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INEQUALITIES FOR TWO SPECIFIC CLASSES OF FUNCTIONS USING CHEBYSHEV FUNCTIONAL

Mohammad Masjed-Jamei

Abstract

In this paper, we introduce two specific classes of functions in L^p - spaces that can generate new and known inequalities in the literature. By using some recent results related to the Chebyshev functional, we then obtain upper bounds for the absolute value of the two introduced functions and consider three particular examples. One of these examples is a suitable tool for finding upper and lower bounds of some incomplete special functions such as incomplete gamma and beta functions.

1 Introduction

Let $L^p[a,b]$ $(1 \le p < \infty)$ denote the space of p-power integrable functions on the interval $[a,b]$ with the standard norm

$$\|f\|_p = \left(\int_a^b |\, f(t)\,|^p \, dt \right)^{1/p},$$

and $L^\infty[a,b]$ show the space of all essentially bounded functions on $[a,b]$ with the norm

$$\|f\|_\infty = \sup_{x \in [a,b]} |\, f(x)\,| \, .$$

For two absolutely continuous functions $f, g : [a,b] \to \mathbf{R}$ and the positive function $w : [a,b] \to \mathbf{R}^+$ such that $wf, wg, wfg \in L^1[a,b]$, the weighted Chebyshev functional [1] is defined by

$$\mathbf{T}\,(w,f,g) = \int_a^b w(x) f(x)\, g(x)\, dx - \left(\int_a^b w(x) f(x)\, dx \right) \left(\int_a^b w(x) g(x)\, dx \right). \quad (1)$$

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If $w(x)$ is uniformly distributed on $[a,b]$ then (1) is reduced to the usual Chebyshev functional

$$\mathbf{T}\,(f,g) = \frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \frac{1}{(b-a)^2}\left(\int_a^b f(x)\,dx\right)\left(\int_a^b g(x)\,dx\right). \quad (2)$$

To date, extensive research has been done on the bounds of Chebyshev functional, see e.g. [1,2,5]. The first work dates back to 1882, when Chebyshev [3] proved that if $f', g' \in L^{\infty}[a,b]$ then

$$|\,\mathbf{T}\,(f,g)\,| \leq \frac{1}{12}(b-a)^2 \,\|\, f' \,\|_{\infty} \,\|\, g' \,\|_{\infty}\,. \quad (3)$$

Later on, in 1934 Gruss [9] showed that

$$|\,\mathbf{T}\,(f,g)\,| \leq \frac{1}{4}(M_1 - m_1)(M_2 - m_2), \quad (4)$$

where m_1, m_2, M_1 and M_2 are real numbers satisfying the conditions

$$m_1 \leq f(x) \leq M_1 \quad \texttt{and} \quad m_2 \leq g(x) \leq M_2 \quad \texttt{for all} \ x \in [a,b]. \quad (5)$$

The constant $1/4$ is the best possible number in (4) in the sense that it cannot be replaced by a smaller quantity.
An inequality related to usual Chebyshev functional is due to Ostrowski [16] in 1938. If $f : [a,b] \to \mathbf{R}$ is a differentiable function with bounded derivative, then

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \left(\frac{1}{4} + \frac{(x-(a+b)/2)^2}{(b-a)^2}\right)(b-a)\,\|\, f' \,\|_{\infty} \quad (6)$$

for all $x \in [a,b]$.
Today this inequality plays a key role in numerical quadrature rules [7,8,13].
A mixture type of inequalities (3) and (4) was introduced in [17] as

$$|\,\mathbf{T}\,(f,g)\,| \leq \frac{1}{8}(b-a)(M_1 - m_1)\,\|\, g' \,\|_{\infty}\,, \quad (7)$$

in which f is a Lebesgue integrable function satisfying (5) and g is absolutely continuous so that $g' \in L^{\infty}[a,b]$. The constant $1/8$ is also the best possible number in (7).
The following theorem, due to Niezgoda [15], is probably the most recent work about finding appropriate bounds for the usual Chebyshev functional.

