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TOPOLOGIES IN GENERALIZED ORLICZ SEQUENCE SPACES

Enno Kolk

Abstract

In 1994 S. D. Parashar and B. Choudhary defined certain paranorms in some Orlicz sequence spaces of Maddox type. Their ideas are applied later by many authors for topologization of various generalized Orlicz sequence spaces. We determine alternative F-seminorms in such spaces by using the standard arguments of modular spaces theory and a result about the topologization of sequence spaces defined by modulus functions.

1 Notation and background

Let $\mathbb{N} = \{1, 2, \dots\}$ and let $\mathbb{K}$ be the field of real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$. We write $\lim_n$, $\sup_n$, $\inf_n$ and $\sum_n$ instead of $\lim_{n\to\infty}$, $\sup_{n\in\mathbb{N}}$, $\inf_{n\in\mathbb{N}}$ and $\sum_{n=1}^{\infty}$, respectively. By the symbol ι we denote the identity mapping $\iota(z) = z$. The superposition of two mappings f and g is denoted by fg, i.e., $(fg)(z) = f(g(z))$. For two sequences $\mathbf{x} = (x_k)$, $\mathbf{y} = (y_k)$ we use the notation $\mathbf{xy} = (x_k y_k)$ provided that $x_k y_k$ is determined for all $k \in \mathbb{N}$. We also use the notation $\mathbb{R}^+ = [0, \infty)$ and $\mathbf{e}_k = (e_{ik})_{i\in\mathbb{N}}$ $(k \in \mathbb{N})$, where $e_{ik} = 1$ if $i = k$ and $e_{ik} = 0$ otherwise. In all definitions which contain infinite series we tacitly assume the convergence of these series.

An F-space is usually understood as a complete metrizable topological vector space over $\mathbb{K}$. It is known that the topology of an F-space E can be given by an F-*norm*, i.e., by the functional $g : E \to \mathbb{R}$ with the axioms (see [20], p. 13)

(N1) $g(0) = 0$;

(N2) $g(x + y) \le g(x) + g(y) \quad (x, y \in E)$;

(N3) $|\alpha| \le 1$ $(\alpha \in \mathbb{K})$, $x \in E \implies g(\alpha x) \le g(x)$;

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(N4) $\lim_n \alpha_n = 0 \ (\alpha_n \in \mathbb{K}),\ x \in E \implies \lim_n g(\alpha_n x) = 0$;

(N5) $g(x) = 0 \implies x = 0$.

A functional g with the axioms (N1)–(N4) is called an F-*seminorm*. A *paranorm* in E is defined as a functional $g : E \to \mathbb{R}$ satisfying the axioms (N1), (N2) and

(N6) $g(-x) = g(x) \quad (x \in E)$;

(N7) $\lim_n \alpha_n = \alpha \ (\alpha_n, \alpha \in \mathbb{K}),\ \lim_n g(x_n - x) = 0 \ (x_n, x \in E) \implies \lim_n g(\alpha_n x_n - \alpha x) = 0$.

A *seminorm* in E is a functional $g : E \to \mathbb{R}$ with the axioms (N1), (N2) and

(N8) $g(\alpha x) = |\alpha| g(x) \quad (\alpha \in \mathbb{K},\ x \in E)$.

An F-seminorm (paranorm, seminorm) g is called *total* if (N5) holds. So, an F-norm (norm) is a total F-seminorm (seminorm).

In the following, unlike the module $|\cdot|$, the seminorm of an element $x \in E$ is sometimes denoted by $\dot{|}x\dot{|}$.

Remark 1. *It is clear that every paranorm g with (N3) is an F-seminorm. Conversely, since (N3) yields (N6), and for $\lim_n \alpha_n = \alpha$, because of (N2) and (N3), we have*

$$g(\alpha_n x_n - \alpha x) \le 2(|\alpha| + 1) g(x_n - x) + g((\alpha_n - \alpha)x)$$

for sufficently large values of n, any F-seminorm is a paranorm which satisfies (N3). Thereby, F-seminorms coincide with paranorms satisfying axiom (N3).

Let $\mathbf{X}$ be a sequence of seminormed linear spaces $\left(X_k, \dot{|}\cdot\dot{|}_k\right)$ $(k \in \mathbb{N})$ over $\mathbb{K}$. Then the set $s(\mathbf{X})$ of all sequences $\mathbf{x} = (x_k)$, $x_k \in X_k \quad (k \in \mathbb{N})$, together with coordinatewise addition and scalar multiplication is linear space (over $\mathbb{K}$). Any linear subspace of $s(\mathbf{X})$ is called a *generalized sequence space*. In the case $\left(X_k, \dot{|}\cdot\dot{|}_k\right) = (X, \dot{|}\cdot\dot{|})$ $(k \in \mathbb{N})$ we write X instead of $\mathbf{X}$, and if $X_k = \mathbb{K}$ $(k \in \mathbb{N})$, then we omit the symbol $\mathbf{X}$ in notation. So, for example, s denotes the linear space of all $\mathbb{K}$-valued sequences. As usual, linear subspaces of s are called *sequence spaces*.

A continuous and non-decreasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is called a φ-*function* (cf. [30], p. 4) if

$$\phi(t) = 0 \iff t = 0.$$

A φ-function ϕ is called a *modulus function* (or, briefly, a *modulus*) if

$$\phi(t + u) \le \phi(t) + \phi(u) \ \ (t, u \in \mathbb{R}^+), \tag{1}$$

and an *Orlicz function* if (1) is replaced by the condition of convexity

$$\phi(\alpha t + (1 - \alpha)u) \le \alpha\phi(t) + (1 - \alpha)\phi(u) \ \ (t, u \in \mathbb{R}^+, 0 \le \alpha \le 1).$$

For example, the function $\iota^p(t) = t^p$ is an unbounded modulus function for $p \leq 1$, and it is an Orlicz function if $p \geq 1$. The function $\phi(t) = t/(1+t)$ is a bounded modulus.

We remark that the modulus functions are the same as the moduli of continuity (see [12], p. 866).

If $\mathbf{p} = (p_k)$ is a sequence with $0 < p_k \leq \sup p_k < \infty$ and $\mathbf{X}$ is a sequence of seminormed linear spaces $(X_k, \dot{|}\cdot\dot{|}_k)$, then we can define the generalized sequence spaces

$$\ell_\infty(\mathbf{p}, \mathbf{X}) = \left\{(x_k) \in s(\mathbf{X}) : \sup_k \dot{|}x_k\dot{|}_k^{p_k} < \infty\right\},$$

$$c_0(\mathbf{p}, \mathbf{X}) = \left\{(x_k) \in s(\mathbf{X}) : \lim_k \dot{|}x_k\dot{|}_k^{p_k} = 0\right\},$$

$$\ell(\mathbf{p}, \mathbf{X}) = \left\{(x_k) \in s(\mathbf{X}) : \sum_k \dot{|}x_k\dot{|}_k^{p_k} < \infty\right\},$$

$$w_0(\mathbf{p}, \mathbf{X}) = \left\{(x_k) \in s(\mathbf{X}) : \lim_n n^{-1} \sum_{k=1}^{n} \dot{|}x_k\dot{|}_k^{p_k} = 0\right\}.$$

In the case $X_k = \mathbb{K}$ $(k \in \mathbb{N})$ these spaces reduce to so-called Maddox sequence spaces $\ell_\infty(\mathbf{p})$, $c_0(\mathbf{p})$, $\ell(\mathbf{p})$ and $w_0(\mathbf{p})$, respectively (see, for example, [19] or [28]). If here $p_k = p$ $(k \in \mathbb{N})$, then we get well-known sequence spaces ℓ_∞, c_0, ℓ_p and w_0^p of all bounded, convergent to zero, absolutely p-summable and strongly summable to zero of index p sequences, respectively. As usual, in the case $p = 1$ we write ℓ and w_0 instead of ℓ_1 and w_0^1.

