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FIXED POINT THEOREMS IN MINIMAL GENERALIZED CONVEX SPACES

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Abstract

This paper deals with coincidence and fixed point theorems in minimal generalized convex spaces. By establishing a kind of KKM Principle in minimal generalized convex space, we obtain some results on coincidence point and fixed point theorems. Generalized versions of Ky Fan's lemma, Fan-Browder fixed point theorem, Nash equilibrium theorem and some Urai's type fixed point theorems in minimal generalized convex spaces are given.

1 Introduction

As it is known, fixed point theory, as a relevant topic both in pure and applied mathematics, is a flourishing branch of nonlinear analysis with many directions of development and has a broad set of applications in other sciences and engineering, for example differential equations, chaos and etc.

Park and Kim introduced the concept of generalized convex space in 1993, which it extends many generalized convex structures on topological vector spaces [18]. Although this new concept generalizes topological vector space, it was mainly developed in connection with the fixed point theory and KKM theory. This new concept comes at the top of a chain of several well known generalizations of convex space that can now be seen as particular forms of G-convex spaces, for details see [4]. A brief survey of some recent generalization of Fan-KKM principle and its applications in coincidence point and fixed point theory can be found in [13] and [23].

The concept of minimal structure and minimal spaces, as a generalization of topology and topological spaces were introduced in [12]. Further results about minimal spaces can be found in [1, 2, 3, 5, 11] and [19]. Recently, authors in [4] introduced the notion of minimal generalized convex space as an extended version

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of generalized convex space. At the present paper, Fan-KKM principle in minimal generalized convex space applied to obtain some coincidence and fixed point theorems. In fact, Ky Fan's lemma, Fan-Browder fixed point theorem, Nash equilibrium theorem and some Urai's type fixed point theorems in minimal generalized convex space are extended.

2 Preparatory results

To ease understanding of the material incorporated in this paper we recall some basic definitions and results. For details on the following notions we refer to [3, 4, 6, 11, 12] and [19] and references therein.

A family $\mathcal{M} \subseteq \mathcal{P}(X)$ is said to be a *minimal structure* on X if $\emptyset, X \in \mathcal{M}$. In this case $(X, \mathcal{M})$ is called a *minimal space*. For some examples in this setting see [11]. In a minimal space $(X, \mathcal{M})$, $A \in \mathcal{P}(X)$ is said to be an *m-open set* if $A \in \mathcal{M}$ and also $B \in \mathcal{P}(X)$ is an *m-closed set* if $B^c \in \mathcal{M}$. We set $m\text{-Int}(A) = \bigcup\{U : U \subseteq A, U \in \mathcal{M}\}$ and $m\text{-Cl}(A) = \bigcap\{F : A \subseteq F, F^c \in \mathcal{M}\}$.

Proposition 1. [11] *For any two sets A and B,*

(a) $m\text{-Int}(A) \subseteq A$ *and* $m\text{-Int}(A) = A$ *if A is an m-open set.*

(b) $A \subseteq m\text{-Cl}(A)$ *and* $A = m\text{-Cl}(A)$ *if A is an m-closed set.*

(c) $m\text{-Int}(A) \subseteq m\text{-Int}(B)$ *and* $m\text{-Cl}(A) \subseteq m\text{-Cl}(B)$ *if* $A \subseteq B$.

(d) $m\text{-Int}(A \cap B) \subseteq (m\text{-Int}(A)) \cap (m\text{-Int}(B))$ *and* $(m\text{-Int}(A)) \cup (m\text{-Int}(B)) \subseteq m\text{-Int}(A \cup B)$.

(e) $m\text{-Cl}(A \cup B) \supseteq (m\text{-Cl}(A)) \cup (m\text{-Cl}(B))$ *and* $m\text{-Cl}(A \cap B) \subseteq (m\text{-Cl}(A)) \cap (m\text{-Cl}(B))$.

(f) $m\text{-Int}(m\text{-Int}(A)) = m\text{-Int}(A)$ *and* $m\text{-Cl}(m\text{-Cl}(B)) = m\text{-Cl}(B)$.

(g) $(m\text{-Cl}(A))^c = m\text{-Int}(A^c)$ *and* $(m\text{-Int}(A))^c = m\text{-Cl}(A^c)$.