1.1. Theorem A. *Let $f, \alpha, \beta \in L^p[a,b]$ and $g \in L^q[a,b]$ $(1/p + 1/q = 1\,,\ 1 \leq p \leq \infty)$ be functions such that $\alpha(t) + \beta(t)$ is a constant function and $\alpha(t) \leq f(t) \leq \beta(t)$ for all $t \in [a,b]$. Then we have*

$$|\,\mathbf{T}\,(f,g)\,| \leq \frac{1}{2(b-a)}\,\|\,\beta - \alpha\,\|_p \left\| g - \frac{1}{b-a}\int_a^b g(t)\,dt \right\|_q . \quad (8)$$

For $p = q = 2$, (8) leads to the well-known inequality [14]

$$|\mathbf{T}(f,g)| \le \frac{1}{2}(M_1 - m_1)\sqrt{\mathbf{T}(g,g)} \;\; \texttt{s.t.} \;\; m_1 \le f(x) \le M_1. \tag{9}$$

On the other hand, in 1997 Dragomir and Wang [6] introduced an inequality of Ostrowski-Grss type, according to the following theorem.

1.2. Theorem B. *If $f : [a,b] \to \mathbf{R}$ is a differentiable function with bounded derivative and $\alpha_0 \le f'(t) \le \beta_0$ for all $t \in [a,b]$, then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(b)-f(a)}{b-a}\left(x - \frac{a+b}{2}\right) \right| \le \frac{1}{4}(b-a)(\beta_0 - \alpha_0). \tag{10}$$

There are many improvements and refinements of the right hand side of inequality (10) in the literature. See e.g. [4,10].
In this paper, we introduce two specific classes of functions in L^p - spaces that can generate many new and known inequalities in the literature and obtain their upper bounds using theorem A. Hence, let us first consider the following kernel defined on $[a,b]$

$$K(x;t,u,v) = \begin{cases} u(t) & t \in [a,x], \\ v(t) & t \in (x,b], \end{cases} \tag{11}$$

where $u(t)$ and $v(t)$ are two arbitrary integrable functions such that $u(t) \in C^1[a,x]$ and $v(t) \in C^1(x,b]$. Based on kernel (11), we now define the two following specific functions

$$\begin{aligned} \mathbf{F}_1(x;f,u,v) = {} & (u(x)-v(x))\,f(x) + v(b)f(b) - u(a)f(a) \\ & - \textstyle\int_a^x u'(t)f(t)\,dt - \int_x^b v'(t)f(t)\,dt\,, \end{aligned} \tag{12}$$

and

$$\mathbf{F}_2(x;f,u,v) = \mathbf{F}_1(x;f,u,v) - \frac{f(b)-f(a)}{b-a}\left(\int_a^x u(t)\,dt + \int_x^b v(t)\,dt\right). \tag{13}$$

It can be verified that the two functions (12) and (13) are respectively produced by

$$\int_a^b f'(t)\,K(x;t,u,v)\;dt \; = \mathbf{F}_1(x;f,u,v)\,, \tag{14}$$

and

$$(b-a)\,\mathbf{T}(f'(t), K(x;t,u,v)) = \mathbf{F}_2(x;f,u,v)\,, \tag{15}$$

where $\mathbf{T}(\,.\,,\,.\,)$ is the same as usual Chebyshev functional.

2. Main Theorem. *Let $f : \mathbf{I} \to \mathbf{R}$, where $\mathbf{I}$ is an interval, be a function differentiable in the interior $\mathbf{I}^0$ of $\mathbf{I}$, and let $[a,b] \subset \mathbf{I}^0$. Suppose that $f', \alpha, \beta \in$*

$L^p[a,b]$ are functions such that $\alpha(t)+\beta(t)$ is a constant function and $\alpha(t)\le f'(t) \le \beta(t)$ for all $t\in[a,b]$. Then we respectively have

$$|\,\mathbf{F}_1\,(x;f,u,v)\,| \le \begin{cases} \left(\int_a^x |\,u(t)|^q\,dt+\int_x^b |\,v(t)|^q\,dt\right)^{1/q} \|f'\|_p & (\,1/p+1/q=1\,),\\ \left(\int_a^x |\,u(t)|\,dt+\int_x^b |\,v(t)|\,dt\right) \|f'\|_\infty & (p=\infty\ ;\ q=1\,),\\ \max\limits_{t\in[a,b]} |\,K(x;t,u,v)\,|\ \ \|f'\|_1 & (q=\infty\ ;\ p=1)\,, \end{cases} \tag{16}$$

