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SHARP MAXIMAL FUNCTION ESTIMATES AND BOUNDEDNESS FOR COMMUTATORS ASSOCIATED WITH GENERAL INTEGRAL OPERATOR

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Abstract

In this paper, we establish the sharp maximal function estimates for the commutator associated with some integral operator with general kernel and the weighted Lipschitz functions. As an application, we obtain the boundedness of the commutator on weighted Lebesgue, Morrey and Triebel-Lizorkin space. The operator includes Littlewood-Paley operators, Marcinkiewicz operators and Bochner-Riesz operator.

1 Introduction and Preliminaries

As the development of singular integral operators(see [12][28][29]), their commutators have been well studied(see [8][26][27]). In [8][26][27], the authors prove that the commutators generated by the singular integral operators and BMO functions are bounded on $L^p(R^n)$ for $1 < p < \infty$. Chanillo (see [4]) proves a similar result when singular integral operators are replaced by the fractional integral operators. In [5][14][23], the boundedness for the commutators generated by the singular integral operators and Lipschitz functions on Triebel-Lizorkin and $L^p(R^n)(1 < p < \infty)$ spaces are obtained. In [1][13], the boundedness for the commutators generated by the singular integral operators and the weighted BMO and Lipschitz functions on $L^p(R^n)(1 < p < \infty)$ spaces are obtained. In [3], some singular integral operators with general kernel are introduced, and the boundedness for the operators and their commutators generated by BMO and Lipschitz functions are obtained(see [3][15]). The purpose of this paper is to prove the sharp maximal function estimates for the commutator associated with some integral operator with general kernel and the weighted Lipschitz functions. As an application, we obtain the boundedness of the commutator on weighted Lebesgue, Morrey and Triebel-Lizorkin space. The operator includes Littlewood-Paley operators, Marcinkiewicz operators and Bochner-Riesz operator.

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First, let us introduce some notations. Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For any locally integrable function f, the sharp maximal function of f is defined by

$$M^{\#}(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy,$$

where, and in what follows, $f_Q = |Q|^{-1} \int_Q f(x)dx$. It is well-known that (see [12][28])

$$M^{\#}(f)(x) \approx \sup_{Q \ni x} \inf_{c \in C} \frac{1}{|Q|} \int_Q |f(y) - c| dy.$$

Let

$$M(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| dy.$$

For $\eta > 0$, let $M_\eta(f)(x) = M(|f|^\eta)^{1/\eta}(x)$.

For $0 < \eta < 1$ and $1 \le r < \infty$, set

$$M_{\eta,r}(f)(x) = \sup_{Q \ni x} \left(\frac{1}{|Q|^{1-r\eta/n}} \int_Q |f(y)|^r dy \right)^{1/r}.$$

The A_p weight is defined by (see [12]), for $1 < p < \infty$,

$$A_p = \left\{ w \in L^1_{loc}(R^n) : \sup_Q \left(\frac{1}{|Q|} \int_Q w(x)dx \right) \left(\frac{1}{|Q|} \int_Q w(x)^{-1/(p-1)} dx \right)^{p-1} < \infty \right\}$$

and

$$A_1 = \{ w \in L^p_{loc}(R^n) : M(w)(x) \le Cw(x), a.e. \}.$$

The $A(p,r)$ weight is defined by (see [22]), for $1 < p, r < \infty$,

$$A(p,r) = \left\{ w > 0 : \sup_Q \left(\frac{1}{|Q|} \int_Q w(x)^r dx \right)^{1/r} \left(\frac{1}{|Q|} \int_Q w(x)^{-p/(p-1)} dx \right)^{(p-1)/p} < \infty \right\}.$$

Given a non-negative weight function w. For $1 \le p < \infty$, the weighted Lebesgue space $L^p(w)$ is the space of functions f such that

$$||f||_{L^p(w)} = \left(\int_{R^n} |f(x)|^p w(x) dx \right)^{1/p} < \infty.$$

For $\beta > 0$, $p > 1$ and the non-negative weight function w, let $\dot{F}^{\beta,\infty}_p(w)$ be the weighted homogeneous Triebel-Lizorkin space(see [2][23]).

For $0 < \beta < 1$ and the non-negative weight function w, the weighted Lipschitz space $Lip_\beta(w)$ is the space of functions b such that

$$||b||_{Lip_\beta(w)} = \sup_Q \frac{1}{w(Q)^{1+\beta/n}} \int_Q |b(y) - b_Q| dy < \infty.$$

Remark. (1). It has been known that, for $b \in Lip_\beta(w)$, $w \in A_1$ and $x \in Q$,

$$|b_Q - b_{2^kQ}|_{Lip_\beta(w)} \le Ck||b||_{Lip_\beta(w)} w(x) w(2^kQ)^{\beta/n}.$$

(2). Let $b \in Lip_\beta(w)$ and $w \in A_1$. By [11], we know that spaces $Lip_\beta(w)$ coincide and the norms $||b||_{Lip_\beta(w)}$ are equivalent with respect to different values $1 \le p \le \infty$.

In this paper, we will study some integral operators as following(see [3]).

Definition 1. Let $F_t(x,y)$ be defined on $R^n \times R^n \times [0,+\infty)$ and b be a locally integrable function on R^n, set

$$F_t(f)(x) = \int_{R^n} F_t(x,y) f(y) dy$$

and

$$F_t^b(f)(x) = \int_{R^n} (b(x) - b(y)) F_t(x,y) f(y) dy$$

for every bounded and compactly supported function f.