Definition 1. [19] A minimal space $(X, \mathcal{M})$ enjoys the *property U* if the arbitrary union of m-open sets is m-open.

Proposition 2. [19] *For a minimal structure $\mathcal{M}$ on a set X, the following are equivalent.*

(a) $(X, \mathcal{M})$ *has the property U.*

(b) *If* $m\text{-Int}(A) = A$, *then* $A \in \mathcal{M}$.

(c) *If* $m\text{-Cl}(B) = B$, *then* $B^c \in \mathcal{M}$.

Definition 2. [19] Suppose $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ are minimal space. A function $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ is called *m-continuous* if $f^{-1}(B) \in \mathcal{M}$ for any $B \in \mathcal{N}$.

Definition 3. Suppose (X, τ) is a topological space and also suppose $(Y, \mathcal{N})$ is a minimal space. A function $f : (X, \tau) \to (Y, \mathcal{N})$ is called *(τ, m)-continuous* if $f^{-1}(B) \in \tau$ for any $B \in \mathcal{N}$.

Theorem 1. [4] *Suppose (X,τ) is a topological space, $(Y,\mathcal{N})$ is a minimal space. Also suppose that $f:(X,\tau)\to(Y,\mathcal{N})$ is a function. Then the following are equivalent.*

(a) *f is (τ,m)-continuous.*

(b) *$f^{-1}(B)$ is closed set for each m-closed set B in Y.*

(c) $\mathrm{Cl}(f^{-1}(B))\subseteq f^{-1}(m\text{-}\mathrm{Cl}(B))$ *for each subset B of Y.*

(d) $f(\mathrm{Cl}(A))\subseteq m\text{-}\mathrm{Cl}(f(A))$ *for each subset A of X.*

(e) $f^{-1}(m\text{-}\mathrm{Int}(B))\subseteq \mathrm{Int}(f^{-1}(B))$ *for each subset B of Y.*

Definition 4. [19] For a minimal space $(X,\mathcal{M})$,

(a) a family of m-open sets $\mathcal{A}=\{A_j : j\in J\}$ in X is called an *m-open cover* of K if $K\subseteq\bigcup_j A_j$. Any subfamily of $\mathcal{A}$ which is also an m-open cover of K is called a *subcover* of $\mathcal{A}$ for K;

(b) a subset K of X is *m-compact* whenever given any m-open cover of K has a finite subcover.

Definition 5. [3] For two minimal spaces $(X,\mathcal{M})$ and $(Y,\mathcal{N})$ we define *minimal product structure* for $X\times Y$ as follows :

$$\mathcal{M}\times\mathcal{N}=\{A\subseteq X\times Y : \forall\ (x,y)\in A,\ \ \exists U\in\mathcal{M},\exists V\in\mathcal{N};(x,y)\in U\times V\subseteq A\}.$$

Definition 6. [3] A *linear minimal structure* on a vector space X over the complex field $\mathbb{F}$ is a minimal structure $\mathcal{M}$ on X such that the two mappings

$$\begin{array}{rcl} + & : & X\times X\to X,\quad (x,y)\mapsto x+y \\ . & : & \mathbb{F}\times X\to X,\quad (t,x)\mapsto tx \end{array}$$

are m-continuous, where $\mathbb{F}$ has the usual topology and both $\mathbb{F}\times X$ and $X\times X$ have the corresponding product minimal structures. A *linear minimal space* (or *minimal vector space*) is a vector space together with a linear minimal structure.

Obviously, any topological vector space is a minimal vector space. Consider the real field $\mathbb{R}$. Clearly $\mathcal{M}=\{(a,b) : a,b\in\mathbb{R}\cup\{\pm\infty\}\}$ is a minimal structure on $\mathbb{R}$. In [4] it is shown that $\mathcal{M}$ is a linear minimal structure on $\mathbb{R}$. This implies there is some linear minimal spaces which are not topological vector space.

A *multimap* $F:X\multimap Y$ is a function from a set X into the power set of Y; that is, a function with the values $F(x)\subseteq Y$ for all $x\in X$. Given $A\subseteq X$, set

$$F(A)=\bigcup_{x\in A}F(x).$$

Definition 7. We say that a multimap $F:X\multimap X$ has a *fixed point* if $x_0\in F(x_0)$, for some $x_0\in X$.