in which

$$\max_{t\in[a,b]} |\,K(x;t,u,v)\,| = \max_{x\in[a,b]} \left\{ \max_{t\in[a,x]} |\,u(t)\,|\ ,\ \max_{t\in(x,b]} |\,v(t)\,| \right\},$$

and

$$\begin{aligned} &|\,\mathbf{F}_2\,(x;f,u,v)\,| \le \tfrac{1}{2}\,\|\,\beta-\alpha\,\|_p \times \\ &\Big(\int_a^x \Big|\,u(t)-\tfrac{1}{b-a}\Big(\int_a^x u(t)\,dt+\int_x^b v(t)\,dt\Big)\Big|^q dt \\ &\quad+\int_x^b \Big|\,v(t)-\tfrac{1}{b-a}\Big(\int_a^x u(t)\,dt+\int_x^b v(t)\,dt\Big)\Big|^q dt\Big)^{1/q}, \end{aligned} \tag{17}$$

where $u(t)$ and $v(t)$ are two arbitrary integrable functions such that $u(t)\in C^1[a,x]$ and $v(t)\in C^1(x,b]$.

Proof. The proof of (16) is straightforward if one applies the well-known Hlder's inequality [14]

$$\|\,fg\,\|_1 \le \|\,f\,\|_p\,\|\,g\,\|_q \qquad (1/p+1/q=1), \tag{18}$$

for identity (14) and then refers to the general kernel (11). To prove (17) one should refer to identity (15) and then use Theorem A, so that we have

$$|\,\mathbf{F}_2\,(x;f,u,v)\,| \le \frac{1}{2}\ \|\,\beta-\alpha\,\|_p \left\| K(.)-\frac{1}{b-a}\int_a^b K(.)\,dt \right\|_q, \tag{19}$$

and since

$$\begin{aligned} &\Big\|\,K(.)-\tfrac{1}{b-a}\int_a^b K(.)\,dt\,\Big\|_q = \\ &\Big(\int_a^x \Big|\,u(t)-\tfrac{1}{b-a}\Big(\int_a^x u(t)\,dt+\int_x^b v(t)\,dt\Big)\Big|^q dt \\ &\quad+\int_x^b \Big|\,v(t)-\tfrac{1}{b-a}\Big(\int_a^x u(t)\,dt+\int_x^b v(t)\,dt\Big)\Big|^q dt\Big)^{1/q}, \end{aligned} \tag{20}$$

the proof is complete.

One of the straightforward cases of theorem 2.1 is when $p=q=2$. In other words, applying the well known Cauchy-Schwartz inequality [12] on (14) and using the main theorem 2.1 for (15) respectively yield

$$|\,\mathbf{F}_1\,(x;f,u,v)\,| \le \left(\int_a^x u^2(t)\,dt+\int_x^b v^2(t)\,dt\right)^{1/2} \|f'\|_2\,, \tag{21}$$

and

$$\begin{aligned} |\,\mathbf{F}_2\,(x;f,u,v)\,| \le\ & \tfrac{1}{2}\,\|\,\beta-\alpha\,\|_2\ \times \\ \Bigg(\int_a^x \Big(u(t) - & \tfrac{1}{b-a}\big(\int_a^x u(t)\,dt + \int_x^b v(t)\,dt\big)\Big)^2\,dt \\ + \int_x^b \Big(v(t) - & \tfrac{1}{b-a}\big(\int_a^x u(t)\,dt + \int_x^b v(t)\,dt\big)\Big)^2\,dt\Bigg)^{1/2}. \end{aligned} \tag{22}$$

Clearly various subclasses can be considered for the main theorem 2. We here study three cases. Other cases can naturally be studied separately.