Let H be the Banach space $H = \{h : ||h|| < \infty\}$. For each fixed $x \in R^n$, we view $F_t(f)(x)$ and $F_t^b(f)(x)$ as the mappings from $[0,+\infty)$ to H. Set

$$T(f)(x) = ||F_t(f)(x)||,$$

which T is bounded on $L^2(R^n)$. The commutator related to F_t^b is defined by

$$T^b(f)(x) = ||F_t^b(f)(x)||$$

and F_t satisfies: there is a sequence of positive constant numbers $\{C_k\}$ such that for any $k \ge 1$,

$$\int_{2|y-z|<|x-y|} (||F_t(x,y) - F_t(x,z)|| + ||F_t(y,x) - F_t(z,x)||) dx \le C,$$

and

$$\left(\int_{2^k|z-y| \le |x-y| < 2^{k+1}|z-y|} (||F_t(x,y) - F_t(x,z)|| + ||F_t(y,x) - F_t(z,x)||)^q dy \right)^{1/q} \le C_k (2^k|z-y|)^{-n/q'},$$

where $1 < q' < 2$ and $1/q + 1/q' = 1$.

Definition 2. Let φ be a positive, increasing function on R^+ and there exists a constant $D > 0$ such that

$$\varphi(2t) \le D\varphi(t) \quad \text{for} \quad t \ge 0.$$

Let w be a non-negative weight function on R^n and f be a locally integrable function on R^n. Set, for $1 \le p < \infty$,

$$||f||_{L^{p,\varphi}(w)} = \sup_{x \in R^n,\ d>0} \left(\frac{1}{\varphi(d)} \int_{Q(x,d)} |f(y)|^p w(y) dy \right)^{1/p},$$

where $Q(x,d)=\{y\in R^n:|x-y|<d\}$. The generalized weighted Morrey space is defined by

$$L^{p,\varphi}(R^n,w)=\{f\in L^1_{loc}(R^n):||f||_{L^{p,\varphi}(w)}<\infty\}.$$

If $\varphi(d)=d^\delta$, $\delta>0$, then $L^{p,\varphi}(R^n,w)=L^{p,\delta}(R^n,w)$, which is the classical Morrey spaces (see [24][25]). If $\varphi(d)=1$, then $L^{p,\varphi}(R^n,w)=L^p(R^n,w)$, which is the weighted Lebesgue spaces (see [6]).

As the Morrey space may be considered as an extension of the Lebesgue space, it is natural and important to study the boundedness of the operator on the Morrey spaces (see [6][9][10][16][21]).

It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors (see [26][27]). In [27], Pérez and Trujillo-Gonzalez prove a sharp estimate for the multilinear commutator. The main purpose of this paper is to prove the sharp maximal inequalities for the commutator. As the application, we obtain the weighted L^p-norm inequality, Morrey and Triebel-Lizorkin spaces boundedness for the commutator.

2 Theorems

We shall prove the following theorems.

Theorem 1. Let T be the integral operator as **Definition 1**, the sequence $\{kC_k\}\in l^1$, $w\in A_1$, $0<\beta<1$, $q'\le s<\infty$ and $b\in Lip_\beta(w)$. Then there exists a constant $C>0$ such that, for any $f\in C_0^\infty(R^n)$ and $\tilde{x}\in R^n$,

$$M^\#(T^b(f))(\tilde{x})\le C||b||_{Lip_\beta(w)}w(\tilde{x})^{1+\beta/n}\left(M_{\beta,s}(f)(\tilde{x})+M_{\beta,s}(T(f))(\tilde{x})\right).$$

Theorem 2. Let T be the integral operator as **Definition 1**, the sequence $\{k2^{\beta k}C_k\}\in l^1$, $w\in A_1$, $0<\beta<1$, $q'\le s<\infty$ and $b\in Lip_\beta(w)$. Then there exists a constant $C>0$ such that, for any $f\in C_0^\infty(R^n)$ and $\tilde{x}\in R^n$,

$$\begin{aligned}&\sup_{Q\ni\tilde{x}}\frac{1}{|Q|^{1+\beta/n}}\int_Q\left|T^b(f)(x)-C_0\right|dx\\ \le\;& C||b||_{Lip_\beta(w)}w(\tilde{x})^{1+\beta/n}\left(M_s(f)(\tilde{x})+M_s(T(f))(\tilde{x})\right).\end{aligned}$$

Theorem 3. Let T be the integral operator as **Definition 1**, the sequence $\{kC_k\}\in l^1$, $w\in A_1$, $0<\beta<\min(1,n/q')$, $q'<p<n/\beta$, $1/r=1/p-\beta/n$ and $b\in Lip_\beta(w)$. Then T^b is bounded from $L^p(w)$ to $L^r(w^{r/p-r(1+\beta/n)})$.

Theorem 4. Let T be the integral operator as **Definition 1**, the sequence $\{kC_k\}\in l^1$, $w\in A_1$, $0<D<2^n$, $0<\beta<\min(1,n/q')$, $q'<p<n/\beta$, $1/r=1/p-\beta/n$ and $b\in Lip_\beta(w)$. Then T^b is bounded from $L^{p,\varphi}(w)$ to $L^{r,\varphi}(w^{r/p-r(1+\beta/n)})$.