Definition 8. Consider two multimaps $F,T:X\multimap Y$. We say that F and T has *coincidence point* if there is some $x_0\in X$ for which $F(x_0)\cap T(x_0)\neq\emptyset$.

Let $\langle D\rangle$ denote the set of all nonempty finite subsets of a set D and let Δ_n be the n-simplex with vertices $e_0, e_1, \ldots, e_n$, Δ_J be the face of Δ_n corresponding to $J \in \langle A\rangle$ where $A \in \langle D\rangle$; for example, if $A = \{a_0, a_1, \ldots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \ldots, a_{i_k}\} \subseteq A$, then $\Delta_J = co\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\}$. A *minimal generalized convex space* (briefly MG-convex space) (X, D, Γ) consists of a minimal space $(X, \mathcal{M})$, a nonempty set D, and a multimap $\Gamma : \langle D\rangle \multimap X$ in which for $A \in \langle D\rangle$ with $n+1$ elements, there exists a (τ, m)-continuous function $\phi_A : \Delta_n \to \Gamma_A := \Gamma(A)$ for which $J \in \langle A\rangle$ implies that $\phi_A(\Delta_J) \subseteq \Gamma_J = \Gamma(J)$ and also $A \subseteq B$ implies that $\Gamma_A \subseteq \Gamma_B$. In case to emphasize $X \supseteq D$, (X, D, Γ) will be denoted by $(X \supseteq D, \Gamma)$; and if $X = D$, then $(X \supseteq X; \Gamma)$ by (X, Γ). For an MG-convex space $(X \supseteq D, \Gamma)$, a subset $Y \subseteq X$ is said to be *MG-convex* if $N \in \langle D\rangle$ and $N \subseteq Y$ implies that $\Gamma_N \subseteq Y$.

Obviously, any G-convex space is an MG-convex space. On the other hand, suppose that $(X, \mathcal{M})$ is a minimal vector space which is not a topological vector space (for example see [4]). Consider the multimap $\Gamma : \langle X\rangle \multimap X$ defined by $\Gamma(\{a_0, a_1, \ldots, a_n\}) = \{\sum_{i=0}^{n} \lambda_i a_i : 0 \leq \lambda_i \leq 1, \sum_{i=0}^{n} \lambda_i = 1\}$. One can deduce that (X, Γ) is a minimal generalized convex space, of course (X, Γ) is not a generalized convex space [4].

Definition 9. Suppose (X, D, Γ) is an MG-convex space and Y is a minimal space. A multimap $F : D \multimap X$ is called a *KKM multimap* if $\Gamma_A \subseteq F(A)$ for any $A \in \langle D\rangle$. $F : X \multimap Y$ is said to have the *minimal KKM property* (briefly MKKM property) if, for any multimap $G : D \multimap Y$ with m-closed (resp. m-open) values satisfying

$$F(\Gamma_A) \subseteq G(A) \texttt{ for all } A \in \langle D\rangle,$$

the family $\{G(z)\}_{z\in D}$ has the finite intersection property. Set

$$\text{MKKM}(X, Y) = \{F : X \multimap Y : F \texttt{ has the MKKM property}\}.$$

MKKMC(X, Y) denotes the class MKKM for m-closed valued multimaps G and also MKKMO(X, Y) for m-open valued multimaps G.

Theorem 2. [4] *Suppose (X, D, Γ) is an MG-convex space and $F : D \multimap X$ is a multimap satisfying*

(a) *F has m-closed values,*

(b) *F is a KKM map.*

Then $\{F(z) : z \in D\}$ has the finite intersection property.
Further, if

(c) *$\bigcap_{z\in M} F(z)$ is m-compact for some $M \in \langle D\rangle$,*

then $\bigcap\limits_{z\in D} F(z) \neq \emptyset$.

The open version of the KKM principle was presented by Kim [9], Shih and Tan [20] and later Lassonde [10] showed that the classical (closed) and open versions of the KKM principle can be derived from each other.