If $M = \max\{1, \sup_k p_k\}$, then the functions $t^{p_k/M}$ $(k \in \mathbb{N})$ are moduli. Therefore, using also the equivalences

$$\mathbf{x} \in c_0(\mathbf{p}, \mathbf{X}) \iff \left(\dot{|}x_k\dot{|}_k^{p_k/M}\right) \in c_0 \quad \text{and} \quad \mathbf{x} \in \ell(\mathbf{p}, \mathbf{X}) \iff \left(\dot{|}x_k\dot{|}_k^{p_k/M}\right) \in \ell_M,$$

by Theorem 2 [23] we can define in $c_0(\mathbf{p}, \mathbf{X})$ and $\ell(\mathbf{p}, \mathbf{X})$ the F-seminorms (or paranorms)

$$g_\infty(\mathbf{x}) = \sup_k \dot{|}x_k\dot{|}_k^{p_k/M} \quad \text{and} \quad g_M(\mathbf{x}) = \left(\sum_k \dot{|}x_k\dot{|}_k^{p_k}\right)^{1/M},$$

respectively. These F-seminorms are total if $\mathbf{X}$ is a sequence of normed spaces (about the special case $X_k = \mathbb{K}$ $(k \in \mathbb{N})$ see [28]).

Let $\Phi = (\phi_k)$ be a sequence of φ-functions ϕ_k $(k \in \mathbb{N})$ and let $\lambda \subset s$ be a sequence spaces. For $\mathbf{x} = (x_k) \in s(\mathbf{X})$, using the notation $|\mathbf{x}| = \left(\dot{|}x_k\dot{|}\right)$, we write

$$\Phi(|\mathbf{x}|) = \left(\phi_k\left(\dot{|}x_k\dot{|}_k\right)\right)$$

and define the set

$$\lambda^{\exists}(\Phi, \mathbf{X}) = \left\{\mathbf{x} \in s(\mathbf{X}) : (\exists \rho > 0)\ \Phi(|\rho^{-1}\mathbf{x}|) \in \lambda\right\}.$$

For a constant sequence Φ with $\phi_k = \phi$ $(k \in \mathbb{N})$, we write ϕ instead of Φ.

Let ϕ be an Orlicz function. By an *Orlicz sequence space* we mean the Banach sequence space $\ell_\phi = \ell^{\exists}(\phi)$ with the norm (see [27] or [30])

$$\|\mathbf{u}\|_\phi = \inf\left\{\rho > 0 : \sum_k \phi(|\rho^{-1}|u_k|) \le 1\right\}. \tag{2}$$

Woo [42] showed that if $\Phi = (\phi_k)$ is a sequence of Orlicz functions, then the set $\ell_\Phi = \ell^{\exists}(\Phi)$ is also a Banach sequence space with the norm

$$\|\mathbf{u}\|_\Phi = \inf\left\{\rho > 0 : \sum_k \phi_k(|\rho^{-1}u_k|) \le 1\right\}. \tag{3}$$

In the mathematical literature we may find a series of papers which deal with various generalizations and modifications of ℓ_ϕ and ℓ_Φ, where the space ℓ is replaced by different sequence spaces including the spaces of Maddox type and domains of various summability methods. For example, Parashar and Choudhary [32] proved that the sets $\ell(\mathbf{p})^{\exists}(\phi)$ and $w_0(\mathbf{p})^{\exists}(\phi)$ are linear spaces if ϕ is an Orlicz function and $\mathbf{p} = (p_k)$ is a bounded sequence of positive numbers. Denoting $M = \max\{1, \sup_k p_k\}$, they define in $\ell(\mathbf{p})^{\exists}(\phi)$ the total paranorm

$$g(\mathbf{u}) = \inf_{\rho>0,\, m\in\mathbb{N}}\left\{\rho^{p_m/M} : \left(\sum_k (\phi(|\rho^{-1}u_k|))^{p_k}\right)^{1/M} \le 1\right\}, \tag{4}$$

and in $w_0(\mathbf{p})^{\exists}(\phi)$ the total paranorm

$$g_0(\mathbf{u}) = \inf_{\rho>0,\, m\in\mathbb{N}}\left\{\rho^{p_m/M} : \sup_n\left(n^{-1}\sum_k (\phi(|\rho^{-1}u_k|))^{p_k}\right)^{1/M} \le 1\right\}. \tag{5}$$

We note that g and g_0 are also F-norms because of Remark 1.

The idea to topologize different generalized Orlicz sequence spaces by the paranorms of types (4) and (5) is used later by many authors (see, for example, [1], [2], [3], [4], [5], [7], [8], [9], [10], [13], [14], [22], [31], [33], [35], [36], [37], [39]). Using the standard arguments of modular spaces theory and a result about the topologization of sequence spaces defined by moduli, we determine some alternative F-seminorms (or paranorms) in these sequence spaces. It seems that our F-seminorms and norms, in comparison with (4) and (5), are better connected to (2) and (3), and also to the norms given in [16], [17], [25], [26], [30], [38], [40] and [41].

2 Main theorems

Recall that a set $E \subset s(\mathbf{X})$ is called *solid* if $\mathbf{y} = (y_k) \in E$ whenever $\mathbf{x} = (x_k) \in E$ and $\dot{|}y_k\dot{|} \leq \dot{|}x_k\dot{|}$ $(k \in \mathbb{N})$. Known solid sequence spaces are m, c_0, ℓ_p and w_0^p.

Let $T : s_{_T}(\mathbf{X}) \to s(\mathbf{X})$ be a linear operator, where $s_{_T}(\mathbf{X})$ is a linear subspace of $s(\mathbf{X})$.

As an extension of $\lambda^{\exists}(\Phi, \mathbf{X})$ we consider the set

$$\lambda^{\exists}(\Phi, T, \mathbf{X}) = \{\mathbf{x} \in s_{_T}(\mathbf{X}) : (\exists \rho > 0)\ \Phi(|\rho^{-1}T\mathbf{x}|) \in \lambda\},$$

where λ is a sequence space and Φ is a sequence of φ-functions. If T is the identity mapping, then we omit the symbol T in notation.

Theorem 1. *Let $\mathbf{X}$ be a sequence of seminormed linear spaces $\left(X_k, \dot{|}\cdot\dot{|}_k\right)$ $(k \in \mathbb{N})$, λ be a solid sequence space and T be a linear operator as defined above. If $\Phi = (\phi_k)$ is a sequence of Orlicz functions and $\Psi = (\Psi_k)$ is a sequence consisting of modulus and Orlicz functions, then $\lambda^{\exists}(\Psi\Phi, T, \mathbf{X})$ is a generalized sequence space. At that, the generalized sequence space $\lambda^{\exists}(\Psi\Phi, \mathbf{X})$ is solid.*

Proof. Let $\mathbf{x} \in \lambda^{\exists}(\Psi\Phi, T, \mathbf{X})$ with $\Psi\Phi(|\rho^{-1}T\mathbf{x}|) \in \lambda$, $\rho > 0$. If $0 \neq \alpha \in \mathbb{K}$, then by $|(\rho|\alpha|)^{-1}T(\alpha\mathbf{x})| = |(\rho^{-1}T\mathbf{x}|$ we see that $\alpha\mathbf{x}$ belongs to $\lambda^{\exists}(\Psi\Phi, T, \mathbf{X})$. Since $\mathbf{x} \in \lambda^{\exists}(\Psi\Phi, T, \mathbf{X})$ clearly implies $0\mathbf{x} \in \lambda^{\exists}(\Psi\Phi, T, \mathbf{X})$, the homogeneity of $\lambda^{\exists}(\Psi\Phi, T, \mathbf{X})$ is proved.

