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# HILLE-YOSIDA THEOREMS FOR LOCAL CONVOLUTED $C$-SEMIGROUPS AND COSINE FUNCTIONS

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**Abstract**

The theory of convoluted $C$-operator families is an active research field. The main purpose of this paper is to prove several Hille-Yosida type theorems for local convoluted $C$-semigroups and cosine functions.

## 1 Introduction and preliminaries

There is an enormous literature studying global integrated $C$-semigroups and their applications to evolution equations. For the excellent and brief introduction to this subject, we refer the reader to the monographs [1] and [14]. A large numbers of papers, starting presumably with [2] and [35], written over the last twenty years, have concerned local integrated $C$-semigroups and cosine functions. Standard references are [26]-[29], [31], [33]-[34] and [36]-[38].

Compared with the above, there is notably little written about convoluted $C$-semigroups and cosine functions. The class of local convoluted $C$-semigroups was introduced in the papers of I. Ciorănescu and G. Lumer [9]-[12], who related them to ($\omega$-)ultradistribution semigroups (see [4]-[5], [7]-[8], [13], [16]-[17] and [21] for this notion). The classes of (local) convoluted $C$-cosine functions and global convoluted $C$-semigroups have been recently introduced in [25] and further analyzed in [23]. The references [3], [19] and [24] are also of importance.

In this note, we will report a few results on spectral properties of subgenerators of local convoluted $C$-semigroups and cosine functions, and continue the researches raised in [2], [12], [29]-[30] and [36]. The spectral characterization of subgenerators of local convoluted $C$-cosine functions relies upon safe and sound passing to the corresponding theory of semigroups. It is also worth noticing that we exploit the profiling of $C$-pseudoresolvents given in the construction of fractional powers of operators [15].

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By $E$ and $L(E)$ are denoted a non-trivial complex Banach space and the Banach algebra of bounded linear operators on $E$. Given a closed linear operator $A$ acting on $E$, we designate by $D(A)$, $N(A)$, $R(A)$ and $\rho(A)$ its domain, kernel, range and resolvent set, respectively.

Henceforth we assume $C \in L(E)$, $C$ is injective, $\tau \in (0,\infty]$, $K$ is a complex valued locally integrable function in $[0,\tau)$ and $K$ is not identical to zero. Let us remind that the $C$ resolvent set of $A$, denoted by $\rho_C(A)$, is the set consisted of all complex numbers $\lambda$ such that the operator $\lambda - A$ is injective and that $R(C) \subseteq R(\lambda - A)$. Put $\Theta(t) := \int_0^t K(s)ds,\ t \in [0,\tau)$; then it is clear that $\Theta$ is an absolutely continuous function in $[0,\tau)$ and that $\Theta'(t) = K(t)$ for a.e. $t \in [0,\tau)$. We employ occasionally the following condition for the function $K$ :

(P1) $K$ is Laplace transformable, i.e., $K \in L^1_{loc}([0,\infty))$ and there exists $\beta \in \mathbb{R}$ so that $\tilde{K}(\lambda) := \mathcal{L}(K)(\lambda) := \lim_{b\to\infty} \int_0^b e^{-\lambda t}K(t)dt := \int_0^\infty e^{-\lambda t}K(t)dt$ exists for all $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda > \beta$. Put $\mathrm{abs}(K)$ :=inf$\{\mathrm{Re}\lambda : \tilde{K}(\lambda)$ exists$\}$ and denote by $\mathcal{L}^{-1}$ the inverse Laplace transform.

Let us recall that a function $K \in L^1_{loc}([0,\tau))$ is called a kernel, if for every $\phi \in C([0,\tau))$, the supposition $\int_0^t K(t-s)\phi(s)ds = 0,\ t \in [0,\tau)$, implies $\phi \equiv 0$; due to the famous E. C. Titchmarsh's theorem ([1]), the condition $0 \in \mathrm{supp}K$ implies that $K$ is a kernel.

**Definition 1.** *([9]-[12], [25]) Suppose $\tau \in (0,\infty]$, $A$ is a closed operator and $K \in L^1_{loc}([0,\tau))$. If there exists a strongly continuous family $(S_K(t))_{t\in[0,\tau)}$ in $L(E)$ such that:*

*(i)* $S_K(t)A \subseteq AS_K(t)$, $t \in [0,\tau)$,

*(ii)* $\int_0^t S_K(s)xds \in D(A),\ x \in E,\ t \in [0,\tau)$ *and*

$$A\int_0^t S_K(s)xds = S_K(t)x - \Theta(t)Cx,\ x \in E,\ t \in [0,\tau),$$

*(iii)* $S_K(t)C = CS_K(t),\ t \in [0,\tau)$,

*then it is said that $A$ is a subgenerator of a (local) $K$-convoluted $C$-semigroup $(S_K(t))_{t\in[0,\tau)}$.*

**Definition 2.** *([23], [25]) Suppose $\tau \in (0,\infty]$, $A$ is a closed operator and $K \in L^1_{loc}([0,\tau))$. If there exists a strongly continuous family $(C_K(t))_{t\in[0,\tau)}$ in $L(E)$ such that:*

*(i)* $C_K(t)A \subseteq AC_K(t)$, $t \in [0,\tau)$,

*(ii)* $\int_0^t (t-s)C_K(s)x\,ds \in D(A),\ x \in E,\ t \in [0,\tau),$

$$A\int_0^t (t-s)C_K(s)x\,ds = C_K(t)x - \Theta(t)Cx,\ x \in E,\ t \in [0,\tau) \text{ and}$$