Theorem 3. [4] *Suppose* (X, D, Γ) *is an* MG*-convex space and* $F : D \multimap X$ *is a multimap satisfying*

(a) F *has* m*-open values,*

(b) F *is KKM map.*

Then $\{F(z) : z \in D\}$ *has the finite intersection property.*
Further, if

(c) $\bigcap_{z\in N} m\text{-Cl}(F(z))$ *is* m*-compact for some* $N \in \langle D\rangle$,

(d) *minimal space* $(X, \mathcal{M})$ *has the property* U.

Then $\bigcap_{z\in D} m\text{-Cl}(F(z)) \neq \emptyset$.

Corollary 1. [4] *Suppose* (X, D, Γ) *is an* MG*-convex space. Then* I_X *defined by* $I_X(x) = \{x\}$ *is an element of* $\text{MKKM}(X, X) = \text{MKKMC}(X, X) \cap \text{MKKMO}(X, X)$.

3 Fixed Point Theorems

The present section is focused on fixed point theory for m-closed and m-open valued multimaps in minimal generalized convex spaces.

Theorem 4. *Suppose* (X, D, Γ) *is an* MG*-convex space,* $(Y, \mathcal{M})$ *is a minimal space,* $S : D \multimap Y$, $T : X \multimap Y$ *and* $F \in \text{MKKMC}(X, Y)$ *are multimaps satisfying*

(a) S *has* m*-open values,*

(b) *for each* $y \in F(X)$, $M \in \langle S^-(y)\rangle$ *implies that* $\Gamma_M \subseteq T^-(y)$,

(c) $Y = S(N)$ *for some* $N \in \langle D\rangle$.

Then F *and* T *have a coincidence point.*

Proof. Since S is m-open valued, so the multimap $G : D \multimap Y$ defined by $G(z) = Y \setminus S(z)$ for all $z \in D$ is an m-closed valued multimap. Condition (c) implies that

$$\bigcap_{z\in N} G(z) = \bigcap_{z\in N} (Y \setminus S(z)) = Y \setminus \bigcup_{z\in N} S(z) = \emptyset;$$

i.e., $\{G(z) : z \in D\}$ doesn't have the finite intersection property. Now, $F \in \text{MKKMC}(X, Y)$ and Definition 9 imply that there exists $A \in \langle D\rangle$ for which $F(\Gamma_A) \nsubseteq G(A)$. Therefore, there is $y_0 \in F(\Gamma_A)$ such that $y_0 \notin G(z) = Y \setminus S(z)$ for all $z \in A$. Thus $y_0 \in S(z)$ for each $z \in A$ and hence $z \in S^-(y_0)$ for all $z \in A$; i.e., $A \in \langle S^-(y_0)\rangle$. Since $y_0 \in F(X)$, so (b) implies that $\Gamma_A \subseteq T^-(y_0)$ and so $y_0 \in F(\Gamma_A) \subseteq F(T^-(y_0))$. Consequently, there is an element $x_0 \in T^-(y_0)$ for which $y_0 \in F(x_0)$ and so $y_0 \in F(x_0) \cap T(x_0)$.

Corollary 2. *Suppose* (X, D, Γ) *is an* MG*-convex space,* $S : D \multimap X$ *and* $T : X \multimap X$ *are two multimaps satisfying*

(a) S *has* m*-open values,*

(b) $x \in X$ *and* $M \in \langle S^-(x)\rangle$ *imply that* $\Gamma_M \subseteq T^-(x)$,

(c) $X = S(N)$ *for some* $N \in \langle D\rangle$.

Then T *has a fixed point.*

Proof. From Corollary 1, $I_X \in \text{MKKMC}(X, X)$. Assume $F = I_X$ and $Y = X$. Theorem 4 implies that, there is $x_0 \in X$ such that $I_X(x_0) \cap T(x_0) \neq \emptyset$; i.e., $x_0 \in T(x_0)$.

Corollary 3. *Suppose (X, D, Γ) is an m-compact MG-convex space, $S : X \multimap D$ and $T : X \multimap X$ are two multimaps satisfying*

(a) *S^- has m-open values,*

(b) *$x \in X$ and $M \in \langle S(x)\rangle$ imply that $\Gamma_M \subseteq T(x)$,*

(c) *S has nonempty values.*

Then T has a fixed point.