To prove the additivity, let $\mathbf{x}, \mathbf{y} \in \lambda^{\exists}(\Psi\Phi, T, \mathbf{X})$. Then there exist positive numbers ρ, σ such that $\Psi\Phi(|\rho^{-1}T\mathbf{x}|)$ and $\Psi\Phi(|\sigma^{-1}T\mathbf{y}|)$ are in λ. Let $T\mathbf{x} = (T_k\mathbf{x})$. Because any ϕ_k is non-decreasing and convex, for $\theta = \max\{2\rho, 2\sigma\}$ and all $k \in \mathbb{N}$ we have that

$$\begin{aligned}\phi_k\left(\dot{|}\theta^{-1}T_k(\mathbf{x}+\mathbf{y})\dot{|}\right) &\leq \phi_k\left(\dot{|}(2\rho)^{-1}T_k(\mathbf{x})\dot{|} + \dot{|}(2\sigma)^{-1}T_k(\mathbf{y}\dot{|}\right)\\ &\leq 1/2\phi_k\left(\dot{|}\rho^{-1}T_k(\mathbf{x})\dot{|}\right) + 1/2\phi_k\left(\dot{|}\sigma^{-1}T_k(\mathbf{y})\dot{|}\right).\end{aligned} \tag{6}$$

Now, if ψ_k is an Orlicz function, then $\psi_k\phi_k$ is also an Orlicz function, so (6) is true (for this index k) with $\psi_k\phi_k$ instead of ϕ_k. But if ψ_k is a modulus function, then from (6), by condition (1), for such index k we get

$$\psi_k\phi_k\left(\dot{|}\theta^{-1}T_k(\mathbf{x}+\mathbf{y})\dot{|}\right) \leq \psi_k\phi_k\left(\dot{|}\rho^{-1}T_k(\mathbf{x})\dot{|}\right) + \psi_k\phi_k\left(\dot{|}\sigma^{-1}T_k(\mathbf{y})\dot{|}\right). \tag{7}$$

Hence (7) is true for all $k \in \mathbb{N}$. Since λ is solid, $\Psi\Phi(|\theta^{-1}T(\mathbf{x}+\mathbf{y})|)$ must be in λ and, consequently, $\mathbf{x}+\mathbf{y} \in \lambda^{\exists}(\Psi\Phi, T, \mathbf{X})$.

The solidity of $\lambda^{\exists}(\Psi\Phi, \mathbf{X})$ follows by

$$\dot{|}y_k\dot{|} \leq \dot{|}x_k\dot{|} \implies \psi_k\phi_k\left(\dot{|}\rho^{-1}y_k\dot{|}\right) \leq \psi_k\phi_k\left(\dot{|}\rho^{-1}x_k\dot{|}\right) \quad (k \in \mathbb{N}),$$

because λ is solid. □

Recall that an F-seminorm g in λ is called *absolutely monotone* if for any two elements $\mathbf{u} = (u_k)$, $\mathbf{v} = (v_k)$ of λ with $|u_k| \leq |v_k|$ $(k \in \mathbb{N})$ the inequality $g(\mathbf{u}) \leq g(\mathbf{v})$ holds.

In the following, using the standard arguments of theory of modular spaces (see [30], proof of Theorem 1.5), we determine some F-seminorms in generalized Orlicz sequence space $\lambda^{\exists}(\Phi, T, \mathbf{X})$.

Theorem 2. *Let $\Phi = (\phi_k)$ be a sequence of Orlicz functions and let λ be a solid sequence space. If λ is topologized by an absolutely monotone F-seminorm g, then*

$$\widehat{h}(\mathbf{x}) = \inf\{\rho > 0 : g(\Phi(|\rho^{-1}T\mathbf{x}|)) \leq \rho\}$$

is an F-seminorm in $\lambda^{\exists}(\Phi, T, \mathbf{X})$. If g is an absolutely monotone seminorm in λ, then

$$h(\mathbf{x}) = \inf\{\rho > 0 : g(\Phi(|\rho^{-1}T\mathbf{x}|)) \leq 1\}$$

is a seminorm in $\lambda^{\exists}(\Phi, T, \mathbf{X})$. Moreover, if all spaces X_k $(k \in \mathbb{N})$ are normed and T satisfies the condition

$$T\mathbf{x} = 0 \implies \mathbf{x} = 0, \tag{8}$$

then $\widehat{h}$ is an F-norm (h is a norm) in $\lambda^{\exists}(\Phi, T, \mathbf{X})$ whenever g is an F-norm (a norm) in λ.

Proof. Let $\mathbf{x} \in \lambda^{\exists}(\Phi, T, \mathbf{X})$ with $\Phi(|\rho^{-1}T\mathbf{x}|) \in \lambda$, $\rho > 0$. Since the functions ϕ_k are convex and g is an absolutely monotone F-seminorm in solid sequence space λ, for $0 < \alpha \leq 1$, $\Phi(|\alpha\rho^{-1}T\mathbf{x}|)$ also belongs to λ and

$$g(\Phi(|\alpha\rho^{-1}T\mathbf{x}|)) \leq g(\alpha\Phi(|\rho^{-1}T\mathbf{x}|)). \tag{9}$$

This shows, because of (N2), that the functionals $\hat{h}$ and h are determined in $l^{\exists}(\Phi, T, \mathbf{X})$. Moreover, it is clear that $\widehat{h}(0) = 0$ and $h(0) = 0$.

Further, since T is linear and the functions ϕ_k are non-decreasing and convex, by (N3) we get

$$g(\Phi(|\rho^{-1}T(\alpha\mathbf{x})|)) \leq g(\Phi(|\rho^{-1}T\mathbf{x}|)) \quad (|\alpha| \leq 1) \tag{10}$$

which implies

$$\{\rho > 0 : g(\Phi(|\rho^{-1}T\mathbf{x}|)) \leq \rho\} \subset \{\rho > 0 : g(\Phi(|\rho^{-1}T(\alpha\mathbf{x})|)) \leq \rho\}.$$

Therefore,

$$\widehat{h}(\alpha\mathbf{x}) \leq \widehat{h}(\mathbf{x}) \quad (|\alpha| \leq 1).$$

Now, let $\mathbf{x}, \mathbf{y} \in \lambda^{\exists}(\Phi, T, \mathbf{X})$ and $\varepsilon > 0$. If $s = \widehat{h}(\mathbf{x}) + \varepsilon$, $t = \widehat{h}(\mathbf{y}) + \varepsilon$, then

$$g(\Phi(|s^{-1}T\mathbf{x}|)) \leq s, \quad g(\Phi(|t^{-1}T\mathbf{y}|)) \leq t,$$

whence

$$\begin{aligned} g\left(\Phi\left(\left|\frac{T(\mathbf{x}+\mathbf{y})}{s+t}\right|\right)\right) &\le g\left(\Phi\left(\left|\frac{s}{s+t}\,\frac{T\mathbf{x}}{s}+\frac{t}{s+t}\,\frac{T\mathbf{y}}{t}\right|\right)\right) \\ &\le g(\Phi(\left|\frac{T\mathbf{x}}{s}\right|))+g(\Phi(\left|\frac{T\mathbf{y}}{t}\right|)) \\ &\le s+t \end{aligned}$$

in the case of F-seminorm g. Hence $\widehat{h}(\mathbf{x}+\mathbf{y}) \le \widehat{h}(\mathbf{x})+\widehat{h}(\mathbf{y})+2\varepsilon$, and we obtain

$$\widehat{h}(\mathbf{x}+\mathbf{y}) \le \widehat{h}(\mathbf{x})+\widehat{h}(\mathbf{y}),$$

because $\varepsilon > 0$ is arbitrarily. But if g is a seminorm, then, denoting $s = h(\mathbf{x})+\varepsilon$, $t = h(\mathbf{y})+\varepsilon$, by

$$g(\Phi(|s^{-1}T\mathbf{x}|)) \le 1, \quad g(\Phi(|t^{-1}T\mathbf{y}|)) \le 1$$

we similary get

$$\begin{aligned} g\left(\Phi\left(\left|\frac{T(\mathbf{x}+\mathbf{y})}{s+t}\right|\right)\right) &\le \frac{s}{s+t}\,g\left(\Phi\left(\left|\frac{T\mathbf{x}}{s}\right|\right)\right) \\ &\quad + \frac{t}{s+t}\,g\left(\Phi\left(\left|\frac{T\mathbf{y}}{t}\right|\right)\right) \\ &\le 1 \end{aligned}$$

which gives

$$h(\mathbf{x}+\mathbf{y}) \le h(\mathbf{x})+h(\mathbf{y}).$$

To prove (N8) for h, we take $\alpha \neq 0$. Then

$$\begin{aligned} h(\alpha\mathbf{x}) &= \inf\left\{\rho>0 : g\left(\Phi\left(\left|\frac{T(\alpha\mathbf{x})}{\rho}\right|\right)\right) \le 1\right\} \\ &= |\alpha|\inf\left\{\frac{\rho}{|\alpha|}>0 : g\left(\Phi\left(\left|\frac{T\mathbf{x}}{\rho/|\alpha|}\right|\right)\right) \le 1\right\} \\ &= |\alpha|\,h(\mathbf{x}). \end{aligned}$$