*(iii)* $C_K(t)C = CC_K(t),\ t \in [0,\tau),$

*then it is said that $A$ is a subgenerator of a (local) $K$-convoluted $C$-cosine function $(C_K(t))_{t\in[0,\tau)}$.*

Plugging $K(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$, where $\alpha > 0$, in Definition 1 and Definition 2, we obtain the well known classes of $\alpha$-times integrated semigroups and cosine functions. The integral generator of $(S_K(t))_{t\in[0,\tau)}$, resp. $(C_K(t))_{t\in[0,\tau)}$, is defined by

$$\Big\{(x,y) \in E \times E : S_K(t)x - \Theta(t)Cx = \int_0^t S_K(s)y\,ds,\ t \in [0,\tau)\Big\},\ \text{resp.,}$$

$$\Big\{(x,y) \in E \times E : C_K(t)x - \Theta(t)Cx = \int_0^t (t-s)C_K(s)y\,ds,\ t \in [0,\tau)\Big\},$$

and it is a closed linear operator which is an extension of any subgenerator of $(S_K(t))_{t\in[0,\tau)}$, resp. $(C_K(t))_{t\in[0,\tau)}$. Suppose $A$ is a subgenerator of $(S_K(t))_{t\in[0,\tau)}$, resp. $(C_K(t))_{t\in[0,\tau)}$. It could be of interest to accent that the integral generator $\hat{A}$ of $(S_K(t))_{t\in[0,\tau)}$, resp. $(C_K(t))_{t\in[0,\tau)}$, satisfies $\hat{A} = C^{-1}\hat{A}C = C^{-1}AC$. Denote by $\wp(S_K)$ the set consisted of all subgenerators of a (local) convoluted $C$-semigroup $(S_K(t))_{t\in[0,\tau)}$. Then $\wp(S_K)$ need not be finite ([23]) and, equipped with corresponding algebraic operations, $\wp(S_K)$ becomes a complete lattice whose partially ordering coincides with the usual set inclusion. In the case $\text{card}(\wp(S_K)) < \infty$, one can prove that $\wp(S_K)$ is a Boolean lattice and that, specifically, $\text{card}(\wp(S_K)) = 2^n$ for some $n \in \mathbb{N}_0$. The preceding assertions possess natural reformulations in the case of (local) convoluted $C$-cosine functions.

Suppose $a > 0$ and $b > 0$. The logarithmic region $\tilde{\Lambda}_{a,b}$ [18] and the exponential region $E(a,b)$ [2] are defined as follows:

$$\tilde{\Lambda}_{a,b} := \Big\{\lambda \in \mathbb{C} : \text{Re}\lambda \geq a + b\ln(1+|\lambda|)\Big\} \text{ and}$$

$$E(a,b) := \Big\{\lambda \in \mathbb{C} : \text{Re}\lambda \geq b,\ |\text{Im}\lambda| \leq e^{a\text{Re}\lambda}\Big\}.$$

Set $\tilde{\Lambda}^2_{a,b} := \{\lambda^2 : \lambda \in \tilde{\Lambda}_{a,b}\}$, $E^2(a,b) := \{\lambda^2 : \lambda \in E(a,b)\}$ and notice that, by the proof of [2, Lemma 2.6] and [22, Remark, p. 759], we have that

$$\tilde{\Lambda}_{a,b} \subseteq E(\frac{1}{b}, a),\ a > 0,\ b > 0. \tag{1}$$

¿From now on, we assume that $(M_p)$ is a sequence of positive real numbers such that $M_0 = 1$ and that the following conditions hold:

(M.1) $M_p^2 \leq M_{p+1}M_{p-1},\ p \in \mathbb{N}$,

(M.2) $M_n \leq AH^n \min_{p,\ q\in\mathbb{N},\ p+q=n} M_pM_q,\ n \in \mathbb{N},$ for some $A > 1$ and $H > 1$, and

(M.3)' $\sum_{p=1}^{\infty} \frac{M_{p-1}}{M_p} < \infty$.

The Gevrey sequences $(p!^s)$, $(p^{ps})$ and $(\Gamma(1+ps))$ satisfy the above conditions, where $\Gamma(\cdot)$ denotes the Gamma function and $s > 1$. Put $m_p := \frac{M_p}{M_{p-1}},\ p \in \mathbb{N}$; then (M.1) implies that $(m_p)$ is increasing and (M.3)' simply means that $\sum_{p=1}^{\infty} \frac{1}{m_p} < \infty$. The associated function of $(M_p)$ is defined by $M(\lambda) := \sup_{p\in\mathbb{N}_0} \ln \frac{|\lambda|^p}{M_p},\ \lambda \in \mathbb{C} \setminus \{0\}$, $M(0) := 0$. It is well known that the function $t \mapsto M(t),\ t \geq 0$ is increasing as well as that $\lim_{|\lambda|\to\infty} M(\lambda) = +\infty$ and that the function $M$ vanishes in some open neighborhood of zero. Denote by $m(\lambda)$ the number of $m_p \leq \lambda$. Since $(M_p)$ satisfies (M.1), it follows that (cf. [20, p. 50]) $M(t) = \int_0^t \frac{m(\lambda)}{\lambda} d\lambda,\ t \geq 0$. This implies that the mapping $t \mapsto M(t),\ t \geq 0$ is absolutely continuous and that the mapping $t \mapsto M(t),\ t \in [0,\infty) \setminus \{m_p : p \in \mathbb{N}\}$ is continuously differentiable with $M'(t) = \frac{m(t)}{t},\ t \in [0,\infty) \setminus \{m_p : p \in \mathbb{N}\}$.