2.1. Subclass 1. If $u(x) - v(x)$ is a constant function

If $u(x)-v(x)$ is a constant number, say $c \neq 0$, then the kernel (11) and functions (12) and (13) are respectively reduced to

$$K(x;t,v+c,v) = \begin{cases} v(t)+c & t \in [a,x], \\ v(t) & t \in (x,b], \end{cases} \tag{23}$$

$$\mathbf{F}_1\,(x;f,v+c,v) = c\,f(x) + v(b)f(b) - (c+v(a))f(a) - \int_a^b v'(t)f(t)\,dt\,, \tag{24}$$

and

$$\mathbf{F}_2\,(x;f,v+c,v) = \mathbf{F}_1\,(x;f,v+c,v) - \frac{f(b)-f(a)}{b-a}\left(c(x-a) + \int_a^b v(t)\,dt\right). \tag{25}$$

In fact, relations (24) and (25) show that

$$\mathbf{F}_1\,(x;f,v+c,v) = c\,f(x) + A \quad \texttt{and} \quad \mathbf{F}_2\,(x;f,v+c,v) = c\,f(x) + A + Bx + D, \tag{26}$$

where c, A, B and D are real constants.
Let us consider a particular example of the first subclass here.

Example 1. Suppose that $[a,b] = [0,1]$, $c = 1$ and $v(t) = t^2$. Under these assumptions, relations (23), (24) and (25) change to

$$K(x;t,1+t^2,t^2) = \begin{cases} 1+t^2 & t \in [0,x], \\ t^2 & t \in (x,1], \end{cases} \tag{27}$$

$$\mathbf{F}_1\left(x;f,1+t^2,t^2\right) = f(x) + f(1) - f(0) - 2\int_0^1 t\,f(t)\,dt\,, \tag{28}$$

and

$$\begin{aligned} \mathbf{F}_2\left(x;f,1+t^2,t^2\right) = f(x) + f(1) - f(0) - 2\int_0^1 t\,f(t)\,dt \\ -\,(f(1)-f(0))\left(x + \int_0^1 t^2 f(t)\,dt\right) \end{aligned} \tag{29}$$

After some calculations, substituting the above relations in each two parts of the main theorem respectively yields

$$\left| f(x) - 2\int_0^1 t\,f(t)\,dt + f(1) - f(0) \right| \le \begin{cases} \left(\int_0^x (1+t^2)^q dt + \frac{1-x^{2q+1}}{2q+1} \right)^{1/q} \|f'\|_p\,, \\ (x+1/3) \ \ \|f'\|_\infty\,, \\ (1+x^2) \ \ \|f'\|_1\,, \end{cases} \tag{30}$$

and

$$\begin{aligned} &\left| f(x) - 2\int_0^1 t\,f(t)\,dt + (f(1)-f(0))\left(1 - x - \int_0^1 t^2 f(t)\,dt\right) \right| \\ &\qquad \le \tfrac{1}{2}\,\|\,\beta-\alpha\,\|_p \left(\int_0^x \left| t^2 - x + 2/3 \right|^q dt + \int_x^1 \left| t^2 - x - 1/3 \right|^q dt \right)^{1/q}, \end{aligned} \tag{31}$$

where $\alpha(t) \le f'(t) \le \beta(t)$, $1/p + 1/q = 1$ and $x, t \in [0,1]$. For instance, $p = q = 2$ in inequality (31) gives

$$\begin{aligned} &\left| f(x) - 2\int_0^1 t\,f(t)\,dt + (f(1)-f(0))\left(1 - x - \int_0^1 t^2 f(t)\,dt\right) \right| \\ &\le \tfrac{\sqrt{5}}{30}\sqrt{30x^3 - 45x^2 + 15x + 4}\ \left(\int_0^1 (\beta(t)-\alpha(t))^2\,dt \right)^{1/2} \quad \text{for all } x \in [0,1]. \end{aligned} \tag{32}$$

2.2. Subclass 2. If $u(x)$ and $v(x)$ are linear functions

Suppose that $u(t) = p_1 t + q_1$ and $v(t) = p_2 t + q_2$ where p_1, q_1 and p_2, q_2 are all real parameters. Therefore we have

$$K(x; t, p_1 t + q_1, p_2 t + q_2) = \begin{cases} p_1 t + q_1 & t \in [a, x], \\ p_2 t + q_2 & t \in (x, b], \end{cases} \tag{33}$$

$$\begin{aligned} \mathbf{F}_1(x; f, p_1 t + q_1, p_2 t + q_2) &= ((p_1 - p_2)x + q_1 - q_2)\,f(x) + (p_2 b + q_2) f(b) \\ &\quad -(p_1 a + q_1) f(a) - p_1 \int_a^x f(t)\,dt - p_2 \int_x^b f(t)\,dt\,, \end{aligned} \tag{34}$$