Theorem 5. Let T be the integral operator as **Definition 1**, the sequence $\{k2^{\beta k}C_k\}\in l^1$, $w\in A_1$, $0<\beta<\min(1,n/q')$, $q'<p<n/\beta$, $1/r=1/p-\beta/n$ and $b\in Lip_\beta(w)$. Then T^b is bounded from $L^p(w)$ to $\dot{F}_r^{\beta,\infty}(w^{r/p-r(1+\beta/n)})$.

3 Proofs of Theorems

To prove the theorems, we need the following lemma.

Lemma 1.(see [3]) Let T be the integral operator as **Definition 1**. Then T is bounded on $L^p(w)$ for $w \in A_\infty$ with $1 < p < \infty$.

Lemma 2.(see [11][13]) For any cube Q, $b \in Lip_\beta(w)$, $0 < \beta < 1$ and $w \in A_1$, we have

$$\sup_{x\in Q} |b(x) - b_Q| \le C||b||_{Lip_\beta(w)} w(Q)^{1+\beta/n}|Q|^{-1}.$$

Lemma 3.(see [2][23]). For $0 < \beta < 1$, $1 < p < \infty$ and $w \in A_\infty$, we have

$$||f||_{\dot{F}_p^{\beta,\infty}(w)} \approx \left|\left|\sup_{Q\ni\cdot} \frac{1}{|Q|^{1+\beta/n}} \int_Q |f(x) - f_Q|dx\right|\right|_{L^p(w)}$$
$$\approx \left|\left|\sup_{Q\ni\cdot} \inf_c \frac{1}{|Q|^{1+\beta/n}} \int_Q |f(x) - c|dx\right|\right|_{L^p(w)}.$$

Lemma 4.(see [12]). Let $0 < p < \infty$ and $w \in \cup_{1\le r<\infty} A_r$. Then, for any smooth function f for which the left-hand side is finite,

$$\int_{R^n} M(f)(x)^p w(x)dx \le C \int_{R^n} M^{\#}(f)(x)^p w(x)dx.$$

Lemma 5.(see [22]). Suppose that $0 \le \eta < n$, $1 < s < p < n/\eta$, $1/r = 1/p - \eta/n$ and $w \in A(p, r)$. Then

$$||M_{\eta,s}(f)||_{L^r(w^r)} \le C||f||_{L^p(w^p)}.$$

Lemma 6.(see [7][12]) If $w \in A_p$, then $w\chi_Q \in A_p$ for $1 \le p \le \infty$ and any cube Q.

Lemma 7. Let $1 < r < \infty$, $0 < \eta < \infty$, $0 < D < 2^n$, $w \in A_\infty$ and $L^{r,\varphi}(R^n, w)$ be the weighted Morrey space as **Definition 2**. Then, for any smooth function f for which the left-hand side is finite,

$$||M(f)||_{L^{r,\varphi}(w)} \le C||M^{\#}(f)||_{L^{r,\varphi}(w)}.$$

Proof. Notice that $w\chi_Q \in A_\infty$ for any cube $Q = Q(x, d)$ by [8] and Lemma 6, thus, for $f \in L^{r,\varphi}(R^n, w)$ and any cube Q, we have, by Lemma 4,

$$\int_Q M(f)(x)^r w(x)dx = \int_{R^n} M(f)(x)^r w(x)\chi_Q(x)dx$$
$$\le C \int_{R^n} M^{\#}(f)(x)^r w(x)\chi_Q(x)dx$$
$$= C \int_Q M^{\#}(f)(x)^r w(x)dx,$$

thus

$$\left(\frac{1}{\varphi(d)}\int_{Q(x,d)} M(f)(x)^r w(x)dx\right)^{1/r} \le C\left(\frac{1}{\varphi(d)}\int_{Q(x,d)} M^{\#}(f)(x)^r w(x)dx\right)^{1/r}$$

and

$$||M(f)||_{L^{r,\varphi}(w)} \le C||M^{\#}(f)||_{L^{r,\varphi}(w)}.$$

This finishes the proof.

Lemma 8. Let $1 < p < \infty$, $0 < D < 2^n$, $w \in A_1$, T be the integral operator as **Definition 1** and $L^{p,\varphi}(R^n, w)$ be the weighted Morrey space as **Definition 2**. Then

$$||T(f)||_{L^{p,\varphi}(w)} \le C||f||_{L^{p,\varphi}(w)}.$$

Lemma 9. Let $0 < D < 2^n$, $1 \le s < p < n/\eta$, $1/r = 1/p - \eta/n$, $w \in A(p,r)$ and $L^{p,\varphi}(R^n, w)$ be the weighted Morrey space as **Definition 2**. Then

$$||M_{\eta,s}(f)||_{L^{r,\varphi}(w^r)} \le C||f||_{L^{p,\varphi}(w^p)}.$$

The proofs of two Lemmas are similar to that of Lemma 7 by Lemma 1 and 5, we omit the details.