Suppose $l \geq 1,\ \alpha > 0,\ \beta \in \mathbb{R}$ and denote by $\Lambda_{\alpha,\ \beta,\ l}$ the ultra-logarithmic region of type $l$ introduced by J. Chazarain in [8] (cf. also [30, Section 2.3]) as follows:

$$\Lambda_{\alpha,\beta,l} := \Big\{\lambda \in \mathbb{C} : \mathrm{Re}\lambda \geq \alpha M(l|\mathrm{Im}\lambda|) + \beta\Big\}.$$

We assume that the boundary of $\Lambda_{\alpha,\ \beta,\ l}$, denoted by $\Gamma_l$, is upwards oriented.

We need the following auxiliary lemma; notice only that the assertion (ii) follows from the arguments given in the proof of [22, Lemma 1.10].

**Lemma 1.**

*(i) ([23]) Suppose $A$ is a closed linear operator, $0 < \tau \leq \infty$ and $K \in L^1_{loc}([0,\tau))$. Then the following assertions are equivalent:*

*(i.1) The operator $A$ is a subgenerator of a $K$-convoluted $C$-cosine function $(C_K(t))_{t\in[0,\tau)}$ in $E$.*

*(i.2) The operator $\mathcal{A} \equiv \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}$ is a subgenerator of a $\Theta$-convoluted $\mathcal{C}$-semigroup $(S_\Theta(t))_{t\in[0,\tau)}$ in $E \times E$, where $\mathcal{C} := \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}$.*

*If (i.1) holds, then*

$$S_\Theta(t) = \begin{pmatrix} \int_0^t C_K(s)ds & \int_0^t (t-s)C_K(s)ds \\ C_K(t) - \Theta(t)C & \int_0^t C_K(s)ds \end{pmatrix}, \; 0 \le t < \tau,$$

*and the integral generators of* $(C_K(t))_{t\in[0,\tau)}$ *and* $(S_\Theta(t))_{t\in[0,\tau)}$*, denoted respectively by* $B$ *and* $\mathcal{B}$*, satisfy* $\mathcal{B} = \begin{pmatrix} 0 & I \\ B & 0 \end{pmatrix}$*. Furthermore, the integral generator of* $(C_K(t))_{t\in[0,\tau)}$*, resp.* $(S_\Theta(t))_{t\in[0,\tau)}$*, is* $C^{-1}AC$*, resp.* $\mathcal{C}^{-1}\mathcal{A}\mathcal{C} \equiv \begin{pmatrix} 0 & I \\ C^{-1}AC & 0 \end{pmatrix}$.

*(ii) Suppose* $A$ *is a closed linear operator,* $CA \subseteq AC$ *and* $\lambda \in \mathbb{C}$*. Then* $\lambda \in \rho_{\mathcal{C}}(\mathcal{A}) \Leftrightarrow \lambda^2 \in \rho_C(A)$*. If this is the case, then the following holds:*

$$(\lambda - \mathcal{A})^{-1}\mathcal{C} = \begin{pmatrix} \lambda(\lambda^2 - A)^{-1}C & (\lambda^2 - A)^{-1}C \\ A(\lambda^2 - A)^{-1}C & \lambda(\lambda^2 - A)^{-1}C \end{pmatrix},$$

$$||(\lambda - \mathcal{A})^{-1}\mathcal{C}|| \le (1 + |\lambda|)\sqrt{1 + |\lambda|^2}||(\lambda^2 - A)^{-1}C|| \text{ and}$$
$$||(\lambda^2 - A)^{-1}C|| \le ||(\lambda - \mathcal{A})^{-1}\mathcal{C}||.$$

*(iii) ([23]) Suppose* $K$ *is a kernel and* $A$ *is a subgenerator of a local* $K$*-convoluted* $C$*-semigroup* $(S_K(t))_{t\in[0,\tau)}$*, resp.* $K$*-convoluted* $C$*-cosine function* $(C_K(t))_{t\in[0,\tau)}$*. Then* $S_K(t)S_K(s) = S_K(s)S_K(t)$*, resp.* $C_K(t)C_K(s) = C_K(s)C_K(t)$ *for all* $(t, s) \in [0, \tau) \times [0, \tau)$.

## 2 Spectral properties of subgenerators of local convoluted $C$-semigroups and cosine functions

The following recollection of results known in the existing literature will be helpful in our further work. We mention in passing that the assertion (i) essentially follows from the argumentation given in the proof of [30, Theorem 1.3.1].

**Proposition 1.** *(i) ([12], [30]) Suppose* $\alpha > 0$, $M > 0$, $\beta \ge 0$, $\Phi : \mathbb{C} \to [0, \infty)$, $|K(t)| \le Me^{\beta t}$, $t \ge 0$, $(S_K(t))_{t\in[0,\tau)}$ *is a local* $K$*-convoluted semigroup generated by* $A$ *and*

$$\frac{1}{|\tilde{K}(\lambda)|} \le e^{\Phi(\alpha\lambda)} \text{ for all } \lambda \in \mathbb{C} \text{ with } Re\lambda > \beta \text{ and } \tilde{K}(\lambda) \ne 0.$$