Next we show that $\hat{h}$ and h are total whenever all spaces X_k $(k \in \mathbb{N})$ are normed with the norms $\|\cdot\|_k$, operator T satisfies (8) and g is total. Since $\widehat{h}(\mathbf{x}) = 0$ implies $h(\mathbf{x}) = 0$, it suffices to prove that h satisfies (N5) if g satisfies. If now $h(\mathbf{x}) = 0$, then

$$(\forall \rho > 0)\; g(\Phi(|\rho^{-1}T\mathbf{x}|)) \le 1. \tag{11}$$

If we suppose $\mathbf{x} \neq 0$, then also $T\mathbf{x} \neq 0$ by (8), and there exists an index j with $T_j\mathbf{x} \neq 0$. So, because λ is solid, the norm g is absolutely monotone and ϕ_j is unbounded, $\phi_j(\rho^{-1}\|T_j\mathbf{x}\|_j)\mathbf{e}_j$ belongs to λ and for sufficiently small ρ we get

$$g(\Phi(|\rho^{-1}T\mathbf{x}|)) \ge \phi_j(\rho^{-1}\|T_j\mathbf{x}\|_j)g(\mathbf{e}_j) > 1$$

contrary to (11). Consequently, it must be $\mathbf{x} = 0$. □

In the case of the identity operator T the following somewhat stronger version of Theorem 2 is true.

Proposition 1. *Let $\Phi = (\phi_k)$ be a sequence of Orlicz functions and let λ be a solid sequence space which is topologized by an absolutely monotone F-seminorm g. Then*

$$\widehat{h}(\mathbf{x}) = \inf\{\rho > 0 : g(\Phi(|\rho^{-1}\mathbf{x}|)) \le \rho\}$$

is an absolutely monotone F-seminorm in generalized sequence space $\lambda^{\exists}(\Phi, \mathbf{X})$. If g is an absolutely monotone seminorm, then

$$h(\mathbf{x}) = \inf\{\rho > 0 : g(\Phi(|\rho^{-1}\mathbf{x}|)) \le 1\}$$

is a seminorm in $\lambda^{\exists}(\Phi, \mathbf{X})$. At that, if all spaces X_k $(k \in \mathbb{N})$ are normed, then $\widehat{h}$ is an F-norm (h is a norm) in $\lambda^{\exists}(\Phi, \mathbf{X})$ whenever g is an F-norm (a norm) in λ.

Proof. By Theorem 2, $\widehat{h}$ is an F-seminorm (or F-norm) and h is a seminorm (or norm) in $\lambda^{\exists}(\Phi, \mathbf{X})$. Since

$$\dot{|}x_k\dot{|} \le \dot{|}y_k\dot{|} \implies \phi_k\left(\dot{|}\rho^{-1}x_k\dot{|}\right) \le \phi_k\left(\dot{|}\rho^{-1}y_k\dot{|}\right) \quad (k \in \mathbb{N}),$$

and g is absolutely monotone, one has

$$\{\rho > 0 : g(\Phi(|\rho^{-1}\mathbf{y}|)) \le \rho\} \subset \{\rho > 0 : g(\Phi(|\rho^{-1}\mathbf{x}|)) \le \rho\},$$
$$\{\rho > 0 : g(\Phi(|\rho^{-1}\mathbf{y}|)) \le 1\} \subset \{\rho > 0 : g(\Phi(|\rho^{-1}\mathbf{x}|)) \le 1\}.$$

Consequently, $\widehat{h}(\mathbf{x}) \le \widehat{h}(\mathbf{y})$ and $h(\mathbf{x}) \le h(\mathbf{y})$ whenever $\dot{|}x_k\dot{|} \le \dot{|}y_k\dot{|}$ $(k \in \mathbb{N})$, i.e., $\widehat{h}$ and h are absolutely monotone. □

We remark that Ghosh and Srivastava [18] defined the norm h in generalized sequence space $\lambda^{\exists}(\phi, \mathbf{X})$, where ϕ is an Orlicz function and $\mathbf{X}$ is a sequence of Banach spaces.

3 Applications and corollaries

Almost all generalized Orlicz sequence spaces from the papers cited at the end of Section 1, are related to concrete methods of summability. The most common summability method is a matrix method defined by an infinite scalar matrix $A = (a_{nk})$. If for a sequence $\mathbf{x} \in s(X)$ the series $A_n\mathbf{x} = \sum_k a_{nk}x_k$ $(n \in \mathbb{N})$ converge and the limit $\lim_n A_n\mathbf{x} = l$ exists in X, then we say that $\mathbf{x}$ *is summable to l by the method A*. A summability method A is called *regular in X* if for all convergent in X sequences $\mathbf{x} = (x_k)$ we have,

$$\lim_k x_k = l \implies \lim_n A_n\mathbf{x} = l.$$

It is known that a matrix method A is regular in $\mathbb{K}$ if and only if (see, for example, [11], Theorem 2.3.7)

$$\|A\| = \sup_n \sum_k |a_{nk}| < \infty, \tag{12}$$

$$\lim_n a_{nk} = 0 \quad (k \in \mathbb{N}), \tag{13}$$

$$\lim_n \sum_k a_{nk} = 1. \tag{14}$$

It turned out that conditions (12), (13) and (14) characterize the regularity of A also in any Banach space X and, generally, in any sequentially complete locally convex Hausdorff topological vector space X (see [21] and [29]). A well-known example of a regular matrix method is Cesàro method C_1 defined by the matrix $C_1 = (c_{nk})$, where, for any $n \in \mathbb{N}$, $c_{nk} = n^{-1}$ if $k \leq n$ and $c_{nk} = 0$ otherwise. A (trivial) regular method is defined by *unit matrix* $I = (i_{nk})$, $i_{nn} = 1$ and $i_{nk} = 0$ for $n \neq k$.

Let $A = (a_{nk})$ be a non-negative matrix, i.e., $a_{nk} \geq 0$ $(n, k \in \mathbb{N})$. We say that A is *column-positive* if for any $k \in \mathbb{N}$ there exists an index n_k such that $a_{n_k,k} > 0$. A sequence $\mathbf{u} = (u_k) \in s$ is called *strongly A-summable to l* if $\lim_n \sum_k a_{nk}|u_k - l| = 0$, and *strongly A-bounded* if $\sup_n \sum_k a_{nk}|u_k| < \infty$. It is clear that the set $w_0(A)$ of all strongly A-summable to zero sequences and the set $w_\infty(A)$ of all strongly A-bounded sequences are linear spaces. Moreover, the functional

$$g_{_A}(\mathbf{u}) = \sup_n \sum_k a_{nk}|u_k|$$

is a seminorm in $w_\infty(A)$ and $w_0(A)$, it is a norm if A is column-positive.

More generally, if μ is a solid sequence space, then the set

$$\mu(A) = \{\mathbf{u} = (u_k) \in s : (A_n|\mathbf{u}|) \in \mu\}$$

is also a solid sequence space. Moreover, if μ is topologized by an absolutely monotone F-seminorm (seminorm) g, then in $\mu(A)$ we may define an absolutely monotone F-seminorm (seminorm) by the equality

$$g_{_A}(\mathbf{u}) = g\left(A|\mathbf{u}|\right),$$

where $A|\mathbf{u}| = (A_n|\mathbf{u}|)$. At it, $g_{_A}$ is an F-norm (norm) in $\mu(A)$ if g is an F-norm (norm) in μ and A is column-positive. It should be noted that the set $\mu(A)$ is non-trivial (i.e., it contains nonzero elements) only if $(a_{nk})_{n\in\mathbb{N}} \in \mu$ for some $k \in \mathbb{N}$. In particular, the set $w_0(A)$ is non-trivial if (13) holds for some k.