Proof of Theorem 1. It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant C_0, the following inequality holds:

$$\frac{1}{|Q|}\int_Q \left|T^b(f)(x) - C_0\right| dx \le C||b||_{Lip_\beta(w)} w(\tilde{x})^{1+\beta/n}\left(M_{\beta,s}(f)(\tilde{x}) + M_{\beta,s}(T(f))(\tilde{x})\right).$$

Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Write, for $f_1 = f\chi_{2Q}$ and $f_2 = f\chi_{(2Q)^c}$,

$$F_t^b(f)(x) = (b(x) - b_{2Q})F_t(f)(x) - F_t((b - b_{2Q})f_1)(x) - F_t((b - b_{2Q})f_2)(x).$$

Then

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q ||F_t^b(f)(x) - F_t((b_{2Q} - b)f_2)(x_0)||dx \\
\le\ & \frac{1}{|Q|}\int_Q ||(b(x) - b_{2Q})F_t(f)(x)||dx + \frac{1}{|Q|}\int_Q ||F_t((b - b_{2Q})f_1)(x)||dx \\
&+ \frac{1}{|Q|}\int_Q ||F_t((b - b_{2Q})f_2)(x) - F_t((b - b_{2Q})f_2)(x_0)||dx \\
=\ & I_1 + I_2 + I_3.
\end{aligned}$$

For I_1, by Hölder's inequality and Lemma 2, we obtain

$$\begin{aligned}
I_1 \le\ & \frac{C}{|Q|}\sup_{x\in 2Q}|b(x) - b_{2Q}||Q|^{1-1/s}\left(\int_Q |T(f)(x)|^s dx\right)^{1/s} \\
\le\ & C||b||_{Lip_\beta(w)}\frac{w(2Q)^{1+\beta/n}}{|2Q|}|Q|^{-1/s}|Q|^{1/s-\beta/n}\left(\frac{1}{|Q|^{1-s\beta/n}}\int_Q |T(f)(x)|^s dx\right)^{1/s} \\
\le\ & C||b||_{Lip_\beta(w)}\left(\frac{w(Q)}{|Q|}\right)^{1+\beta/n} M_{\beta,s}(T(f))(\tilde{x}) \\
\le\ & C||b||_{Lip_\beta(w)} w(\tilde{x})^{1+\beta/n} M_{\beta,s}(T(f))(\tilde{x}).
\end{aligned}$$

For I_2, by the boundedness of T, we get

$$\begin{aligned}
I_2 &\le \left(\frac{1}{|Q|}\int_{R^n}|T((b-b_{2Q})f_1)(x)|^s dx\right)^{1/s} \\
&\le C\left(\frac{1}{|Q|}\int_{R^n}|(b(x)-b_{2Q})f_1(x)|^s dx\right)^{1/s} \\
&\le C|Q|^{-1/s}\sup_{x\in 2Q}|b(x)-b_{2Q}||2Q|^{1/s-\beta/n}\left(\frac{1}{|2Q|^{1-s\beta/n}}\int_{2Q}|f(x)|^s dx\right)^{1/s} \\
&\le C||b||_{Lip_\beta(w)}\left(\frac{w(Q)}{|Q|}\right)^{1+\beta/n} M_{\beta,s}(f)(\tilde{x}) \\
&\le C||b||_{Lip_\beta(w)} w(\tilde{x})^{1+\beta/n} M_{\beta,s}(f)(\tilde{x}).
\end{aligned}$$

For I_3, recalling that $s > q'$, we have

$$\begin{aligned}
I_3 &\le \frac{1}{|Q|}\int_Q\int_{(2Q)^c}|b(y)-b_{2Q}||f(y)|||F_t(x,y)-F_t(x_0,y)||dydx \\
&\le \frac{1}{|Q|}\int_Q\sum_{k=1}^{\infty}\int_{2^k d\le|y-x_0|<2^{k+1}d}||F_t(x,y)-F_t(x_0,y)|||b(y)-b_{2^{k+1}Q}||f(y)|dydx \\
&\quad +\frac{1}{|Q|}\int_Q\sum_{k=1}^{\infty}\int_{2^k d\le|y-x_0|<2^{k+1}d}||F_t(x,y)-F_t(x_0,y)|||b_{2^{k+1}Q}-b_{2Q}||f(y)|dydx \\
&\le \frac{C}{|Q|}\int_Q\sum_{k=1}^{\infty}\left(\int_{2^k d\le|y-x_0|<2^{k+1}d}||F_t(x,y)-F_t(x_0,y)||^q dy\right)^{1/q} \\
&\quad \times\sup_{y\in 2^{k+1}Q}|b(y)-b_{2^{k+1}Q}|\left(\int_{2^{k+1}Q}|f(y)|^{q'}dy\right)^{1/q'}dx \\
&\quad +\frac{C}{|Q|}\int_Q\sum_{k=1}^{\infty}|b_{2^{k+1}Q}-b_{2Q}|\left(\int_{2^k d\le|y-x_0|<2^{k+1}d}||F_t(x,y)-F_t(x_0,y)||^q dy\right)^{1/q} \\
&\quad \times\left(\int_{2^{k+1}Q}|f(y)|^{q'}dy\right)^{1/q'}dx \\
&\le C\sum_{k=1}^{\infty}C_k(2^k d)^{-n/q'}\frac{w(2^{k+1}Q)^{1+\beta/n}}{|2^{k+1}Q|}||b||_{Lip_\beta(w)}|2^{k+1}Q|^{1/q'-1/s}|2^{k+1}Q|^{1/s-\beta/n} \\
&\quad \times\left(\frac{1}{|2^{k+1}Q|^{1-s\beta/n}}\int_{2^{k+1}Q}|f(y)|^s dy\right)^{1/s} \\
&\quad +C\sum_{k=1}^{\infty}k||b||_{Lip_\beta(w)}w(\tilde{x})w(2^kQ)^{\beta/n}C_k(2^kd)^{-n/q'}|2^{k+1}Q|^{1/q'-1/s}|2^{k+1}Q|^{1/s-\beta/n} \\
&\quad \times\left(\frac{1}{|2^{k+1}Q|^{1-s\beta/n}}\int_{2^{k+1}Q}|f(y)|^s dy\right)^{1/s}
\end{aligned}$$