and

$$\begin{aligned} &\mathbf{F}_2(x; f, p_1 t + q_1, p_2 t + q_2) = \mathbf{F}_1(x; f, p_1 t + q_1, p_2 t + q_2) \\ &\quad -\tfrac{f(b)-f(a)}{b-a}\left(\tfrac{p_1 - p_2}{2} x^2 + (q_1 - q_2)x + \tfrac{1}{2}(p_2 b^2 - p_1 a^2) + q_2 b - q_1 a \right). \end{aligned} \tag{35}$$

For the sake of simplicity, if we rearrange (34) and (35) by taking

$$p_1 - p_2 = r_1, \quad q_1 - q_2 = r_2, \quad p_2 b + q_2 = r_3 \ \text{ and } \ -p_1 a - q_1 = r_4, \tag{36}$$

then these assumptions would change relations (33), (34) and (35) as follows

$$K\left(x; t, \{r_i\}_{i=1}^4\right) = \begin{cases} \frac{br_1 + r_2 + r_3 + r_4}{b-a}\ t - \frac{abr_1 + ar_2 + ar_3 + br_4}{b-a} & t \in [a, x], \\ \frac{ar_1 + r_2 + r_3 + r_4}{b-a}\ t - \frac{abr_1 + br_2 + ar_3 + br_4}{b-a} & t \in (x, b], \end{cases} \tag{37}$$

$$\begin{aligned} \mathbf{F}_1\left(x; f, \{r_i\}_{i=1}^4\right) &= (r_1 x + r_2)\ f(x) + r_3 f(b) + r_4 f(a) \\ &\quad -\tfrac{br_1 + r_2 + r_3 + r_4}{b-a} \int_a^x f(t)\,dt - \tfrac{ar_1 + r_2 + r_3 + r_4}{b-a} \int_x^b f(t)\,dt\,, \end{aligned} \tag{38}$$

and

$$\begin{array}{l}\mathbf{F}_2\left(x;f,\{r_i\}_{i=1}^4\right)=\mathbf{F}_1\left(x;f,\{r_i\}_{i=1}^4\right)\;-\;\frac{f(b)-f(a)}{2\,(b-a)}\\ \times\left(r_1x^2+2r_2x-abr_1-(a+b)r_2+(b-a)(r_3-r_4)\right)\;.\end{array}\tag{39}$$

By noting that there are four free parameters r_1, r_2, r_3 and r_4 in the kernel (37), many subclasses exist for (38) and (39). The following example shows one of them.

Example 2. Let $r_1 = 0$ in (38) and (39). Then by referring to the main theorem we have

$$\begin{array}{l}\left|\,\mathbf{F}_1\left(x;f,\{r_i\}_{i=2}^4\right)\right|=\left|r_2f(x)+r_3f(b)+r_4f(a)\;-\;\frac{r_2+r_3+r_4}{b-a}\int_a^b f(t)\,dt\right|\\ \le\begin{cases}\frac{|\,r_2+r_3+r_4|}{b-a}\left(\int_a^x\left|t-\frac{ar_2+ar_3+br_4}{r_2+r_3+r_4}\right|^q dt+\int_x^b\left|t-\frac{br_2+ar_3+br_4}{r_2+r_3+r_4}\right|^q dt\right)^{1/q}\|f'\|_p\,,\\ \frac{|\,r_2+r_3+r_4|}{b-a}\left(\int_a^x\left|t-\frac{ar_2+ar_3+br_4}{r_2+r_3+r_4}\right|dt+\int_x^b\left|t-\frac{br_2+ar_3+br_4}{r_2+r_3+r_4}\right|dt\right)\;\|f'\|_\infty\,,\\ \max\limits_{t\in[a,b]}\left|\,K\left(x;t,\{r_i\}_{i=2}^4\right)\right|\quad\|f'\|_1\,,\end{cases}\end{array}\tag{40}$$