As a generalized Orlicz sequence space connected with strong summability we consider the set

$$\mu^{\exists}(A, \Psi\Phi, \mathbf{X}) = \{\mathbf{x} \in s(\mathbf{X}) : (\exists \rho > 0)\ \Psi\Phi(|\rho^{-1}\mathbf{x}|) \in \mu(A)\},$$

where $\mu \subset s$ is a solid sequence space, A is a non-negative matrix, Φ is a sequence of Orlicz functions and Ψ is a sequence of φ-functions. Moreover, if $B = (b_{ki})$ is an arbitrary summability matrix, then we define also the set

$$\mu^{\exists}(A, \Psi\Phi, B, X) = \{\mathbf{x} \in s(X) : (\exists \rho > 0)\ \Psi\Phi(|\rho^{-1} B\mathbf{x}|) \in \mu(A)\}.$$

Let us denote by

$$s_{_B}(X) = \left\{ \mathbf{x} \in s(X) : \text{series } \ B_n\mathbf{x} = \sum_k b_{nk}x_k \quad (n \in \mathbb{N}) \ \text{ converge in } \ X \right\}$$

the *application domain* of the matrix method B. Then the operator $B : s_{_B}(X) \to s(X)$, $B\mathbf{x} = (B_n\mathbf{x})$, is linear. Consequently, since $\mu^{\exists}(A, \Psi\Phi, \mathbf{X})$ and $\mu^{\exists}(A, \Psi\Phi, B, X)$ are defined as the sets $\lambda^{\exists}(\Psi\Phi, \mathbf{X})$ and $\lambda^{\exists}(\Psi\Phi, T, \mathbf{X})$ with $\lambda = \mu(A)$ and $T = B$, from Theorem 1 we immediately get the following proposition about the linearity of $\mu^{\exists}(A, \Psi\Phi, \mathbf{X})$ and $\mu^{\exists}(A, \Psi\Phi, B, X)$.

Proposition 2. *Let Φ be a sequence of Orlicz functions, and let Ψ be a sequence consisting of modulus and Orlicz functions. If $\mu \subset s$ is a solid sequence space and $A = (a_{nk})$ is a non-negative matrix, then $\mu^{\exists}(A, \Psi\Phi, \mathbf{X})$ is a solid generalized sequence space. Moreover, if B is an arbitrary summability matrix, then $\mu^{\exists}(A, \Psi\Phi, B, X)$ is a generalized sequence space.*

We apply Proposition 2 in the case if Ψ is the sequence $\mathcal{I}^{\mathbf{p}} = (\iota^{p_k})$ of φ-functions $\iota^{p_k}(t) = t^{p_k}$, where (p_k) is a sequence of positive numbers.

Corollary 1. *Let $\mathbf{p} = (p_k)$ be a sequence of positive numbers. If Φ, μ, A and B are the same as in Proposition 2, then*

$$\mu^{\exists}(A, \mathbf{p}, \Phi, \mathbf{X}) = \left\{ \mathbf{x} \in s(\mathbf{X}) : (\exists \rho > 0)\ \left(\sum_k a_{nk} \left(\phi_k \left(\left| \rho^{-1} x_k \right| \right) \right)^{p_k} \right) \in \mu \right\}$$

is a solid generalized sequence space and

$$\mu^{\exists}(A, \mathbf{p}, \Phi, B, X) = \left\{ \mathbf{x} \in s(X) : (\exists \rho > 0)\ \left(\sum_k a_{nk} \left(\phi_k \left(\left| \rho^{-1} \sum_i b_{ki} x_i \right| \right) \right)^{p_k} \right) \in \mu \right\}$$

is a generalized sequence space.

In [2], [4], [5], [7], [9], [10], [13], [14], [22], [31], [32], [35] and [36] the authors prove the linearity of $\mu^{\exists}(A, \mathbf{p}, \Phi, \mathbf{X})$ and $\mu^{\exists}(A, \mathbf{p}, \Phi, B, X)$ (mostly for $\mu \in \{\ell, \ell_\infty, c_0\}$ and some concrete matrices A, B) under the assumption that the sequence $\mathbf{p}$ is bounded. Our results show that these sets are linear spaces also for unbounded sequence $\mathbf{p}$.

To describe the topologies for the sequence spaces given above we need some results about the topologization of generalized sequence space

$$\mu(\Psi, \mathbf{X}) = \{\mathbf{x} \in s(\mathbf{X}) : \Psi(|\mathbf{x}|) \in \mu\} \tag{15}$$

defined by a sequence space μ and by a sequence $\Psi = (\psi_k)$ of modulus functions. Also a lemma on the AK-property of $\mu(A)$ is necessary.

Recall that the mth *section* of a sequence $\mathbf{u} = (u_k)$ is defined by $\mathbf{u}^{[m]} = \sum_{k=1}^{m} u_k \mathbf{e}_k$. An F-seminormed sequence space (λ, g) is called an AK-*space* if λ contains the sequences $\mathbf{e}_k$ $(k \in \mathbb{N})$ and $\lim_m \mathbf{u}^{[m]} = \mathbf{u}$ for any $\mathbf{u} \in \lambda$. Well-known AK-spaces are c_0 with the norm $\|\mathbf{u}\|_\infty = \sup_k |u_k|$ and ℓ_p $(p \geq 1)$ with the norm $\|\mathbf{u}\|_p = \left(\sum_k |u_k|^p\right)^{1/p}$.

The following proposition was proved in [34] and [23].

Proposition 3. *Let (μ, g) be an F-seminormed (or a paranormed) solid sequence space. If g is absolutely monotone and the sequence of modulus functions $\Psi = (\psi_k)$ satisfies one of equivalent conditions*

(M1) There exist a function ν and a number $\delta > 0$ such that $\psi_k(ut) \leq \nu(u)\psi_k(t)$ $(0 \leq u < \delta,\ t \geq 0)$ and $\lim_{u \to 0+} \nu(u) = 0$;

(M2) $$\lim_{u \to 0+} \sup_{t>0} \sup_k \frac{\psi_k(ut)}{\psi_k(t)} = 0,$$

then

$$g_\Psi(\mathbf{x}) = g(\Psi(|\mathbf{x}|)) \quad (\mathbf{x} \in \mu(\Psi, \mathbf{X}))$$

defines an absolutely monotone F-seminorm (or a paranorm) in solid generalized sequence space $\mu(\Psi, \mathbf{X})$. If (μ, g) is an AK-space, then g_Ψ is an absolutely monotone F-seminorm in $\mu(\Psi, \mathbf{X})$ for an arbitrary sequence Ψ of moduli, and $(\mu(\Psi, \mathbf{X}), g_\Psi)$ is an AK-space. If the spaces X_k $(k \in \mathbb{N})$ are normed and g is an F-norm, then g_Ψ is an F-norm in $\mu(\Psi, \mathbf{X})$.

Lemma 1. *Let A be a non-negative matrix such that $(a_{nk})_{n \in \mathbb{N}} \in \mu$ for any $k \in \mathbb{N}$. If (μ, g) is an F-seminormed AK-space such that g is absolutely monotone, then $(\mu(A), g_A)$ is an F-seminormed AK-space. At it, if g is an F-norm (a norm) and A is column-positive, then $(\mu(A), g_A)$ is an F-normed (a normed) AK-space.*

Proof. The assumption $(a_{nk})_n \in \mu$ $(k \in \mathbb{N})$ is equivalent to $\{\mathbf{e}_k : k \in \mathbb{N}\} \subset \mu(A)$.