$$\begin{aligned}
&\leq C||b||_{Lip_\beta(w)}\sum_{k=1}^{\infty}C_k\left(\frac{w(2^{k+1}Q)}{|2^{k+1}Q|}\right)^{1+\beta/n}M_{\beta,s}(f)(\tilde{x})\\
&\quad +C||b||_{Lip_\beta(w)}w(\tilde{x})\sum_{k=1}^{\infty}kC_k\left(\frac{w(2^{k+1}Q)}{|2^{k+1}Q|}\right)^{\beta/n}M_{\beta,s}(f)(\tilde{x})\\
&\leq C||b||_{Lip_\beta(w)}w(\tilde{x})^{1+\beta/n}M_{\beta,s}(f)(\tilde{x})\sum_{k=1}^{\infty}(k+1)C_k\\
&\leq C||b||_{Lip_\beta(w)}w(\tilde{x})^{1+\beta/n}M_{\beta,s}(f)(\tilde{x}).
\end{aligned}$$

These complete the proof of Theorem 1.

Proof of Theorem 2. It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant C_0, the following inequality holds:

$$\frac{1}{|Q|^{1+\beta/n}}\int_Q\left|T^b(f)(x)-C_0\right|dx \leq C||b||_{Lip_\beta(w)}w(\tilde{x})\left(M_s(f)(\tilde{x})+M_s(T(f))(\tilde{x})\right).$$

Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Write, for $f_1 = f\chi_{2Q}$ and $f_2 = f\chi_{(2Q)^c}$,

$$F_t^b(f)(x) = (b(x)-b_{2Q})F_t(f)(x) - F_t((b-b_{2Q})f_1)(x) - F_t((b-b_{2Q})f_2)(x).$$

Then

$$\begin{aligned}
&\frac{1}{|Q|^{1+\beta/n}}\int_Q||F_t^b(f)(x)-F_t((b_{2Q}-b)f_2)(x_0)||dx\\
\leq\ &\frac{1}{|Q|^{1+\beta/n}}\int_Q||(b(x)-b_{2Q})F_t(f)(x)||dx+\frac{1}{|Q|^{1+\beta/n}}\int_Q||F_t((b-b_{2Q})f_1)(x)||dx\\
&+\frac{1}{|Q|^{1+\beta/n}}\int_Q||F_t((b-b_{2Q})f_2)(x)-F_t((b-b_{2Q})f_2)(x_0)||dx\\
=\ &I_1+I_2+I_3.
\end{aligned}$$

By using the same argument as in the proof of Theorem 1, we get

$$\begin{aligned}
I_1 &\leq \frac{C}{|Q|^{1+\beta/n}}\sup_{x\in 2Q}|b(x)-b_{2Q}||Q|^{1-1/s}\left(\int_Q|T(f)(x)|^s dx\right)^{1/s}\\
&\leq C||b||_{Lip_\beta(w)}\frac{w(2Q)^{1+\beta/n}}{|2Q|}|Q|^{-1/s}|Q|^{1/s-\beta/n}\left(\frac{1}{|Q|}\int_Q|T(f)(x)|^s dx\right)^{1/s}\\
&\leq C||b||_{Lip_\beta(w)}\left(\frac{w(Q)}{|Q|}\right)^{1+\beta/n}M_s(T(f))(\tilde{x})\\
&\leq C||b||_{Lip_\beta(w)}w(\tilde{x})^{1+\beta/n}M_s(T(f))(\tilde{x}),
\end{aligned}$$