*Then, for every* $t \in (0, \tau)$*, there exist* $\beta(t) > 0$ *and* $M(t) > 0$ *such that*

$$\Lambda_{t,\alpha,\beta(t)} := \Big\{\lambda \in \mathbb{C} : \tilde{K}(\lambda) \ne 0,\ Re\lambda \ge \frac{\Phi(\alpha\lambda)}{t} + \beta(t)\Big\} \subseteq \rho(A) \text{ and}$$

$$||R(\lambda : A)|| \le M(t)e^{\Phi(\alpha\lambda)},\ \lambda \in \Lambda_{t,\alpha,\beta(t)},\ \tilde{K}(\lambda) \ne 0.$$

*Furthermore, the existence of a sequence* $(t_n)$ *in* $[0,\tau)$ *satisfying* $\lim_{n\to\infty} t_n = \tau$ *and* $\sup_{n\in\mathbb{N}} \ln ||S_K(t_n)|| < \infty$ *implies that there exist* $\beta' > 0$ *and* $M' > 0$ *such that* $\Lambda_{\tau,\alpha,\beta'} \subseteq \rho(A)$ *and that* $||R(\lambda : A)|| \leq M' e^{\Phi(\alpha\lambda)},\ \lambda \in \Lambda_{\tau,\alpha,\beta'}$.

*(ii) ([2], [29]) Suppose* $\alpha > 0$, $\tau \in (0,\infty]$ *and* $A$ *generates an* $\alpha$*-times integrated semigroup* $(S_\alpha(t))_{t\in[0,\tau)}$. *Then, for every* $a \in (0, \frac{\tau}{\alpha})$, *there exist* $b > 0$ *and* $M > 0$ *such that:*

$$E(a,b) \subseteq \rho(A) \text{ and } ||R(\lambda : A)|| \leq M(1+|\lambda|)^\alpha,\ \lambda \in E(a,b). \tag{2}$$

*(iii) ([2], [29]) Suppose* $\alpha > 0$, $a > 0$, $b > 0$, $M > 0$,

$$E(a,b) \subseteq \rho(A) \text{ and } ||R(\lambda : A)|| \leq M(1+|\lambda|)^\alpha,\ \lambda \in E(a,b).$$

*Then, for every* $\beta \in (\alpha+1,\infty)$, *the operator* $A$ *generates a local* $\beta$*-times integrated semigroup* $(S_\beta(t))_{t\in[0,a(\beta-\alpha-1))}$.

Before proceeding further, of concern is to stress that there exist examples of local twice integrated $C$-semigroups (once integrated $C$-cosine functions) whose integral generators possess empty $C$-resolvent sets ([28]).

The main objective in the next theorem is to prove the converse of Proposition 1(i) as well as to transfer the assertion of Proposition 1(iii) to local integrated $C$-semigroups.

**Theorem 1.** *(i) Suppose* $CA \subseteq AC$, $K$ *satisfies (P1),* $r_0 \geq \max(0, abs(K))$ *and* $\Phi : [r_0,\infty) \to [0,\infty)$ *is a continuously differentiable, strictly increasing mapping. Suppose, further,* $\lim_{t\to\infty} \Phi(t) = +\infty$, $\Phi'(\cdot)$ *is bounded on* $[r_0,\infty)$ *and there exist* $\alpha > 0$, $\gamma > 0$ *and* $\beta > r_0$ *such that*

$$\Psi_{\alpha,\beta,\gamma} := \Big\{\lambda \in \mathbb{C} : Re\lambda \geq \frac{\Phi(\alpha|Im\lambda|)}{\gamma} + \beta\Big\} \subseteq \rho_C(A).$$

*Designate by* $\Gamma_{\alpha,\beta,\gamma}$ *the upwards oriented boundary of* $\Psi_{\alpha,\beta,\gamma}$ *and by* $\Omega_{\alpha,\beta,\gamma}$ *the open region which lies to the right of* $\Gamma_{\alpha,\beta,\gamma}$. *Let the following conditions hold.*

*(i.1) The mapping* $\lambda \mapsto \tilde{K}(\lambda)(\lambda - A)^{-1}C$ *is analytic on* $\Omega_{\alpha,\beta,\gamma}$ *and continuous on* $\Gamma_{\alpha,\beta,\gamma}$.

*(i.2) There exist* $M > 0$ *and* $\sigma > 0$ *such that:*

$$\Big\|\tilde{K}(\lambda)(\lambda - A)^{-1}C\Big\| \leq Me^{-\Phi(\sigma|\lambda|)},\ \lambda \in \overline{\Omega_{\alpha,\beta,\gamma}}.$$

*(i.3) There exists a function* $m : [0,\infty) \to (0,\infty)$ *such that* $m(s) = 1,\ s \in [0,1]$ *and that, for every* $s > 1$, *there exists a number* $r_s > r_0$ *so that:*

$$\frac{\Phi(t)}{\Phi(st)} \geq m(s),\ t \geq r_s.$$

*(i.4)* $\lim_{t\to\infty} te^{-\Phi(\sigma t)} = 0.$

*(i.5)* $(\exists a \geq 0)(\exists r'_a > r_0)(\forall t > r'_a)\frac{\ln t}{\Phi(t)} \geq a.$

*Then the operator $A$ is a subgenerator of a local $K$-convoluted $C$-semigroup on $[0, a + m(\frac{\alpha}{\sigma\gamma}))$.*

*(ii) Suppose $\alpha > 0$, $a > 0$, $b > 0$, $M > 0$, $CA \subseteq AC$,*

$$E(a,b) \subseteq \rho_C(A),\ \|(\lambda - A)^{-1}C\| \leq M(1+|\lambda|)^{\alpha},\ \lambda \in E(a,b),$$