Let $\mathbf{u} \in \mu(A)$. Then the sequence $A|\mathbf{u}|$ is in μ and, since (μ, g) is AK-space,

$$\begin{aligned} &\lim_m g\left(A|\mathbf{u}| - A|\mathbf{u}|^{[m]}\right) \\ &\quad = \lim_m g\left(\left(\overbrace{0, \ldots, 0}^{m}, \sum_k a_{m+1,k}|u_k|, \ldots\right)\right) = 0. \end{aligned} \tag{16}$$

To prove that $\lim_m \mathbf{u}^{[m]} = \mathbf{u}$, we use the inequality

$$\begin{aligned} g_A\left(\mathbf{u} - \mathbf{u}^{[m]}\right) &\leq \sum_{n=1}^{r} g\left(\sum_{k=m+1}^{\infty} a_{nk}|u_k|\mathbf{e}_n\right) \\ &\quad + g\left(\left(\overbrace{0, \ldots, 0}^{r}, \sum_{k=m+1}^{\infty} a_{r+1,k}|u_k|, \ldots\right)\right) \\ &= G^1_{rm} + G^2_{rm}. \end{aligned}$$

Let $\varepsilon > 0$. Because g is absolutely monotone,

$$G^2_{mm} \le g\left(A|\mathbf{u}| - A|\mathbf{u}|^{[m]}\right),$$

and by (16) we get $\lim_m G^2_{mm} = 0$. Thus there exists a number $m_0 \in \mathbb{N}$ with

$$G^2_{m_0,m_0} < \varepsilon/2.$$

Now, since the series $\sum_k a_{nk}|u_k|$ converge, we can fix $m_1 \ge m_0$ such that, for any $m \ge m_1$,

$$G^1_{m_0,m} < \varepsilon/2.$$

Hence, using also the inequalities $G^2_{m_0,m} \le G^2_{m_0,m_0}$ $(m \ge m_0)$, we have that

$$g_A\left(\mathbf{u} - \mathbf{u}^{[m]}\right) \le G^1_{m_0,m} + G^2_{m_0,m} < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

if $m \ge m_1$. Consequently, $\lim_m g_A\left(\mathbf{u} - \mathbf{u}^{[m]}\right) = 0$, i.e., $\lim_m \mathbf{u}^{[m]} = \mathbf{u}$ in $\mu(A)$. □

Now we can determine F-seminorms in generalized Orlicz sequence spaces $\mu^{\exists}(A, \Psi\Phi, B, X)$ and $\mu^{\exists}(A, \Psi\Phi, \mathbf{X})$.

Proposition 4. *Let Φ be a sequence of Orlicz functions, μ be a solid sequence space, $A = (a_{nk})$ be a non-negative matrix and B be an arbitrary matrix. For a sequence Ψ of modulus functions which satisfies one of conditions (M1) and (M2), the following is true:*

(i) If g is an absolutely monotone F-seminorm in μ, then

$$\widehat{h}_{\Psi}(\mathbf{x}) = \inf\left\{\rho > 0 : g\left(A\left(\Psi\Phi(|\rho^{-1}B\mathbf{x}|)\right)\right) \le \rho\right\}$$

is an F-seminorm in $\mu^{\exists}(A, \Psi\Phi, B, X)$;

(ii) If the space X is normed, matrix A is column-positive and g is an absolutely monotone F-norm in μ, then $\widehat{h}_{\Psi}$ is an F-norm in $\mu^{\exists}(A, \Psi\Phi, B, X)$ whenever the operator B satisfies (8).

If (μ, g) is an AK-space with respect to the F-seminorm (F-norm) g and A is such that $(a_{nk})_{n\in\mathbb{N}} \in \mu$ $(k \in \mathbb{N})$, then the statements (i) and (ii) are true for an arbitrary sequence Ψ of modulus functions.

In the case $B = I$, all previous statements hold for the space $\mu^{\exists}(A, \Psi\Phi, \mathbf{X})$ with absolutely monotone $\widehat{h}_{\Psi}$.

Proof. The set $\mu^{\exists}(A, p, \Psi\Phi, B, X)$ we may consider as the set $\lambda^{\exists}(\Phi, T, X)$, where $\lambda = \mu(A)(\Psi)$ is the sequence space of type (15) and $T = B$. So, if we topologize λ on the basis of Proposition 3, then our statements follow by Theorem 2 and Proposition 1 in view of Lemma 1. □

Next we apply Proposition 4 to the sequence spaces $\mu^{\exists}(A,\mathbf{p},\Phi,B,X)$ and $\mu^{\exists}(A,\mathbf{p},\Phi,\mathbf{X})$ in the case of bounded sequence $\mathbf{p}=(p_k)$.

Proposition 5. *Let Φ, μ, A and B be the same as in Proposition 4. Let $\mathbf{p}=(p_k)$ be a bounded sequence of positive numbers. If $\inf_k p_k > 0$, then the following is true:*

(i) If g is an absolutely monotone F-seminorm in μ, then

$$\widehat{h}_{\mathbf{p}}(\mathbf{x}) = \inf\{\rho>0 : g\left(A\left(\Phi\left(|\rho^{-1}B\mathbf{x}|\right)\right)^{\mathbf{p}}\right)\le\rho\}$$
$$= \inf\{\rho>0 : g\left(\left(\sum_k a_{nk}\left(\phi_k\left(\left|\rho^{-1}\sum_i b_{ki}x_i\right|\right)\right)^{p_k}\right)\right)\le\rho\}$$

is an F-seminorm in $\mu^{\exists}(A,\mathbf{p},\Phi,B,X)$;

(ii) If the space X is normed, matrix A is column-positive and g is an absolutely monotone F-norm in μ, then $\widehat{h}_{\mathbf{p}}$ is an F-norm in $\mu^{\exists}(A,\mathbf{p},\Phi,B,X)$ whenever the operator B satisfies (8).

If (μ,g) is an AK-space with respect to the F-seminorm (F-norm) g and A is such that $(a_{nk})_{n\in\mathbb{N}}\in\mu$ $(k\in\mathbb{N})$, then the statements (i) and (ii) are true without the restriction $\inf_k p_k>0$.

In the case $B=I$, our statements hold for $\mu^{\exists}(A,\mathbf{p},\Phi,\mathbf{X})$ with absolutely monotone $\widehat{h}_{\mathbf{p}}$.

Proof. Since the functions ι^{p_k} are not moduli for $p_k>1$, we introduce new sequence $\mathbf{r}=(r_k)$ of numbers $r_k=p_k/M$ $(k\in\mathbb{N})$, where $M=\max\{1,\sup_k p_k\}$. Then the functions ι^{r_k} are moduli for any $k\in\mathbb{N}$. In addition, the sequence $\mathcal{I}^{\mathbf{r}}=(\iota^{r_k})$ satisfies (M2) if and only if $\inf_k p_k>0$, since (M2) reduces to $\lim_{u\to0+}\sup_k u^{r_k}=0$ in this case. Thus, since $\Phi^M=(\Psi_k^M)$, $\Psi_k^M(t)=(\Psi_k(t))^M$, is the sequence of Orlicz functions and $\mu^{\exists}(A,\mathbf{p},\Phi,B,X)$ is precisely the space $\mu^{\exists}(A,\mathcal{I}^{\mathbf{r}}\Phi^M,B,X)$, it suffices to apply Proposition 4. □

Corollary 2. *Let Φ, $\mathbf{p}$, A and B be the same as in Proposition 5. If g is an absolutely monotone F-seminorm in μ, then*

$$\widehat{h}_1(\mathbf{x}) = \inf\{\rho>0 : \sum_n\sum_k a_{nk}\left(\phi_k\left(\left|\rho^{-1}\sum_i b_{ki}x_i\right|\right)\right)^{p_k}\le\rho\}$$

is an F-seminorm in $\ell^{\exists}(A,\mathbf{p},\Phi,B,X)$ whenever the series $\sum_n a_{nk}$ $(k\in\mathbb{N})$ converge, and

$$\widehat{h}_\infty(\mathbf{x}) = \inf\{\rho>0 : \sup_n\sum_k a_{nk}\left(\phi_k\left(\left|\rho^{-1}\sum_i b_{ki}x_i\right|\right)\right)^{p_k}\le\rho\}$$

is an F-seminorm in ${c_0}^{\exists}(A,\mathbf{p},\Phi,B,X)$ whenever A satisfies (13). *Moreover, if $\inf_k p_k>0$, then $\widehat{h}_\infty$ is an F-seminorm in ${\ell_\infty}^{\exists}(A,\mathbf{p},\Phi,B,X)$.*

If the space X is normed, g is an absolutely monotone F-norm in μ, matrix A is column-positive and B satisfies (8), *then $\widehat{h}_1$ and $\widehat{h}_\infty$ are F-norms.*

Proof. Since $\ell, \|\cdot\|_1)$ and $(c_0, \|\cdot\|_\infty)$ are AK-spaces but $(\ell_\infty, \|\cdot\|_\infty)$ is not, the statements of corollary follow from Proposition 5. □

The authors of [1], [3], [4], [5], [6], [7], [8], [9], [10], [13], [14], [32], [35] and [36] defined paranorms of type (4) and (5) in some spaces $\mu^{\ni}(A, \mathbf{p}, \Phi, \mathbf{X})$ and $\mu^{\ni}(A, \mathbf{p}, \Phi, B, X)$ with $\mu \in \{\ell, \ell_\infty, c_0\}$. Özdemir and Solak [31] defined the same type paranorm in $\mu^{\ni}(D, \mathbf{p}, \phi, \mathbf{X})$, where $D = (d_{ki})$ with $d_{kk} = k^{-s}$, $s > 0$ and $d_{ki} = 0$ otherwise. Alternatively, Proposition 5 and Corollary 2 allow us to use F-seminorms of type $\widehat{h}_{\mathbf{p}}$ by topologization of these spaces.