and

$$\begin{array}{l}\left|\,\mathbf{F}_2\left(x;f,\{r_i\}_{i=2}^4\right)\right|=\\ =\left|\,\mathbf{F}_1\left(x;f,\{r_i\}_{i=2}^4\right)-\frac{f(b)-f(a)}{2\,(b-a)}\left(2r_2x-(a+b)r_2+(b-a)(r_3-r_4)\right)\right|\\ \qquad\le\frac{1}{2}\,\|\,\beta-\alpha\,\|_p\left(\int_a^x\left|\frac{r_2+r_3+r_4}{b-a}t-\frac{r_2}{b-a}x-\frac{(3a+b)\,r_2+(a+b)(r_3+r_4)}{2(b-a)}\right|^q dt\right.\\ \qquad\qquad\left.+\int_x^b\left|\frac{r_2+r_3+r_4}{b-a}t-\frac{r_2}{b-a}x-\frac{(a+3b)\,r_2+(a+b)(r_3+r_4)}{2(b-a)}\right|^q dt\right)^{1/q}.\end{array}\tag{41}$$

Note, to compute the integrals of relations (40) and (41), we can use the following general identity

$$\int_{c^*}^{d^*}|\,t-\theta\,|^q\,dt=\begin{cases}\frac{(d^*-\theta)^{q+1}+(c^*-\theta)^{q+1}}{q+1} & \text{if}\quad c^*<\theta<d^*\,,\\ \frac{(d^*-\theta)^{q+1}-(c^*-\theta)^{q+1}}{q+1} & \text{if}\quad \theta<c^*<d^*\,,\\ \frac{-(d^*-\theta)^{q+1}+(c^*-\theta)^{q+1}}{q+1} & \text{if}\quad c^*<d^*<\theta\,,\end{cases}\tag{42}$$

in which $c^* < d^*$ and $\theta \in \mathbf{R}$.

Remark 1. For $(r_1, r_2, r_3, r_4) = (0, 1, 0, 0)$ inequality (41) generates inequality (10).

Remark 2. For $r_1 = 0$ and $r_2 + r_3 + r_4 = 0$, since the kernel (37) is reduced to

$$K\,(x;t,-r_4,r_3)=\begin{cases}-r_4 & t\in[a,x],\\ r_3 & t\in(x,b],\end{cases}\tag{43}$$

so inequalities (38) and (39) are respectively transformed to

$$|\, r_3 f(b) + r_4 f(a) \, - (r_3 + r_4)\, f(x)\,| \le \begin{cases} \left(|\, r_4|^q \,(x-a) + |\, r_3|^q \,(b-x)\right)^{1/q} \|f'\|_p \,, \\ \left(|\, r_4|\, (x-a) + |\, r_3|\, (b-x)\right) \ \|f'\|_\infty \,, \\ \max \left\{ |\, r_4| \,, |\, r_3| \right\} \ \|f'\|_1 \,, \end{cases} \tag{44}$$

and

$$\left| \tfrac{bf(a)-af(b)}{b-a} + \tfrac{f(b)-f(a)}{b-a}\, x - f(x) \right| \le \tfrac{\|\, \beta-\alpha\, \|_p}{2|\, r_3+r_4|} \times$$
$$\left(\left| r_4 + \tfrac{r_3 b + r_4 a - (r_3+r_4)\, x}{b-a} \right|^q (x-a) + \left| r_3 - \tfrac{r_3 b + r_4 a - (r_3+r_4)\, x}{b-a} \right|^q (b-x) \right)^{1/q}. \tag{45}$$

Remark 3. For $r_1 = 0$, $r_2 + r_3 + r_4 = 0$ and $a = 0$, since the kernel (37) is reduced to

$$K\,(x; t, t - r_4, -r_3) = \begin{cases} t - r_4 & t \in [0, x], \\ -r_3 & t \in (x, b], \end{cases} \tag{46}$$

so we have

$$|\, \mathbf{F}_1\,(x; f, t - r_4, -r_3)\,| = \left| (x - (r_3 + r_4)) f(x) + r_3 f(b) + r_4 f(0) \, - \textstyle\int_0^x f(t)\, dt \right|$$

$$\le \begin{cases} \left(\int_0^x |\, t - r_4|^q \, dt + |\, r_3|^q \,(b-x) \right)^{1/q} \|f'\|_p \,, \\ \left(\int_0^x |\, t - r_4|\, dt + |\, r_3|\, (b-x) \right) \ \|f'\|_\infty \,, \\ \max\limits_{x\in[0,b]} \left\{ |\, r_3| \,, \max\limits_{t\in[0,x]} |\, t - r_4| \right\} \ \|f'\|_1 \,, \end{cases} \tag{47}$$