*and the mapping $\lambda \mapsto (\lambda - A)^{-1}C$, $\lambda \in E(a,b)$ is continuous. Then, for every $\beta \in (\alpha+1, \infty)$, the operator $A$ is a subgenerator of a local $\beta$-times integrated $C$-semigroup $(S_\beta(t))_{t\in[0,a(\beta-\alpha-1))}$.*

PROOF. To prove (i), set

$$S_K(t) := \frac{1}{2\pi i} \int\limits_{\Gamma_{\alpha,\beta,\gamma}} e^{\lambda t}\tilde{K}(\lambda)(\lambda - A)^{-1}C d\lambda,\ t \in [0, a + m(\frac{\alpha}{\sigma\gamma})). \tag{3}$$

Let us show that the improper integral appearing in (3) converges for all $t \in [0, a + m(\frac{\alpha}{\sigma\gamma}))$. Denote by $\Gamma^1_{\alpha,\beta,\gamma} := \{\lambda \in \Gamma_{\alpha,\beta,\gamma} :\ \mathrm{Im}\lambda \geq 0\}$ and $\Gamma^2_{\alpha,\beta,\gamma} := \{\lambda \in \Gamma_{\alpha,\beta,\gamma} :\ \mathrm{Im}\lambda \leq 0\}$. Clearly, $\Gamma^1_{\alpha,\beta,\gamma} = \{\frac{\Phi(\alpha s)}{\gamma} + \beta + is : s \geq 0\}$. Taking into account the equality $\lim_{t\to\infty}\Phi(t) = +\infty$ as well as (i.3) and (i.5), we easily infer that there exist a sufficiently large real number $r' \geq r_0\sigma^{-1} + 1$ and a number $\zeta > 1$ so that $t\frac{\Phi(\alpha s)}{\gamma} - \Phi(\sigma s) \leq \ln M - \zeta \ln s,\ s \geq r'$. Hence, there exists $M' > 0$ such that:

$$e^{t\frac{\Phi(\alpha s)}{\gamma} - \Phi(\sigma s)} \leq M' s^{-\zeta},\ s \geq r'. \tag{4}$$

Then the choice of $r_0$ gives that, for every $s \geq r'$, $\Phi(\sigma|\frac{\Phi(\alpha s)}{\gamma} + \beta + is|) \geq \Phi(\sigma s)$. Thanks to the estimate (4), one gets:

$$\Big\| \int\limits_{\Gamma^1_{\alpha,\beta,\gamma}\ \cap\ \{\lambda\in\mathbb{C}\ :\ \mathrm{Im}\lambda\geq r'\}} e^{\lambda t}\tilde{K}(\lambda)(\lambda - A)^{-1}C d\lambda \Big\|$$
$$\leq \int\limits_{r'}^{\infty} e^{(\frac{\Phi(\alpha s)}{\gamma} + \beta)t} e^{-\Phi(\sigma s)}\Big(1 + \frac{\alpha\Phi'(\alpha s)}{\gamma}\Big) ds$$
$$\leq \mathrm{Const}.e^{\beta t} \int\limits_{r'}^{\infty} e^{t\frac{\Phi(\alpha s)}{\gamma} - \Phi(\sigma s)} ds \leq \mathrm{Const}.e^{\beta t} \int\limits_{1}^{\infty} \frac{ds}{s^{\zeta}} < \infty.$$

This implies the convergence of the curve integral over $\Gamma^1_{\alpha,\beta,\gamma}$; the convergence of the curve integral over $\Gamma^2_{\alpha,\beta,\gamma}$ can be proved analogically. Hence, $S_K(t) \in L(E)$, $S_K(t)A \subseteq AS_K(t)$ and $S_K(t)C = CS_K(t),\ t \in [0, a + m(\frac{\alpha}{\sigma\gamma}))$. Further on, notice that [15, Proposition 2.6, Remark 2.7] and the assertion (i.1) imply that the mapping $\lambda \mapsto (\lambda - A)^{-1}C,\ \lambda \in \Omega_{\alpha,\beta,\gamma}$ is analytic. Using the Cauchy formula, (i.2) and (i.4), one can simply prove that

$$\int\limits_{\Gamma_{\alpha,\beta,\gamma}} \tilde{K}(\lambda)(\lambda - A)^{-1}C d\lambda = 0.$$

Arguing as in the final part of the proof of [30, Theorem 1.3.2], one gets that $A\int\limits_0^t S_K(s)x = S_K(t)x - \Theta(t)Cx,\ x \in E,\ t \in [0, a + m(\frac{\sigma\gamma}{\alpha}))$, which completes the proof of (i). Having in mind [15, Proposition 2.6, Remark 2.7], the proof of (ii) follows immediately from that of [2, Theorem 2.2]. ( Notice also that the proof of (ii) can be derived from the assertion (i) with $\Phi(t) = \ln(1 + t),\ t \geq 0$ and a non-trivial computation. )