$$
\begin{aligned}
I_2 &\le \frac{1}{|Q|^{1+\beta/n}}|Q|^{1-1/s}\left(\int_{R^n}|T((b-b_{2Q})f_1)(x)|^s dx\right)^{1/s}\\
&\le C\frac{1}{|Q|^{1+\beta/n}}|Q|^{1-1/s}\left(\int_{R^n}|(b(x)-b_{2Q})f_1(x)|^s dx\right)^{1/s}\\
&\le C\frac{1}{|Q|^{1+\beta/n}}|Q|^{1-1/s}\sup_{x\in 2Q}|b(x)-b_{2Q}||2Q|^{1/s}\left(\frac{1}{|2Q|}\int_{2Q}|f(x)|^s dx\right)^{1/s}\\
&\le C||b||_{Lip_\beta(w)}\left(\frac{w(Q)}{|Q|}\right)^{1+\beta/n}M_s(f)(\tilde{x})\\
&\le C||b||_{Lip_\beta(w)}w(\tilde{x})^{1+\beta/n}M_s(f)(\tilde{x}),\\
I_3 &\le \frac{1}{|Q|^{1+\beta/n}}\int_Q\int_{(2Q)^c}|b(y)-b_{2Q}||f(y)|||F_t(x,y)-F_t(x_0,y)||dydx\\
&\le \frac{1}{|Q|^{1+\beta/n}}\int_Q\sum_{k=1}^\infty\int_{2^kd\le|y-x_0|<2^{k+1}d}||F_t(x,y)-F_t(x_0,y)|||b(y)-b_{2^{k+1}Q}||f(y)|dydx\\
&\quad+\frac{1}{|Q|^{1+\beta/n}}\int_Q\sum_{k=1}^\infty\int_{2^kd\le|y-x_0|<2^{k+1}d}||F_t(x,y)-F_t(x_0,y)|||b_{2^{k+1}Q}-b_{2Q}||f(y)|dydx\\
&\le \frac{C}{|Q|^{1+\beta/n}}\int_Q\sum_{k=1}^\infty\left(\int_{2^kd\le|y-x_0|<2^{k+1}d}||F_t(x,y)-F_t(x_0,y)||^q dy\right)^{1/q}\\
&\quad\times\sup_{y\in 2^{k+1}Q}|b(y)-b_{2^{k+1}Q}|\left(\int_{2^{k+1}Q}|f(y)|^{q'}dy\right)^{1/q'}dx\\
&\quad+\frac{C}{|Q|^{1+\beta/n}}\int_Q\sum_{k=1}^\infty|b_{2^{k+1}Q}-b_{2Q}|\left(\int_{2^kd\le|y-x_0|<2^{k+1}d}||F_t(x,y)-F_t(x_0,y)||^q dy\right)^{1/q}\\
&\quad\times\left(\int_{2^{k+1}Q}|f(y)|^{q'}dy\right)^{1/q'}dx\\
&\le C|Q|^{-\beta/n}\sum_{k=1}^\infty C_k(2^kd)^{-n/q'}\frac{w(2^{k+1}Q)^{1+\beta/n}}{|2^{k+1}Q|}||b||_{Lip_\beta(w)}|2^{k+1}Q|^{1/q'}\\
&\quad\times\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|f(y)|^s dy\right)^{1/s}\\
&\quad+C|Q|^{-\beta/n}\sum_{k=1}^\infty k||b||_{Lip_\beta(w)}w(\tilde{x})w(2^kQ)^{\beta/n}C_k(2^kd)^{-n/q'}|2^{k+1}Q|^{1/q'}\\
&\quad\times\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|f(x)|^s dx\right)^{1/s}\\
&\le C||b||_{Lip_\beta(w)}\sum_{k=1}^\infty 2^{\beta k}C_k\left(\frac{w(2^{k+1}Q)}{|2^{k+1}Q|}\right)^{1+\beta/n}M_s(f)(\tilde{x})\\
&\quad+C||b||_{Lip_\beta(w)}w(\tilde{x})\sum_{k=1}^\infty k2^{\beta k}C_k\left(\frac{w(2^{k+1}Q)}{|2^{k+1}Q|}\right)^{\beta/n}M_s(f)(\tilde{x})\\
&\le C||b||_{Lip_\beta(w)}w(\tilde{x})^{1+\beta/n}M_s(f)(\tilde{x})\sum_{k=1}^\infty(k+1)2^{\beta k}C_k\\
&\le C||b||_{Lip_\beta(w)}w(\tilde{x})^{1+\beta/n}M_s(f)(\tilde{x}).
\end{aligned}
$$

This completes the proof of Theorem 2.

Proof of Theorem 3. Choose $q' < s < p$ in Theorem 1, notice $w^{r/p-r(1+\beta/n)} \in A_\infty$ and $w^{1/p} \in A(p,r)$, we have, by Lemma 1, 4 and 5,

$$
\begin{aligned}
& ||T^b(f)||_{L^r(w^{r/p-r(1+\beta/n)})} \\
\le\ & \|M(T^b(f))\|_{L^r(w^{r/p-r(1+\beta/n)})} \\
\le\ & C\|M^{\#}(T^b(f))^{\#}\|_{L^r(w^{r/p-r(1+\beta/n)})} \\
\le\ & C||b||_{Lip_\beta(w)}(\|M_{\beta,s}(T(f))w^{1+\beta/n}\|_{L^r(w^{r/p-r(1+\beta/n)})} \\
& +\|M_{\beta,s}(f)w^{1+\beta/n}\|_{L^r(w^{r/p-r(1+\beta/n)})}) \\
=\ & C||b||_{Lip_\beta(w)}\left(\|M_{\beta,s}(T(f))\|_{L^r(w^{r/p})} + \|M_{\beta,s}(f)\|_{L^r(w^{r/p})}\right) \\
\le\ & C||b||_{Lip_\beta(w)}(\|T(f)\|_{L^p(w)} + \|f\|_{L^p(w)}) \\
\le\ & C||b||_{Lip_\beta(w)}\|f\|_{L^p(w)}.
\end{aligned}
$$

This completes the proof of Theorem 3.

Proof of Theorem 4. Choose $q' < s < p$ in Theorem 1, notice $w^{r/p-r(1+\beta/n)} \in A_\infty$ and $w^{1/p} \in A(p,r)$, we have, by Lemma 7-9,

$$
\begin{aligned}
& ||T^b(f)||_{L^{r,\varphi}(w^{r/p-r(1+\beta/n)})} \\
\le\ & \|M(T^b(f))\|_{L^{r,\varphi}(w^{r/p-r(1+\beta/n)})} \\
\le\ & C\|M^{\#}(T^b(f))^{\#}\|_{L^{r,\varphi}(w^{r/p-r(1+\beta/n)})} \\
\le\ & C||b||_{Lip_\beta(w)}(\|M_{\beta,s}(T(f))w^{1+\beta/n}\|_{L^{s,\varphi}(w^{r/p-r(1+\beta/n)})} \\
& +\|M_{\beta,s}(f)w^{1+\beta/n}\|_{L^{r,\varphi}(w^{r/p-r(1+\beta/n)})}) \\
=\ & C||b||_{Lip_\beta(w)}\left(\|M_{\beta,s}(T(f))\|_{L^{r,\varphi}(w^{r/p})} + \|M_{\beta,s}(f)\|_{L^{r,\varphi}(w^{r/p})}\right) \\
\le\ & C||b||_{Lip_\beta(w)}(\|T(f)\|_{L^{p,\varphi}(w)} + \|f\|_{L^{p,\varphi}(w)}) \\
\le\ & C||b||_{Lip_\beta(w)}\|f\|_{L^{p,\varphi}(w)}.
\end{aligned}
$$

This completes the proof of Theorem 4.

