} \alpha_i - 1) + 1) f'(z)} \right|$$

$$\le |c| + \frac{1}{(n(\sum\limits_{i=1}^{n} \alpha_i - 1) + 1)} \sum_{i=1}^{n} (\alpha_i - 1)(2M_i + 1)$$

$$\le |c| + \frac{n}{(n(\sum\limits_{i=1}^{n} \alpha_i - 1) + 1)} \max_{1 \le i \le n} (\alpha_i - 1)(2M_i + 1).$$

According with (6), we have

$$\left| c|z|^{2(n(\sum\limits_{i=1}^{n} \alpha_i - 1) + 1)} + (1 - |z|^{2(n(\sum\limits_{i=1}^{n} \alpha_i - 1) + 1)}) \frac{zf''(z)}{(n(\sum\limits_{i=1}^{n} \alpha_i - 1) + 1) f'(z)} \right| \le 1 \qquad (z \in \mathcal{U})$$

Now, applying Theorem 1.1 we obtain that the function $G_{\alpha_1,\dots,\alpha_n,n}(z)$ defined by (8) is in $\mathcal{S}$. $\square$

For $n = 1$ in Theorem 2.1 we obtain:

Corollary 2.1. *Let $g \in \mathcal{A}$ a function that satisfies the inequality (4) and $M \geq 1$. We consider $\alpha \in \mathbb{R}\left(\alpha \in \left[1, \frac{2M+1}{2M} + 1\right]\right)$ and $c \in \mathbb{C}$.*
If

$$|c| \leq 1 + \frac{\alpha - 1}{\alpha}(2M + 1)$$

and $|g(z)| \leq M$ for all $z \in \mathcal{U}$, then the function

$$G_\alpha(z) = \left(\alpha \int_0^z (g(t))^{\alpha-1} dt\right)^{\frac{1}{\alpha}}$$

is in the univalent function class $\mathcal{S}$.

For $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \alpha$ we obtain

Corollary 2.2. *Let $g_i \in \mathcal{A}$ for $i \in \{1, \ldots, n\}$ all the functions which satisfies the inequality (4) and $M_i \geq 1$.*
We consider $\alpha \in \mathbb{R}\left(\alpha \in \left[1, \frac{1}{n} \cdot \max\left\{\frac{(2M_i+1)n}{(2M_i+1)n-1} - 1\right\}_{i=\overline{1,n}} + 1\right]\right)$ and $c \in \mathbb{C}$.
If

$$|c| \leq 1 + \frac{n}{n(\alpha - 1) + 1} \cdot \max_{1 \leq i \leq n} (\alpha - 1)(2M_i + 1)$$

and

$$|g_i(z)| \leq M_i \qquad (z \in \mathcal{U}, i \in \{1, \ldots, n\}),$$

then the function $G_{n,\alpha}(z)$ defined by (5) is in $\mathcal{S}$.

The integral operator

$$J_{\alpha_1,\alpha_2,\ldots,\alpha_n,\gamma} = \left(\gamma \int_0^z t^{\gamma-1} \prod_{j=1}^n \left(\frac{f_j(t)}{t}\right)^{\alpha_j} dt\right)^{\frac{1}{\gamma}} \tag{9}$$

was introduced and studied by D. Breaz and N. Breaz in [1].

Theorem 2.2. *Let $f_j \in \mathcal{A}$ for $j \in \{1, \ldots, n\}$ all the functions which satisfies the inequality (4) and $M_j \geq 1$.*
We consider $\alpha_j \in \mathbb{R}$, $\left(\alpha_j \in \left[1, \max\left\{\frac{(2M_j+1)n}{(2M_j+1)n-1}\right\}_{j=\overline{1,n}}\right]\right)$ and $\gamma, c \in \mathbb{C}$.
If

$$|c| \leq 1 + \frac{n}{|\gamma|} \cdot \max_{1 \leq j \leq n} \alpha_j(2M_j + 1) \tag{10}$$

and

$$|f_j(z)| \leq M_j \qquad (z \in \mathcal{U}, j \in \{1, \ldots, n\}), \tag{11}$$

then the function $J_{\alpha_1,\ldots,\alpha_n,\gamma}(z)$ given by (9) is in the univalent function class $\mathcal{S}$.

Proof. We define the function

$$h(z) = \int_0^z \prod_{j=1}^{n} \left(\frac{f_j(t)}{t} \right)^{\alpha_j} dt.$$

Then from here we have that

$$\frac{h''(z)}{h'(z)} = \sum_{j=1}^{n} \alpha_j \left(\frac{z f_j'(z) - f_j(z)}{f_j(z)} \right)$$

So

$$\left| c|z|^{2\gamma} + (1-|z|^{2\gamma}) \frac{zh''(z)}{\gamma h'(z)} \right| = \left| c|z|^{2\gamma} + (1-|z|^{2\gamma}) \frac{1}{\gamma} \sum_{j=1}^{n} \alpha_j \left(\frac{z f_j'(z) - f_j(z)}{f_j(z)} \right) \right|$$

$$\leq |c| + \frac{1}{|\gamma|} \sum_{j=1}^{n} \alpha_j \left(\left| \frac{z^2 f_j'(z)}{(f_j(z))^2} \right| \cdot \frac{|f_j(z)|}{|z|} + 1 \right)$$

Because from (11), $|f_j(z)| \leq M_j$ for $z \in \mathcal{U}$ and $j \in \{1, \ldots, n\}$, using the inequality (4), we obtain that

$$\left| c|z|^{2\gamma} + (1-|z|^{2\gamma}) \frac{zh''(z)}{\gamma h'(z)} \right| \leq |c| + \frac{1}{|\gamma|} \sum_{j=1}^{n} \alpha_j (2M_j + 1)$$

$$\leq |c| + \frac{n}{|\gamma|} \max_{1 \leq j \leq n} \alpha_j (2M_j + 1)$$

Now, using the hypothesis (10) results

$$\left| c|z|^{2\gamma} + (1-|z|^{2\gamma}) \frac{zh''(z)}{\gamma h'(z)} \right| \leq 1, \qquad z \in \mathcal{U}$$

Applying the Theorem 1.1 we obtain that the function $J_{\alpha_1,\ldots,\alpha_n,\gamma}(z)$ is in the univalent functions class $\mathcal{S}$. □

Remark 2.1. *For $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \alpha$ and $\frac{1}{\gamma} = \frac{1}{n(\alpha-1)+1}$ in Theorem 2.2 we obtain the theorem proved by Breaz et al. in [3] for the function $G_{n,\alpha}(z)$.*

Corollary 2.3. *Let $f \in \mathcal{A}$ the function that satisfies the inequality (4) and $M \geq 1$. We suppose that $\alpha \in \mathbb{R}, \left(\alpha \in \left[1, \frac{2M+1}{2M}\right]\right)$ and $\gamma, c \in \mathbb{C}$.*
If

$$|c| \leq 1 + \frac{\alpha}{|\gamma|}(2M+1)$$

and

$$|f(z)| \leq M, \qquad z \in \mathcal{U}$$

then the function $J_{\alpha,\gamma}(z) = \left(\gamma \int_0^z t^{\gamma-1} \left(\frac{f(t)}{t} \right)^{\alpha} dt \right)^{\frac{1}{\gamma}}$ is in the univalent functions class $\mathcal{S}$.

Proof. In Theorem 2.2 we consider $n = 1$. □

Acknowledgment. This work was partially supported by the strategic project POSDRU 107/1.5/S/77265, inside POSDRU Romania 2007-2013 co-financed by the European Social Fund-Investing in People.

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