**Remark 1.** *The suppositions of Theorem 1(i) are satisfied for the function* $\Phi(t) = ct^{\frac{1}{s}} + d$, *where* $s > 1$, $c > 0$ *and* $d \in \mathbb{R}$. *For example, the item (i.3) holds for the function* $m(\varsigma) = \frac{\varepsilon}{\varsigma^{\frac{1}{s}}}$, *where* $\varepsilon > 0$ *can be chosen arbitrarily, and the item (i.5) holds with* $a = 0$. *If* $K(t) = \mathcal{L}^{-1}(e^{-\lambda^{\frac{1}{s}}})(t),\ t \geq 0$ *and* $||R(\lambda : A)|| = O(e^{(\cos(\frac{\pi}{2s}) - c\sigma^{\frac{1}{s}})|\lambda|^{\frac{1}{s}}}),\ \lambda \in \overline{\Omega_{\alpha,\beta,\gamma}}$, *then we may apply Theorem 1(i) to deduce that* $A$ *generates a local* $K$*-convoluted semigroup on* $[0, \frac{\sigma^{\frac{1}{s}}\gamma}{\alpha^{\frac{1}{s}}})$. *Further on, the assumption on continuous differentiability of the function* $\Phi(\cdot)$ *given in the formulation of Theorem 1(ii), and Theorem 2(iii) given below, can be slightly weakened. In fact, one can assume that there exists an increasing sequence* $(n_p)$ *in* $[r_0, \infty)$ *such that the function* $\Phi(\cdot)$ *is of class* $C^1$ *in* $[r_0, \infty) \setminus \{n_p : p \in \mathbb{N}\}$. *Suppose now that there exist numbers* $\alpha > 0$, $\beta \in \mathbb{R}$ *and* $l \geq 1$ *such that* $\Lambda_{\alpha,\beta,l} \subseteq \rho(A)$ *and that* $||R(\lambda : A)|| = O(e^{M(l|\lambda|)})$. *Since, for every* $L \geq 1$, *there exist constants* $K > 1$ *and* $B > 0$, *and a number* $E_L > 0$, *such that* $M(Lt) \leq \frac{3}{2}LM(t) + K,\ t \geq 0$ *and that* $LM(t) \leq M(B^{L-1}t) + E_L$ *(cf. [6, Lemma 2.1.3] and [12]), it can be proved by means of Theorem 1(i) (with* $a = 0$ *and* $m(s) = \frac{1}{\frac{3}{2}s + \varepsilon},\ s > 1,\ 0 < \varepsilon$ *given in advance) that, for every* $\varsigma > 0$, $A$ *generates a local* $\mathcal{L}^{-1}(1/\Pi_{i=1}^{\infty}(1 + \frac{B(l+\varsigma)\lambda}{m_p}))$*-convoluted semigroup on* $[0, \frac{2}{3}\frac{\sigma}{l\alpha})$. *By Theorem 2, the previous example can be simply reformulated in the case of local convoluted cosine functions.*

**Theorem 2.** *(i) Suppose* $K$ *is a kernel,* $M > 0$, $\beta \geq 0$, $\alpha > 0$, $\Phi : \mathbb{C} \to [0, \infty)$, $|\Theta(t)| \leq Me^{\beta t},\ t \geq 0$, $(C_K(t))_{t \in [0,\tau)}$ *is a local* $K$*-convoluted cosine function generated by* $A$ *and*

$$\frac{1}{|\tilde{\Theta}(\lambda)|} \leq e^{\Phi(\alpha\lambda)} \text{ for all } \lambda \in \mathbb{C} \text{ with } Re\lambda > \beta \text{ and } \tilde{K}(\lambda) \neq 0.$$

*Then, for every* $t \in (0, \tau)$, *there exist* $\beta(t) > 0$ *and* $M(t) > 0$ *such that*

$$\Lambda^2_{t,\alpha,\beta(t)} := \Big\{\lambda^2 \in \mathbb{C} : \tilde{K}(\lambda) \neq 0,\ Re\lambda \geq \frac{\Phi(\alpha\lambda)}{t} + \beta(t)\Big\} \subseteq \rho(A) \text{ and}$$

$$\Big\| R(\lambda^2 : A) \Big\| \leq M(t) \frac{e^{\Phi(\alpha\lambda)}}{|\lambda|}, \ \lambda \in \Lambda_{t,\alpha,\beta(t)}, \ \tilde{K}(\lambda) \neq 0.$$

*Furthermore, the existence of a sequence* $(t_n)$ *in* $[0, \tau)$ *satisfying* $\lim_{n\to\infty} t_n = \tau$ *and* $\sup_{n\in\mathbb{N}} \ln ||C_K(t_n)|| < \infty$ *implies that there exist* $\beta' > 0$ *and* $M' > 0$ *such that* $\Lambda^2_{\tau,\alpha,\beta'} \subseteq \rho(A)$ *and that* $||R(\lambda^2 : A)|| \leq M' \frac{e^{\Phi(\alpha\lambda)}}{|\lambda|}, \ \lambda \in \Lambda_{\tau,\alpha,\beta'}$.

*(ii) Suppose* $\alpha > 0$ *and* $A$ *generates a (local)* $\alpha$*-times integrated cosine function* $(C_\alpha(t))_{t\in[0,\tau)}$. *Then, for every* $a \in (0, \frac{\tau}{\alpha+1})$, *there exist* $b > 0$ *and* $M > 0$ *such that:*

$$E^2(a, b) \subseteq \rho(A) \ \textit{and} \ ||R(\lambda^2 : A)|| \leq M(1 + |\lambda|)^\alpha, \ \lambda \in E(a, b). \tag{5}$$

*(iii) Suppose* $CA \subseteq AC$, $K(\cdot)$ *satisfies (P1),* $r_0 \geq \max(0, abs(K))$ *and* $\Phi : [r_0, \infty) \to [0, \infty)$ *is a continuously differentiable, strictly increasing mapping. Suppose, further,* $\lim_{t\to\infty} \Phi(t) = +\infty$, $\Phi'(\cdot)$ *is bounded on* $[r_0, \infty)$ *and there exist* $\alpha > 0$, $\gamma > 0$ *and* $\beta > r_0$ *such that*

$$\Psi^2_{\alpha,\beta,\gamma} := \Big\{ \lambda^2 : \lambda \in \Psi_{\alpha,\beta,\gamma} \Big\} \subseteq \rho_C(A). \tag{6}$$