A sharpened version of Proposition 5 is possible in the case if A is the unit matrix I.

Proposition 6. *Let Φ be a sequence of Orlicz functions, Ψ be a sequence of φ-functions and B be a summability matrix. Let μ and λ be solid sequence spaces, where λ is topologized by an absolutely monotone seminorm g'. If*

$$\psi_k(1) = 1 \quad (k \in \mathbb{N}), \tag{17}$$

$$\mu(\Psi) \subset \lambda \tag{18}$$

and

$$g'(\mathbf{u}) \le 1 \iff |u_k| \le 1 \ \ (k \in \mathbb{N}), \tag{19}$$

then

$$h_{\mathbf{p}}(\mathbf{x}) = \inf\left\{\rho > 0 : g'\left(\Psi\Phi\left(|\rho^{-1}B\mathbf{x}|\right)\right) \le 1\right\}$$

and

$$h(\mathbf{x}) = \inf\left\{\rho > 0 : g'\left(\Phi\left(|\rho^{-1}B\mathbf{x}|\right)\right) \le 1\right\}$$

are seminorms, with $h_{\mathbf{p}} = h$, in $\mu^{\ni}(\Psi\Phi, B, X)$. Thereat, $h_{\mathbf{p}}$ and h are norms if X is normed, g' is an absolutely monotone norm in λ and B satisfies (8).

Proof. By Theorem 2 (with $T = B$) we see that h is a seminorm in $\lambda^{\ni}(\Phi, B, X)$. At it, h is a norm in $\mu^{\ni}(\Phi, B, X)$ if g' is an absolutely monotone norm in λ, X is normed and B satisfies (8). Further, by (17) we have

$$\psi_k(t) \le 1 \iff t \le 1 \ \ (t \in \mathbb{R}^+, k \in \mathbb{N}).$$

So, using also (19), we get that $h_{\mathbf{p}}(\mathbf{x}) = h(\mathbf{x})$ for any $\mathbf{x} \in \lambda^{\ni}(\Phi, B, X)$. Since

$$\mu^{\ni}(\Psi\Phi, B, X) \subset \lambda^{\ni}(\Phi, B, X)$$

because of (18), $h_{\mathbf{p}}$ and h are seminorms (norms) also in $\mu^{\ni}(\Psi\Phi, B, X)$. □

Corollary 3. *Let Φ be a sequence of Orlicz functions, B be a summability matrix and $\mathbf{p} = (p_k)$ be a sequence of positive numbers. Then*

$$h_{\infty,\mathbf{p}}(\mathbf{x}) = \inf\left\{\rho > 0 : \sup_k \left(\phi_k\left(\left|\rho^{-1}\sum_i b_{ki}x_i\right|\right)\right)^{p_k} \leq 1\right\}$$

and

$$h_\infty(\mathbf{x}) = \inf\left\{\rho > 0 : \sup_k \phi_k\left(\left|\rho^{-1}\sum_i b_{ki}x_i\right|\right) \leq 1\right\}$$

are seminorms, with $h_{\infty,\mathbf{p}} = h_\infty$, in generalized Orlicz sequence space $c_0^{\exists}(\mathbf{p}, \Phi, B, X)$. Thereat, $h_{\infty,\mathbf{p}}$ and h_∞ are norms if X is normed and B satisfies (8).

Proof. By Corollary 1 the set $c_0^{\exists}(\mathbf{p}, \Phi, B, X)$ is a generalized sequence space. Further, Theorem 4 [24] shows, because of

$$\iota^{p_k}(1) = 1^{p_k} = 1 \ \ (k \in \mathbb{N}),$$

that

$$c_0(\mathbf{p}) \subset \ell_\infty.$$

Since, besides this, the norm $\|\cdot\|_\infty$ in ℓ_∞ satisfies (19), we can apply Proposition 6 with $\mu = c_0$, $\lambda = \ell_\infty$ and $\Psi = (\iota^{p_k})$. □

In summability theory an infinite matrix $A = (a_{nk})$ is called *normal* if $a_{nk} = 0$ for $k > n$ and $a_{nn} \neq 0$. For example, unit matrix I and Cesàro matrix C_1 are normal. If (v_k) is a given scalar sequence, then the matrix $D(v_k) = (d_{ki})$, where $d_{kk} = v_k$ and $d_{ki} = 0$ otherwise, is known as *diagonal matrix*. It is clear that a diagonal matrix $D(v_k)$ is normal if $v_k \neq 0$ for any $k \in \mathbb{N}$. In connection with applications of Propositions 4 – 6 and Corollaries 2 and 3 it is essential to remark that every non-negative normal matrix A is column-positive and the summability operator B, defined by a normal matrix B, satisfies (8).

In [4], [7], [14], [35], [36] and [39] the authors considered the spaces of type $\mu^{\exists}(A, \mathbf{p}, \Phi, B, X)$, where the matrix A is column-positive and the matrix B is related to difference sequences. For fixed $m, r \in \mathbb{N}$ the *difference operator* Δ_r^m is defined by (see [36])

$$\Delta_r^m\mathbf{x} = (\Delta_r^m x_k), \ \ \Delta_r^m x_k = \Delta_r^{m-1}x_k - \Delta_r^{m-1}x_{k+r}, \ \ \Delta_r^0 x_k = x_k \quad (k \in \mathbb{N}).$$

If $r = 1$, then Δ_r^m reduces to the difference operator Δ^m introduced in [15].

Since

$$\Delta_r^m x_k = \sum_{i=0}^{m}(-1)^i\binom{n}{i}x_{k+ri} \quad (k \in \mathbb{N}),$$

Δ_r^m is the summability operator defined by the *difference matrix* $\Delta_r^m = (\delta_{kj})$, where $\delta_{kj} = (-1)^i\binom{m}{i}$ if $j = k + ri$, $(0 \leq i \leq m, k \in \mathbb{N})$ and $\delta_{kj} = 0$ otherwise. It is not

difficult to see that (19) fails for difference operators Δ_r^m. Therefore, Proposition 5 allows not to define F-norms in generalized sequence spaces $\mu^{\exists}(A, \mathbf{p}, \Phi, \Delta_r^m, X)$. To overcome this difficulty we introduce a new class of summability matrices which contains all difference matrices.

Let $m \in \mathbb{N}$ be fixed. We say that an infinite scalar matrix $B = (b_{ki})$ is *m-normal* if, for any $k \in \mathbb{N}$, $b_{k,k+m} \neq 0$ and $b_{ki} = 0$ if $i > k + m$. For example, the difference matrix Δ_r^m is rm-normal and Δ^m is m-normal. Now, if the matrix B is m-normal, then $B\mathbf{x} = 0$, $\mathbf{x} \in s_B(X)$, implies $\mathbf{x} = 0$ whenever $x_1 = \cdots = x_m = 0$. This approach and the definitions of paranorms from [14], [35], [36] and [39] lead us to the following modification of Proposition 5.