Proof. It follows from (c) and (a) that $\{S^-(z) : z \in D\}$ is an m-open cover of X, also m-compactness of X implies that, there is $N \in \langle D\rangle$ for which $X = S^-(N)$. Therefore, since $(S^-)^- = S$, so all conditions in Corollary 2 hold for (S^-, T^-) instead of (S, T). Consequently, there is an element $x_0 \in X$ for which $x_0 \in T^-(x_0)$; i.e., $x_0 \in T(x_0)$.

Corollary 4. *Suppose (X, D, Γ) is an MG-convex space with the property U, $S : X \multimap D$ and $T : X \multimap X$ are two multimaps satisfying*

(a) *$x \in X$ and $M \in \langle S(x)\rangle$ imply that $\Gamma_M \subseteq T(x)$,*

(b) *$X = \bigcup\{m\text{-Int}(S^-(z)) : z \in N\}$ for some $N \in \langle D\rangle$.*

Then T has a fixed point.

Proof. Consider the multimap $m\text{-Int}(S^-) : D \multimap X$. It follows from Proposition 2 that $m\text{-Int}(S^-)$ is m-open valued. Since $\langle (m\text{-Int}(S^-))^-(x)\rangle \subseteq \langle S(x)\rangle$, so condition (b) in Corollary 2 satisfies for $(m\text{-Int}(S^-), T^-)$ instead of (S, T). Now, applying Corollary 2 completes the proof.

Corollary 5. *Suppose (X, D, Γ) is an m-compact MG-convex space with the property U, $S : X \multimap D$ and $T : X \multimap X$ are two multimaps satisfying*

(a) *$\bigcup\{S^-(z) : z \in D\} = \bigcup\{m\text{-Int}(S^-(z)) : z \in D\}$,*

(b) *$x \in X$ and $M \in \langle S(x)\rangle$ imply that $\Gamma_M \subseteq T(x)$,*

(c) *S has nonempty value.*

Then T has a fixed point.

Proof. Since S has nonempty value, so for each $x \in X$, there is an element $z \in D$ such that $z \in S(x)$ and so $x \in S^-(z)$; i.e., $X = \bigcup\{S^-(z) : z \in D\}$. Therefore, (a) implies that $X = \bigcup\{m\text{-Int}(S^-(z)) : z \in D\}$. It follows from property U that $m\text{-Int}(S^-)$ is m-open valued. Thus $\{m\text{-Int}(S^-(z)) : z \in D\}$ is an m-open cover of X. Consequently, there is $N \in \langle D\rangle$ for which $X = \bigcup\{m\text{-Int}(S^-(z)) : z \in N\}$. Now, by Corollary 4, T has a fixed point.

Remark 1. Note that

(a) Theorem 4 for G-convex space is due to Park [14] and also in [16] it applied to various form of the Fan-Browder theorem, Ky Fan intersection theorem and Nash equilibrium theorem for G-convex space.

(b) Corollary 2 is an extension of the open version of Theorem 4.1 in [14] which it gives various forms of the Fan-Browder Theorem, Ky Fan intersection theorem, and the Nash equilibrium theorem for G-convex spaces [15].

(c) Corollary 3 is an extended version of Theorem 4.3 in [14] and Theorem 3 in [17] and since it is a generalization of the Fan-Browder fixed point theorem for topological vector spaces, which it applied to the existence of maximal elements in mathematical economics by Borglin and Keiding [7] and Yannelis and Prabhakar [22].

(d) Corollary 4 and Corollary 5 extend Theorem $3'$ and Theorem $3''$ in [17] respectively.

Theorem 5. *Suppose* (X, D, Γ) *is an* MG*-convex space,* $(Y, \mathcal{M})$ *is a minimal space,* $S : D \multimap Y$*,* $T : X \multimap Y$ *and* $F \in \mathrm{MKKMO}(X, Y)$ *satisfying*

(a) *S has m-closed value,*

(b) *for each $y \in F(X)$, $M \in \langle S^-(y)\rangle$ implies that $\Gamma_M \subseteq T^-(y)$,*

(c) *$Y = S(N)$ for some $N \in \langle D\rangle$.*

Then F and T have a coincidence point.