Proof Theorem 5. Choose $q' < s < p$ in Theorem 2, notice that $w^{r/p-r(1+\beta/n)} \in A_\infty$ and $w^{1/p} \in A(p,r)$. By using Lemma 3, we obtain

$$
\begin{aligned}
& ||T^b(f)||_{\dot{F}_r^{\beta,\infty}(w^{r/p-r(1+\beta/n)})} \\
\le\ & C\left|\left|\sup_{Q\ni\cdot}\frac{1}{|Q|^{1+\beta/n}}\int_Q\left|T^b(f)(x) - T((b_{2Q}-b)f_2)(x_0)\right|dx\right|\right|_{L^r(w^{r/p-r(1+\beta/n)})} \\
\le\ & C||b||_{Lip_\beta(w)}(\|M_s(T(f))w^{1+\beta/n}\|_{L^r(w^{r/p-r(1+\beta/n)})} \\
& +\|M_s(f)w^{1+\beta/n}\|_{L^r(w^{r/p-r(1+\beta/n)})}) \\
=\ & C||b||_{Lip_\beta(w)}\left(\|M_s(T(f))\|_{L^r(w^{r/p})} + \|M_s(f)\|_{L^r(w^{r/p})}\right) \\
\le\ & C||b||_{Lip_\beta(w)}(\|T(f)\|_{L^p(w)} + \|f\|_{L^p(w)}) \\
\le\ & C||b||_{Lip_\beta(w)}||f||_{L^p(w)}.
\end{aligned}
$$

This completes the proof of the theorem.

4 Applications

In this section we shall apply the Theorems 1-5 of the paper to some particular operators such as the Littlewood-Paley operators, Marcinkiewicz operator and Bochner-Riesz operator.

Application 1. Littlewood-Paley operators.

Fixed $\varepsilon > 0$ and $\mu > (3n+2)/n$. Let ψ be a fixed function which satisfies:

(1) $\int_{R^n} \psi(x)dx = 0$,

(2) $|\psi(x)| \le C(1+|x|)^{-(n+1)}$,

(3) $|\psi(x+y) - \psi(x)| \le C|y|^\varepsilon (1+|x|)^{-(n+1+\varepsilon)}$ when $2|y| < |x|$;

We denote $\Gamma(x) = \{(y,t) \in R_+^{n+1} : |x-y| < t\}$ and the characteristic function of $\Gamma(x)$ by $\chi_{\Gamma(x)}$. The Littlewood-Paley commutators are defined by

$$g_\psi^b(f)(x) = \left(\int_0^\infty |F_t^b(f)(x)|^2 \frac{dt}{t}\right)^{1/2},$$

$$S_\psi^b(f)(x) = \left[\int\int_{\Gamma(x)} |F_t^b(f)(x,y)|^2 \frac{dydt}{t^{n+1}}\right]^{1/2}$$

and

$$g_\mu^b(f)(x) = \left[\int\int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{n\mu} |F_t^b(f)(x,y)|^2 \frac{dydt}{t^{n+1}}\right]^{1/2},$$

where

$$F_t^b(f)(x) = \int_{R^n} (b(x) - b(y))\psi_t(x-y)f(y)dy,$$

$$F_t^b(f)(x,y) = \int_{R^n} (b(x) - b(z))f(z)\psi_t(y-z)dz$$

and $\psi_t(x) = t^{-n}\psi(x/t)$ for $t > 0$. Set $F_t(f)(y) = f * \psi_t(y)$. We also define

$$g_\psi(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t}\right)^{1/2},$$

$$S_\psi(f)(x) = \left(\int\int_{\Gamma(x)} |F_t(f)(y)|^2 \frac{dydt}{t^{n+1}}\right)^{1/2}$$

and

$$g_\mu(f)(x) = \left(\int\int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{n\mu} |F_t(f)(y)|^2 \frac{dydt}{t^{n+1}}\right)^{1/2},$$

which are the Littlewood-Paley operators (see [29]). Let H be the space

$$H = \left\{h : ||h|| = \left(\int_0^\infty |h(t)|^2 dt/t\right)^{1/2} < \infty\right\}$$

or

$$H = \left\{ h : ||h|| = \left(\int\int_{R_+^{n+1}} |h(y,t)|^2 dydt/t^{n+1} \right)^{1/2} < \infty \right\},$$

then, for each fixed $x \in R^n$, $F_t^b(f)(x)$ and $F_t^b(f)(x,y)$ may be viewed as the mapping from $[0,+\infty)$ to H, and it is clear that

$$g_\psi^b(f)(x) = ||F_t^b(f)(x)||, \quad g_\psi(f)(x) = ||F_t(f)(x)||,$$

$$S_\psi^b(f)(x) = \left|\left|\chi_{\Gamma(x)} F_t^b(f)(x,y)\right|\right|, \quad S_\psi(f)(x) = \left|\left|\chi_{\Gamma(x)} F_t(f)(y)\right|\right|$$

and

$$g_\mu^b(f)(x) = \left|\left| \left(\frac{t}{t+|x-y|} \right)^{n\mu/2} F_t^b(f)(x,y) \right|\right|, \; g_\mu(f)(x) = \left|\left| \left(\frac{t}{t+|x-y|} \right)^{n\mu/2} F_t(f)(y) \right|\right|.$$