Proposition 7. *Let Φ, μ, A and B be the same as in Proposition 4. Assume that B is m-normal and $\mathbf{p} = (p_k)$ is a bounded sequence of positive numbers. If $\inf_k p_k > 0$, then the following is true:*

(i) If g is an absolutely monotone F-seminorm in μ, then

$$\widetilde{h}_{\mathbf{p}}(\mathbf{x}) = \sum_{j=1}^{m} \left|x_j\right| + \inf\{\rho > 0 : g\left(A\left(\left(\Phi\left(|\rho^{-1}B\mathbf{x}|\right)\right)^{\mathbf{p}}\right)\right) \leq \rho\}$$

is an F-seminorm in $\mu^{\exists}(A, \mathbf{p}, \Phi, B, X)$;

(ii) If the space X is normed, matrix A is column-positive and g is an absolutely monotone F-norm in μ, then $\widetilde{h}_{\mathbf{p}}$ is an F-norm in $\mu^{\exists}(A, \mathbf{p}, \Phi, B, X)$.

If (μ, g) is an AK-space with respect to the F-seminorm (F-norm) g and A is such that $(a_{nk})_{n\in\mathbb{N}} \in \mu$ for any $k \in \mathbb{N}$, then the statements (i) and (ii) are true without the restriction $\inf_k p_k > 0$.

Proposition 7 and Corollary 4 determine alternative topologies for the difference sequence spaces from [2], [7], [14], [22], [35], [36] and [39].

Stronger versions of Propositions 5 and 7 hold in the case if $\mathbf{p} = (p_k)$ is a constant sequence with $p_k = 1$ $(k \in \mathbb{N})$.

Proposition 8. *Let Φ be a sequence of Orlicz functions, μ be a solid sequence space, $A = (a_{nk})$ be a non-negative matrix and B be an arbitrary matrix. Then the following is true:*

(i) If g is an absolutely monotone seminorm in μ, then

$$\begin{aligned} h_{\Phi}(\mathbf{x}) &= \inf\left\{\rho > 0 : g\left(A\left(\Phi\left(|\rho^{-1}B\mathbf{x}|\right)\right)\right) \leq 1\right\} \\ &= \inf\left\{\rho > 0 : g\left(\left(\sum_k a_{nk}\phi_k\left(\left|\rho^{-1}\sum_i b_{ki}x_i\right|\right)\right)\right) \leq 1\right\} \end{aligned}$$

is a seminorm in $\mu^{\exists}(A, \Phi, B, X)$;

(ii) If the space X is normed, matrix A is column-positive and g is an absolutely monotone norm in μ, then h_{Φ} is a norm in $\mu^{\exists}(A, \Phi, B, X)$ whenever the operator B satisfies (8).

If $B = I$, then all previous statements hold for the space $\mu^{\exists}(A, \Phi, \mathbf{X})$ with absolutely monotone h_{Φ}.

Proof. As it was remarked above, $\mu(A)$ is a solid sequence space which is topologized by an absolutely monotone seminorm (norm) $g_{A}(\mathbf{u}) = g\left(A|\mathbf{u}|\right)$ if g is an absolutely monotone seminorm (norm and A is column-positive). Similarly to the proof of Proposition 4, interpreting $\mu^{\exists}(A, \Phi, B, \mathbf{X})$ as the space $\lambda^{\exists}(\Phi, T, X)$ with $\lambda = \mu(A)$ and $T = B$, we may apply Theorem 2. □

Proposition 9. *Let Φ, μ and A be the same as in Proposition 8, and let B be an m-normal matrix. Then the following is true:*

(i) If g is an absolutely monotone seminorm in μ, then

$$\widetilde{h}_{_{\Phi}}(\mathbf{x}) = \sum_{j=1}^{m} \left|x_j\right| + \inf\{\rho > 0 : g\left(A\left(\Phi\left(|\rho^{-1}B\mathbf{x}|\right)\right)\right) \leq 1\}$$
$$= \sum_{j=1}^{m} \left|x_j\right| + \inf\{\rho > 0 : g\left(\left(\sum_{k} a_{nk}\phi_k\left(\left|\rho^{-1}\sum_{i} b_{ki}x_i\right|\right)\right)\right) \leq 1\}$$

is a seminorm in $\mu^{\exists}(A, \Phi, B, X)$;

(ii) If the space X is normed, matrix A is column-positive and g is an absolutely monotone norm in μ, then $\widetilde{h}_{_{\Phi}}$ is an norm in $\mu^{\exists}(A, \Phi, B, X)$.

Based on Proposition 9, we may state a modified variant of Corollary 3.

Corollary 4. *Let Φ be a sequence of Orlicz functions, B be a summability matrix and $\mathbf{p} = (p_k)$ be a sequence of positive numbers. If B is m-normal, then*

$$\widetilde{h}_{\infty,\mathbf{p}}(\mathbf{x}) = \sum_{j=1}^{m} \left|x_j\right| + \inf\left\{\rho > 0 : \sup_{k}\left(\phi_k\left(\left|\rho^{-1}\sum_{i} b_{ki}x_i\right|\right)\right)^{p_k} \leq 1\right\}$$

and

$$\widetilde{h}_{\infty}(\mathbf{x}) = \sum_{j=1}^{m} \left|x_j\right| + \inf\left\{\rho > 0 : \sup_{k}\phi_k\left(\left|\rho^{-1}\sum_{i} b_{ki}x_i\right|\right) \leq 1\right\}$$

are seminorms, with $\widetilde{h}_{\infty,\mathbf{p}} = \widetilde{h}_{\infty}$, in generalized Orlicz sequence space $c_0{}^{\exists}(\mathbf{p}, \Phi, B, X)$. Thereat, $\widetilde{h}_{\infty,\mathbf{p}}$ and $\widetilde{h}_{\infty}$ are norms if X is normed.

Proposition 8 generalizes the results about the topologies in various Orlicz sequence spaces given in [16], [17], [26], [40], [41] and [42]. A space of type $\mu^{\exists}(A, \phi, B, X)$ with the m-normal matrix $B = \Delta^m$ is studied in [38]. Corollary 4 determines alternative topologies in a generalized difference space of type $c_0{}^{\exists}(\mathbf{p}, \Phi, B, X)$ with $B = \Delta_r^m$ from [36].

Remark 2. *If both A and B are non-negative matrices, then we may define the generalized Orlicz sequence space of absolute type*

$$\lambda^{\exists}[A, \Psi\Phi, B, X] = \{\mathbf{x} \in s(X) : (\exists\rho > 0)\ A(\Psi\Phi(\rho^{-1}B|\mathbf{x}|)) \in \lambda\}$$

and its special variant $\mu^{\exists}[A, \mathbf{p}, \Phi, B, X]$. *For* $B = I$ *it is possible to define also the spaces of absolute type* $\lambda^{\exists}[A, \Psi\Phi, \mathbf{X}]$ *and* $\lambda^{\exists}[A, \mathbf{p}, \Phi, X]$, *but they clearly coincide with the spaces* $\lambda^{\exists}(A, \Psi\Phi, \mathbf{X})$ *and* $\mu^{\exists}(A, \mathbf{p}, \Phi, \mathbf{X})$ *considered below. It is not difficult to see that all our results remain true for these space of absolute type. For example, if* $(X_k, \|\cdot\|_k)$ $(k \in \mathbb{N})$ *are normed spaces,* A *is column-positive and* B *is normal, then the generalized Orlicz sequence space*

$$\ell^{\exists}[A, \Phi, B, X] = \left\{ \mathbf{x} \in s(X) : (\exists \rho > 0) \sum_n \sum_k a_{nk} \phi_k \left(\sum_i b_{ki} \|\rho^{-1} x_i\|_k \right) < \infty \right\}$$

is a normed space with the norm

$$h_\Phi(\mathbf{x}) = \inf \left\{ \rho > 0 : \sum_n \sum_k a_{nk} \phi_k \left(\sum_i b_{ki} \|\rho^{-1} x_i\|_k \right) \le 1 \right\}.$$

Kubiak [25] considered the space $\ell^{\exists}[I, \phi, C_1]$ *named as Cesàro–Orlicz sequence space.*

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Enno Kolk:
Institute of Mathematics, University of Tartu, 50090 Tartu, Estonia
E-mail: enno.kolk@ut.ee