Proof. It is suffice to interchange the position of the terms "m-open" and "$F \in \mathrm{MKKMC}(X, Y)$" by "$m$-closed" and "$F \in \mathrm{MKKMO}(X, Y)$" in the proof of Theorem 4.

Similar to the Corollary 2, applying Corollary 1 and Theorem 5 we have the following result.

Corollary 6. *Suppose* (X, D, Γ) *is an* MG*-convex space,* $S : D \multimap X$ *and* $T : X \multimap X$ *are two multimaps satisfying*

(a) *S has m-closed values,*

(b) *$x \in X$ and $M \in \langle S^-(x)\rangle$ imply that $\Gamma_M \subseteq T^-(x)$,*

(c) *$X = S(N)$ for some $N \in \langle D\rangle$.*

Then T has a fixed point.

Proof. From Corollary 1, $I_X \in \mathrm{MKKMO}(X, X)$. Assume $F = I_X$ and $Y = X$. Theorem 5 implies that, there is $x_0 \in X$ such that $I_X(x_0) \cap T(x_0) \neq \emptyset$; i.e., $x_0 \in T(x_0)$.

Remark 2. It should be noticed that,

(a) Theorem 5 improves a result due to Park (Theorem $3.1'$ in [14]).

(b) Corollary 6 is an extension of the closed version of Theorem 4.1 in [14].

Theorem 6. *Suppose* (X, D, Γ) *is an* MG*-convex space also suppose* $S : D \multimap X$ *and* $T : X \multimap X$ *are two multimaps satisfying*

(a) *S has m-open (resp. m-closed) values,*

(b) *$y \in X$ and $M \in \langle S^-(y)\rangle$ imply that $\Gamma_M \subseteq T^-(y)$,*

(c) *$T(X) \subseteq S(N)$ for some $N \in \langle D\rangle$,*

(d) *T^- has nonempty values.*
Then T has a fixed point.

Proof. Since T^- is a nonempty valued multimap, so for each $x \in X$, there is $y \in X$ in which $y \in T^-(x)$ and hence $x \in T(y)$; it implies that $X = T(X)$. It follows from condition (c) that $X = T(X) = S(N)$. That T has a fixed point follows from Corollary 2 (resp. Corollary 6).

Theorem 7. *Suppose $(X \supseteq D, \Gamma)$ is an MG-convex space and $T : X \multimap X$ is a multimap such that $T(x)$ is an MG-convex set in X for each $x \in X$. If there exist $z_1, z_2, \dots, z_n \in D$ and nonempty m-open (resp. m-closed) subsets $A_i \subseteq T^-(z_i)$ for $i = 1, 2, \dots, n$ such that $X = \bigcup_{i=1}^{n} A_i$, then T has a fixed point.*

Proof. Set $N := \{z_1, z_2, \dots, z_n\}$ and define the multimap $S : D \multimap X$ by $S(z_i) = A_i$ for each $z_i \in N$, and $S(z) = \emptyset$ for each $z \in D \setminus N$. It is easy to see that $X = S(N)$ and $S(z)$ is m-open (resp. m-closed) for all $z \in D$. Hence, (a) and (c) of Corollary 2 (resp. Corollary 6) satisfy. From definition of the multimap S we have $S(z) \subseteq T^-(z)$ for all $z \in D$, so for any $y \in X$ and $z \in D$, $y \in S(z)$ implies that $y \in T^-(z)$ consequently, $z \in S^-(y)$ implies that $z \in T(y)$; i.e., $S^-(y) \subseteq T(y)$ for all $y \in X$. Assume $F : X \multimap X$ is defined by $F(x) = T^-(x)$ for each $x \in X$. Then $S^-(y) \subseteq T(y) = F^-(y)$. Since $F^-(y)$ is MG-convex for each $y \in X$, so (b) in Corollary 2 (resp. Corollary 6) holds and so there is $x_0 \in X$ such that $x_0 \in F^-(x_0) = T(x_0)$.

Remark 3. Theorem 6 and Theorem 7 are extended versions of Theorem 4.4 and Theorem 4.2 in [14] respectively.