*Designate by* $\Gamma_{\alpha,\beta,\gamma}$ *the upwards oriented boundary of* $\Psi_{\alpha,\beta,\gamma}$ *(cf. also the formulation of Theorem 1) and by* $\Omega_{\alpha,\beta,\gamma}$ *the open region which lies to the right of* $\Gamma_{\alpha,\beta,\gamma}$. *Let the following conditions hold.*

*(iii.1) The mapping* $\lambda \mapsto \tilde{K}(\lambda)(\lambda^2 - A)^{-1}C$ *is analytic on* $\Omega_{\alpha,\beta,\gamma}$ *and continuous on* $\Gamma_{\alpha,\beta,\gamma}$.

*(iii.2) There exist* $M > 0$ *and* $\sigma > 0$ *such that:*

$$\Big\| \tilde{K}(\lambda) \Big[ (\lambda^2 - A)^{-1} C + \frac{C}{\lambda} \Big] \Big\| \leq M e^{-\Phi(\sigma|\lambda|)}, \ \lambda \in \Omega_{\alpha,\beta,\gamma}.$$

*(iii.3) The conditions (i.3), (i.4) and (i.5) given in the formulation of Theorem 1 hold.*

*Then* $A$ *is a subgenerator of a local* $K$*-convoluted* $C$*-cosine function on* $[0, a + m(\frac{\alpha}{\sigma\gamma}))$.

*(iv) Suppose* $\alpha > 0$, $a > 0$, $b > 0$, $M > 0$, $CA \subseteq AC$,

$$E^2(a, b) \subseteq \rho_C(A), \ ||(\lambda^2 - A)^{-1}C|| \leq M(1 + |\lambda|)^\alpha, \ \lambda \in E(a, b),$$

*and the mapping* $\lambda \mapsto (\lambda^2 - A)^{-1}C, \ \lambda \in E(a, b)$ *is continuous. Then, for every* $\beta \in (\alpha + 2, \infty)$, $A$ *is a subgenerator of a local* $\beta$*-times integrated* $C$*-cosine function* $(C_\beta(t))_{t\in[0,a(\beta-\alpha-1))}$.

Proof. Suppose $t \in (0,\tau)$, $\sigma \in (0,1)$ and $\mathrm{proj}_1 : E \times E \to E$ is defined by $\mathrm{proj}_1\binom{x}{y} := x$, $x, y \in E$. Then it is clear from Lemma 1(i) that $\mathcal{A}$ generates a (local) $\Theta$-convoluted semigroup $(S_\Theta(s))_{s\in[0,\tau)}$ in $E\times E$ and that, thanks to Theorem 1(i), Lemma 1(ii) and the proof of [30, Theorem 2.3.1], there exist $\beta(t) > 0$ and $M(t) > 0$ such that, for every $x \in E$,

$$R(\lambda^2 : A)x = \mathrm{proj}_1[R(\lambda : \mathcal{A})\tbinom{0}{x}]$$

$$= \mathrm{proj}_1\Big[\frac{1}{\tilde{\Theta}(\lambda)}\int_0^t e^{-\lambda s}\begin{pmatrix} \int_0^s C_K(v)dv & \int_0^s (s-v)C_K(v)dv \\ C_K(s)-\Theta(s)C & \int_0^s C_K(v)dv\end{pmatrix}(I-B_t(\lambda))^{-1}\tbinom{0}{x}ds\Big],$$

for all $\lambda \in \Lambda_{t,\alpha,\beta(t)}$, where $B_t(\lambda) = \frac{1}{\tilde{\Theta}(\lambda)}\Big(e^{-\lambda t}S_\Theta(t)I + \int_t^\infty e^{-\lambda s}\Theta(s)Ids\Big)$, $||B_t(\lambda)|| \leq \sigma$ and $||(I - B_t(\lambda))^{-1}|| \leq \frac{1}{1-\sigma}$, $\lambda \in \Lambda_{t,\alpha,\beta(t)}$. Since $K$ is a kernel, Lemma 1(iii) yields $C_K(t)C_K(s) = C_K(s)C_K(t)$, $0 \leq t$, $s < \tau$ and the last equality gives $(I-B_t(\lambda))^{-1}S_\Theta(s) = S_\Theta(s)(I-B_t(\lambda))^{-1}$, $0 \leq t$, $s < \tau$. Then the partial integration implies:

$$R(\lambda^2 : A)x$$

$$= \mathrm{proj}_1\Big[\frac{1}{\tilde{\Theta}(\lambda)}\int_0^t e^{-\lambda s}(I-B_t(\lambda))^{-1}\begin{pmatrix} \int_0^s C_K(v)dv & \int_0^s (s-v)C_K(v)dv \\ C_K(s)-\Theta(s)I & \int_0^s C_K(v)dv\end{pmatrix}\begin{pmatrix}0\\x\end{pmatrix}\Big]$$

$$= \mathrm{proj}_1\Big[-\frac{(I-B_t(\lambda))^{-1}}{\tilde{K}(\lambda)}e^{-\lambda t}\begin{pmatrix}\int_0^t (t-s)C_K(s)xds \\ \int_0^t C_K(s)xds\end{pmatrix}\Big]$$