It is easily to see that g_ψ^b, S_ψ^b and g_μ^b satisfy the conditions of Theorem 1-5(see [17-19]), thus theorem 1-5 hold for g_ψ^b, S_ψ^b and g_μ^b.

Application 2. Marcinkiewicz operators.

Fixed $\lambda > \max(1, 2n/(n+2))$ and $0 < \gamma \leq 1$. Let Ω be homogeneous of degree zero on R^n with $\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0$. Assume that $\Omega \in Lip_\gamma(S^{n-1})$. The Marcinkiewicz commutators are defined by

$$\mu_\Omega^b(f)(x) = \left(\int_0^\infty |F_t^b(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

$$\mu_S^b(f)(x) = \left[\int\int_{\Gamma(x)} |F_t^b(f)(x,y)|^2 \frac{dydt}{t^{n+3}} \right]^{1/2}$$

and

$$\mu_\lambda^b(f)(x) = \left[\int\int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|} \right)^{n\lambda} |F_t^b(f)(x,y)|^2 \frac{dydt}{t^{n+3}} \right]^{1/2},$$

where

$$F_t^b(f)(x) = \int_{|x-y|\leq t} (b(x)-b(y)) \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)dy$$

and

$$F_t^b(f)(x,y) = \int_{|y-z|\leq t} (b(x)-b(z)) \frac{\Omega(y-z)}{|y-z|^{n-1}} f(z)dz.$$

Set

$$F_t(f)(x) = \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)dy.$$

We also define

$$\mu_\Omega(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

$$\mu_S(f)(x) = \left(\int\int_{\Gamma(x)} |F_t(f)(y)|^2 \frac{dydt}{t^{n+3}}\right)^{1/2}$$

and

$$\mu_\lambda(f)(x) = \left(\int\int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{n\lambda} |F_t(f)(y)|^2 \frac{dydt}{t^{n+3}}\right)^{1/2},$$

which are the Marcinkiewicz operators(see [30]). Let H be the space

$$H = \left\{h : ||h|| = \left(\int_0^\infty |h(t)|^2 dt/t^3\right)^{1/2} < \infty\right\}$$

or

$$H = \left\{h : ||h|| = \left(\int\int_{R_+^{n+1}} |h(y,t)|^2 dydt/t^{n+3}\right)^{1/2} < \infty\right\}.$$

Then, it is clear that

$$\mu_\Omega^b(f)(x) = ||F_t^b(f)(x)||, \quad \mu_\Omega(f)(x) = ||F_t(f)(x)||,$$

$$\mu_S^b(f)(x) = \left|\left|\chi_{\Gamma(x)} F_t^b(f)(x,y)\right|\right|, \quad \mu_S(f)(x) = \left|\left|\chi_{\Gamma(x)} F_t(f)(y)\right|\right|$$

and

$$\mu_\lambda^b(f)(x) = \left|\left|\left(\frac{t}{t+|x-y|}\right)^{n\lambda/2} F_t^b(f)(x,y)\right|\right|, \ \mu_\lambda(f)(x) = \left|\left|\left(\frac{t}{t+|x-y|}\right)^{n\lambda/2} F_t(f)(y)\right|\right|.$$

It is easily to see that μ_Ω^b, μ_S^b and μ_λ^b satisfy the conditions of Theorem 1-5 (see [17-18][30]), thus Theorem 1-5 hold for μ_Ω^b, μ_S^b and μ_λ^b.

Application 3. Bochner-Riesz operator.

Let $\delta > (n-1)/2$, $B_t^\delta(f)\hat{}(\xi) = (1-t^2|\xi|^2)_+^\delta \hat{f}(\xi)$ and $B_t^\delta(z) = t^{-n}B^\delta(z/t)$ for $t > 0$. Set

$$F_{\delta,t}^b(f)(x) = \int_{R^n} (b(x)-b(y)) B_t^\delta(x-y) f(y) dy,$$

The maximal Bochner-Riesz commutator is defined by

$$B_{\delta,*}^b(f)(x) = \sup_{t>0} |B_{\delta,t}^b(f)(x)|.$$

We also define that

$$B_{\delta,*}(f)(x) = \sup_{t>0} |B_t^\delta(f)(x)|$$

which is the maximal Bochner-Riesz operator(see [20]). Let H be the space $H = \{h : ||h|| = \sup_{t>0} |h(t)| < \infty\}$, then

$$B_{\delta,*}^b(f)(x) = ||B_{\delta,t}^b(f)(x)||, \quad B_*^\delta(f)(x) = ||B_t^\delta(f)(x)||.$$

It is easily to see that $B_{\delta,*}^b$ satisfies the conditions of Theorem 1-5(see [17]), thus Theorem 1-5 hold for $B_{\delta,*}^b$.

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