2.3. Subclass 3. If $v(x)$ is a constant function

If $v(x)$ is a constant number, say $d \neq 0$, then the kernel (11) and functions (12) and (13) respectively take the forms

$$K(x; t, u, d) = \begin{cases} u(t) & t \in [a, x], \\ d & t \in (x, b], \end{cases} \tag{48}$$

$$\mathbf{F}_1\,(x; f, u, d) = (u(x) - d)\, f(x) + d f(b) - u(a) f(a) - \int_a^x u'(t) f(t)\, dt \,, \tag{49}$$

and

$$\mathbf{F}_2\,(x; f, u, d) = \mathbf{F}_1\,(x; f, u, d) - \frac{f(b) - f(a)}{b - a} \left(\int_a^x u(t)\, dt + d\,(b - x) \right). \tag{50}$$

Hence, by applying the main theorem for the two above functions (49) and (50) one can get

$$|\, \mathbf{F}_1\,(x; f, u, d)\,| \le \begin{cases} \left(\int_a^x |\, u(t)\,|^q \, dt + |\, d\,|^q \,(b-x) \right)^{1/q} \|f'\|_p \,, \\ \left(\int_a^x |\, u(t)\,|\, dt + |\, d\,|\, (b-x) \right) \ \|f'\|_\infty \,, \\ \max\limits_{x\in[a,b]} \left\{ |\, d\,| \,, \max\limits_{t\in[a,x]} |\, u(t)\,| \right\} \ \|f'\|_1 \,, \end{cases} \tag{51}$$

and

$$
\begin{aligned}
&|\,\mathbf{F}_2\,(x;f,u,d)\,| \le \tfrac{1}{2}\,\|\,\beta-\alpha\,\|_p \times \\
&\Big(\int_a^x \Big|\,u(t)-\tfrac{1}{b-a}\left(\int_a^x u(t)\,dt+d(b-x)\right)\Big|^q dt \\
&+\int_x^b \Big|\,d-\tfrac{1}{b-a}\left(\int_a^x u(t)\,dt+d(b-x)\right)\Big|^q dt\Big)^{1/q}.
\end{aligned} \tag{52}
$$

One of the advantages of inequalities (51) and (52) is to find two upper bounds for the absolute value of some incomplete special functions. In other words, since many incomplete special functions have an integral form as $\int_a^x g(t)\,dt$, the two latter inequalities can be used for this purpose, see also [11] in this regard. For example, since the incomplete gamma function can be represented as

$$
\mathbf{\Gamma}(x;\alpha)=\int_0^x t^{\alpha-1}\exp(-t)\,dt \quad (\alpha>1)\,, \tag{53}
$$

so by choosing $u(t)=t^{\alpha}/\alpha$, $f(t)=exp(-t)$ and $a=0$ in (49) and employing the first inequality of (51) we obtain

$$
\left|\frac{1}{\alpha}x^{\alpha}e^{-x}-\mathbf{\Gamma}(x;\alpha)+d\left(e^{-b}-e^{-x}\right)\right|\le\left(\frac{x^{\alpha q+1}}{\alpha^q(\alpha q+1)}+!\ \ |d|^q\,(b-x)\right)^{\frac{1}{q}}\left(\frac{1-e^{-bp}}{p}\right)^{\frac{1}{p}}, \tag{54}
$$

where $0<x\le b$, $d\neq 0$ and $1/p+1/q=1$ for $p\in[1,\infty)$. Also, since the incomplete beta function can be represented as

$$
\mathbf{B}\,(x;\alpha,\beta)=\int_0^x t^{\alpha-1}(1-t)^{\beta-1}dt \quad (\alpha,\beta>1\,)\,, \tag{55}
$$

so by choosing $u(t)=t^{\alpha}/\alpha$, $f(t)=(1-t)^{\beta-1}$ and $a=0$ in (49) and employing the first inequality of (51) we obtain