$$+ \mathrm{proj}_1\Big[\frac{(I-B_t(\lambda))^{-1}}{\tilde{K}(\lambda)}\int_0^t e^{-\lambda s}\begin{pmatrix}\int_0^s C_K(r)xdr \\ C_K(s)x\end{pmatrix}ds\Big]$$

$$\leq \frac{1}{1-\sigma}\Big(\Big\|\int_0^t (t-s)C_K(s)xds\Big\| + \Big\|\int_0^t C_K(s)xds\Big\|\Big)\frac{e^{-\mathrm{Re}\lambda t}}{|\lambda||\tilde{\Theta}(\lambda)|}$$

$$+ \frac{1}{1-\sigma}\frac{1}{|\lambda||\tilde{\Theta}(\lambda)|}\int_0^t e^{-\mathrm{Re}\lambda s}\Big(\Big\|\int_0^s C_K(r)xdr\Big\| + \Big\|C_K(s)x\Big\|\Big)ds$$

$$\leq \frac{1}{1-\sigma}\Big(\Big\|\int_0^t (t-s)C_K(s)xds\Big\| + \Big\|\int_0^t C_K(s)xds\Big\|\Big)\frac{e^{\Phi(\alpha\lambda)}}{|\lambda|}$$

$$+ \frac{1}{1-\sigma}\frac{e^{\Phi(\alpha\lambda)}}{|\lambda|}\int_0^t\Big(\Big\|\int_0^s C_K(r)xdr\Big\| + \Big\|C_K(s)x\Big\|\Big)ds$$

and this, in turn, implies that (i) holds. The proof of (ii) follows from the assertion (i) of this theorem with $K(t)=\frac{t^{\alpha-1}}{\Gamma(\alpha)},\ t>0$ and $\Phi(\lambda)=(\alpha+1)\ln(1+|\lambda|),\ \lambda\in\mathbb{C}$. Strictly speaking, one gets from (i) that, for every $t\in(0,\tau)$, we have the existence of numbers $\beta(t)>0$ and $M(t)>0$ such that $\tilde{\Lambda}^2_{\beta(t),\frac{\alpha+1}{t}}\subseteq\rho(A)$ and that $||R(\lambda^2:A)||\leq M(t)|\lambda|^{\alpha},\ \lambda\in\tilde{\Lambda}_{\beta(t),\frac{\alpha+1}{t}}$. The proof of (ii) completes an application of (1). The proof of (iii) can be obtained by passing to the theory of semigroups. Indeed, the assumption (6) and Lemma 1(ii) imply that $\Omega_{\alpha,\beta,\gamma}\subseteq\rho_{\mathcal{C}}(\mathcal{A})$ and (iii.1) gives that the mapping $\lambda\mapsto\tilde{\Theta}(\lambda)(\lambda-\mathcal{A})^{-1}\mathcal{C},\ \lambda\in\Omega_{\alpha,\beta,\gamma}$ is analytic on $\Omega_{\alpha,\beta,\gamma}$ and continuous on $\Gamma_{\alpha,\beta,\gamma}$. By (iii.2), we easily infer that there exists $M'>0$ such that $||\tilde{\Theta}(\lambda)(\lambda-\mathcal{A})^{-1}\mathcal{C}||\leq M'e^{-\Phi(\sigma|\lambda|)},\ \lambda\in\overline{\Omega_{\alpha,\beta,\gamma}}$. Since (iii.3) holds, we obtain that the operator $\mathcal{A}$ is a subgenerator of a local $\Theta$-convoluted $\mathcal{C}$-semigroup on $[0,a+m(\frac{\alpha}{\sigma\gamma}))$. The proof of (iii) completes an employment of Lemma 1(i). Notice only that we have the following structural equality:

$$C_K(t)=\frac{1}{2\pi i}\int\limits_{\Gamma_{\alpha,\beta,\gamma}}e^{\lambda t}\lambda\tilde{K}(\lambda)(\lambda^2-A)^{-1}Cd\lambda,\ t\in[0,a+m(\frac{\alpha}{\sigma\gamma})).$$

In order to prove (iv), let us set, for every $t\in[0,a(\beta-\alpha-1))$,

$$C_\beta(t):=\frac{1}{2\pi i}\int\limits_{\Gamma}e^{\lambda t}\frac{(\lambda^2-A)^{-1}C}{\lambda^{\beta-1}}d\lambda,$$

where $\Gamma$ denotes the upwards oriented boundary of $E(a,b)$. Having in mind Lemma 1(ii) and Theorem 1(iv), the proof of (iv) follows from that of (iii).

**Remark 2.** *Suppose $\alpha>0,\ 0<\tau<\infty$ and $A$ generates an $\alpha$-times integrated semigroup $(S_\alpha(t))_{t\in[0,\tau)}$, resp. an $\alpha$-times integrated cosine function $(C_\alpha(t))_{t\in[0,\tau)}$. If there exists a sequence $(t_n)$ in $[0,\tau)$ satisfying $\lim_{n\to\infty}t_n=\tau$ and $\sup_{n\in\mathbb{N}}\ln||S_\alpha(t_n)||<\infty$, resp. $\sup_{n\in\mathbb{N}}\ln||C_\alpha(t_n)||<\infty$, then, for every $a\in(0,\frac{\tau}{\alpha}]$, resp. $a\in(0,\frac{\tau}{\alpha+1}]$, there exist $b>0$ and $M>0$ such that (2), resp. (5), holds.*

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