Lemma 1. *Suppose (X, D, Γ) is an MG-convex space and D' is a nonempty subset of D. Then $(\Gamma_{D'}, D', \Gamma|_{\langle D' \rangle})$ is an MG-convex space too, where $\Gamma_{D'} = \bigcup_{N \in \langle D' \rangle} \Gamma_N$.*

Proof. Consider an arbitrary set A in $\langle D' \rangle$ having $n + 1$ elements. Since $A \in \langle D \rangle$, so from the assumption there is a (τ, m)-continuous function $\phi_A : \Delta_n \longrightarrow \Gamma(A) = \Gamma|_{\langle D' \rangle}(A)$ for which $J \in \langle A \rangle$ implies that $\phi_A(\Delta_J) \subseteq \Gamma(J) = \Gamma|_{\langle D' \rangle}(J)$. Therefore, $(\Gamma_{D'}, D', \Gamma|_{\langle D' \rangle})$ is an MG-convex space.

Suppose (X, D, Γ) is an MG-convex space. For a multimap $T : X \multimap D$, we define $\Gamma \diamond T : X \multimap X$ as the following

$$\Gamma \diamond T(x) = \bigcup_{N \in \langle T(x) \rangle} \Gamma_N.$$

Theorem 8. *Suppose (X, D, Γ) is an MG-convex space, $T : X \multimap D$ is a multimap and also suppose that there exist $z_1, \dots, z_n \in D$ and nonempty m-open (resp. m-closed) subsets $G_i \subseteq T^-(z_i)$ for each $i = 1, \dots, n$. If $\Gamma_{\{z_1, \dots, z_n\}} \subseteq \bigcup_{i=1}^{n} G_i$, then $\Gamma \diamond T$ has a fixed point.*

Proof. Set $Y = \Gamma_{D'}$ where $D' = \{z_1, \ldots, z_n\}$. It follows from Lemma 1 that $(\Gamma(D'), D', \Gamma|_{\langle D' \rangle})$ is an MG-convex space. Consider the multimap $F : D' \multimap Y$ defined by $F(z_i) = Y \setminus G_i$ for each $z_i \in D'$. Therefore, $F(z_i)$ is m-closed (resp. m-open) in Y and $\bigcap\limits_{i=1}^{n} F(z_i) = Y \setminus \bigcup\limits_{i=1}^{n} G_i = \emptyset$ and so the family $\{F(z) : z \in D'\}$ does not have the finite intersection property. It follows from Theorem 2 (resp. Theorem 3) that F is not a KKM map; i.e., there exists $N = \{z_{i_1}, \ldots, z_{i_k}\} \in \langle D' \rangle$ for which $\Gamma_N \nsubseteq F(N) = \bigcup\limits_{j=1}^{n} (Y \setminus G_{i_j})$. There exits $x_0 \in \Gamma_N$ in which $x_0 \in G_{i_j} \subseteq T^-(z_{i_j})$ for each $j \in \{1, \ldots, k\}$, consequently, $z_{i_j} \in T(x_0)$ for all $z_{i_j} \in N$. Therefore, $N \subseteq T(x_0)$ which implies that $x_0 \in \Gamma_N \subseteq \Gamma \diamond T(x_0)$.

Corollary 7. *Suppose Y is a nonempty m-compact MG-convex subset of an MG-convex space (X, Γ) and also $T : Y \multimap Y$ is a nonempty MG-convex valued multimap. If T^- is m-open valued, then T has a fixed point.*

Proof. Since Y is MG-convex subset of X, so one can verify that $(Y, Y, \Gamma|_{\langle Y \rangle})$ is an MG-convex space. We claim that all conditions of Theorem 8 satisfy for $(Y, Y, \Gamma|_{\langle Y \rangle})$. To see this, since T is nonempty valued and T^- is m-open valued, so $\{T^-(y) : y \in Y\}$ is an m-open cover of the m-compact set Y. Therefore, there are $z_1, \ldots, z_n$ in Y for which $\Gamma_{\{z_1, \ldots, z_n\}} \subseteq Y = \bigcup\limits_{i=1}^{n} T^-(z_i)$. Put $G_i = T^-(z_i)$. According to Theorem 8 we have $\Gamma \diamond T$ has a fixed point. That T has a fixed point follows from the definition of $\Gamma \diamond T$ and the fact that T is an MG-convex valued multimap.