$$
\begin{aligned}
&\left|\,\tfrac{1}{\alpha}x^{\alpha}(1-x)^{\beta-1}-\mathbf{B}(x;\alpha,\beta)+d\left((1-b)^{\beta-1}-(1-x)^{\beta-1}\right)\right| \\
&\le(\beta-1)\left(\tfrac{x^{\alpha q+1}}{\alpha^q(\alpha q+1)}+|d|^q\,(b-x)\right)^{\frac{1}{q}}\left(\tfrac{1-(1-b)^{1+(\beta-2)p}}{1+(\beta-2)p}\right)^{\frac{1}{p}},
\end{aligned} \tag{56}
$$

where $0<x\le b\le 1$, $d\neq 0$ and $1/p+1/q=1$ for $p\in[1,\infty)$.

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References

[1] F. Ahmad, N.S. Barnett, S.S. Dragomir, *New weighted Ostrowski and Chebyshev type inequalities*, Nonlinear Analysis: Theory, Methods & Applications, 71 (12) (2009) 1408-1412.

[2] P. Cerone, S.S. Dragomir, *New bounds for the Chebyshev functional*, Appl. Math. Lett., 18 (2005) 603-611.

[3] P.L.Chebyshev, *Sur les expressions approximative des integrals par les auters prises entre les mmes limites*, Proc. Math. Soc. Charkov., 2 (1882) 93-98.

[4] X. L. Cheng, *Improvement of some Ostrowski-Gruss type inequalities*, Comput. Math. Appl. 42 (2001) 109-114.

[5] S.S. Dragomir, *Bounds for some perturbed Chebyshev functionals*, J. Inequal. Pure Appl. Math. 9 (3) (2008) Art. 64.

[6] S.S. Dragomir, S. Wang, *An inequality of Ostrowski-Gruss type and its applications to the estimation of error bounds for some special means and for some numerical quadrature rules*, Comput. Math. Appl. 33 (11) (1997) 15-20.

[7] S.S. Dragomir, S. Wang, *Applications of Ostrowski's inequality to the estimation of error bounds for some special means and for some numerical quadrature rules*, Appl. Math. Lett. 11 (1) (1998) 105-109.

[8] I. Fedotov, S.S. Dragomir, *An inequality of Ostrowski type and its applications for Simpson's rule and special means*, Math. Inequal. Appl. 2 (4) (1999) 491-499.

[9] G. Gruss, *Uber das Maximum des absoluten Betrages von* $\frac{1}{(b-a)}\int_a^b f(x)\,g(x)\,dx - \frac{1}{(b-a)^2}\int_a^b f(x)\,dx \int_a^b g(x)\,dx$, Math. Z., 39 (1935) 215-226.

[10] Z. Liu, *Some Ostrowski-Gruss type inequalities and applications*, Comput. Math. Appl. 53 (2007) 73-79.

[11] M. Masjed-Jamei, *A Main inequality for several special functions*, Comput. Math. Appl., 60 (2010) 1280-1289.

[12] M. Masjed-Jamei, Feng Qi, H.M Srivastava, *Generalizations of some classical inequalities via a special functional property*, Integral Transforms Spec. Funct, 21 (2010) 327-336.

[13] M. Masjed-Jamei, S. S. Dragomir, *A new generalization of the Ostrowski inequality and applications*, Filomat, 25 (2011) 115-123.

[14] D.S. Mitrinovic, J.E. Pecaric and A.M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht/ Boston/ London, 1993.

[15] M. Niezgoda, *A new inequality of Ostrowski-Gruss type and applications to some numerical quadrature rules*, Comput. Math. Appl, 58 (2009) 589-596.

[16] A. Ostrowski, *Uber die absolutabweichung einer differentiebaren funktion van ihrem inte-gralmittelwert*, Comment Math. Helv 10 (1938) 226-227.

[17] A.M. Ostrowski, *On an integral inequality*, Aequat. Math. 4 (1970) 358-373.

Mohammad Masjed-Jamei:
Department of Mathematics, K. N. Toosi University of Technology, P.O. Box: 16315-1618, Tehran, Iran

School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box: 19395-5746, Tehran, Iran

E-mail: mmjamei@kntu.ac.ir; mmjamei@yahoo.com.