Here and in the sequel, for a multimap $T : X \multimap X$, let $K_T = \{x \in X : x \notin T(x)\}$.

Corollary 8. *Suppose (X, Γ) is an MG-convex space and also $T : X \multimap X$ is an MG-convex valued multimap. If there exist $z_1, \ldots, z_n \in X$ and nonempty m-open (resp. m-closed) subsets $G_i \subseteq T^-(z_i) (1 \leq i \leq n)$ for which $K_T \subseteq \bigcup\limits_{i=1}^{n} G_i$, then T has a fixed point.*

Proof. On the contrary, suppose T has no fixed point. It follows from the assumption that $X = K_T \subseteq \bigcup\limits_{i=1}^{n} G_i$. Theorem 8 implies that T has a fixed point, which is a contradiction.

Corollary 9. *Suppose (X, Γ) is an MG-convex space and $T : X \multimap X$ is an MG-convex valued multimap such that for all $x \in K_T$, there is an m-open (resp. m-closed) set G_x in X containing x and there is $y_x \in X$ for which $G_x \cap K_T \subseteq T^-(y_x)$ and K_T is covered by a finite subset of $\{G_x : x \in X\}$. Then T has a fixed point.*

Proof. On the contrary, suppose T has no fixed point. Then $K_T = X$ and so for all $x \in X$ there is an m-open (resp. m-closed) set G_x in X and there is $y_x \in X$ for which $G_x \cap K_T = G_x \subseteq T^-(y_x)$. Therefore,

$$X = \bigcup_{i=1}^{n} G_{x_i} = \bigcup_{i=1}^{n} T^-(y_{x_i}),$$

for some $\{y_{x_1}, \ldots, y_{x_n}\} \in X$. Corollary 8 implies that T has a fixed point, which is a contradiction.

Corollary 10. *Suppose (X, Γ) is an MG-convex space and $T : X \multimap X$ is a multimap satisfying*

(a) *T has nonempty values,*

(b) *there is a multimap $S : X \multimap X$ with MG-convex values such that for all $x \in K_T$ there exists an m-open set G_x in X containing x and there is $y_x \in X$ for which $G_x \cap K_T \subseteq K_S \cap S^-(y_x)$,*

(c) *K_T is covered by a finite subset of $\{G_x : x \in X\}$.*
Then T has a fixed point.

Proof. Suppose T has no fixed point; i.e., $K_T = X$. One can verify that all conditions of Corollary 9 satisfy for the multimap S. Therefore, S has a fixed point which it contradicts with $G_x \subseteq K_S \cap S^-(y_x)$ for all $x \in X$.

Corollary 11. *Suppose that (X, Γ) is an MG-convex space and $T, F : X \multimap X$ are two multimaps satisfying*

(a) *$K_F \subseteq K_T$ and $T(x) \neq \emptyset$ for any $x \in K_F$,*

(b) *there is a multimap $S : X \multimap X$ with MG-convex values such that for all $x \in K_F$ there exists an m-open set G_x in X containing x and there is y_x for which $G_x \cap K_F \subseteq K_S \cap S^-(y_x)$,*

(c) *K_F is covered by a finite subset of $\{G_x : x \in K_T\}$.*
Then F has a fixed point.

Proof. On the contrary, suppose that F has no fixed point. Then (a) implies that $X = K_F = K_T$ and so T has no fixed point. Now, one can deduce that all conditions of corollary 10 satisfy for T. Therefore, T has a fixed point, which is a contradiction with the fact that $K_F \subseteq K_T$.

Remark 4. Note that

(a) The particular form of Theorem 8 goes back to S. Park [16] for convex spaces.

(b) Corollary 7 is a generalization of Browder's Theorem (Theorem 1 in [8]) and also it is an extended version of a result due to Park [16].

(c) Corollary 8 extends Theorem 2.5 in [16] and so it improves a result of Urai [21].

(d) Corollary 9, Corollary 10 and Corollary 11 for convex space were considered in [16] which they extend corresponding results of Urai [16, 21].

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