Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Filomat **25:4** (2011), 37–53 DOI: 10.2298/FIL1104037K

INVARIANT SUBSPACES OF MATRIX GROUPS AND ELEMENTARY-ABELIAN COVERS OF $K_{4,4}$

Boštjan Kuzman

Abstract

We study lifting conditions for groups of automorphisms of the complete bipartite graph $K_{4,4}$. In particular, for $p \neq 2$ a prime we construct, up to isomorphism of projections, all minimal p-elementary abelian covers of $K_{4,4}$ such that the respective covering projections admit a lift of some arc-transitive subgroup of $\mathrm{Aut}(K_{4,4})$.

1 Introduction

Large symmetric graphs, that is, those with automorphism groups acting transitively on the set of arcs (directed edges), are typically constructed as regular covers of smaller graphs. Indeed, by a result of Djoković [4], a regular cover inherits certain symmetry properties of the base graph whenever a "well-behaved" group of automorphisms of the base graph lifts along the covering projection to a group of automorphisms of the respective cover. Along these lines, Djoković [4] constructed the first infinite families of graphs with small valencies and high degree of transitivity (some of these results are also attributed to Conway, see [2]).

In certain cases the process can be reversed. If N is a semiregular normal subgroup of an arc-transitive group $G \leq \mathrm{Aut}(X)$, then the quotient projection $X \to X/N$ is a regular covering projection with N as the group of covering transformations, and the group G/N, acting on X/N, inherits properties of G, acting on X. Clearly, the group G/N lifts to G. It follows that large families of symmetric graphs can be reduced to a (usually small) set of "irreducible" symmetric graphs by recursively factoring out the action of N.

For graphs admitting solvable automorphism groups, the above reduction method is naturally applied to their minimal normal subgroups. Recall that these are elementary abelian, that is, isomorphic to $\mathbb{Z}_p^k$ for some prime p. Hence lifting automorphisms along regular elementary abelian covering projections is essential for the

2000 *Mathematics Subject Classifications.* 05C25, 20B25.

Key words and Phrases. Regular covers, elementary abelian covers, voltage graphs, lifting automorphisms.
Received: July 2, 2010
Communicated by Dragan Stevanović

classification of such classes of symmetric graphs. The automorphism lifting problem in the context of elementary abelian covers was studied by Malnič, Marušič and Potočnik [15]. Their results have been succesfully applied in order to classify elementary abelian covers with specific symmetric properties for a number of small cubic or tetravalent graphs, namely, the complete graphs K_4 [9] and K_5 [12], the Petersen graph [16], the Heawood graph [15], the Möbius-Kantor graph [8], the Octahedron graph [13], and the Pappus graph [18]. In this paper we construct, up to isomorphism of projections, all minimal arc-transitive elementary abelian covers of the complete bipartite graph $K_{4,4}$ (Theorems 3.1 and 3.2), thereby expanding the Census of edge-transitive tetravalent graphs by Potočnik and Wilson [19].

2 Preliminaries

Throughout this paper, graphs are finite, simple, connected and undirected. For a graph X we denote its vertex, edge and arc set by $V(X)$, $E(X)$, $A(X)$, respectively. For a vertex $v \in V(X)$, denote by $N(v)$ the set of vertices adjacent to v.

A *covering projection* is a graph homomorphism $q\colon \tilde{X} \to X$ which is onto and localy bijective, that is, for any pair of vertices $v \in V(X)$ and $\tilde{v} \in q^{-1}(v)$, the restriction $q\colon N(\tilde{v}) \to N(v)$ is a bijection. In this setting we call X a *base graph*, $\tilde{X}$ a *covering graph* or *cover*, and the preimages $q^{-1}(v)$, $v \in V(X)$, and $q^{-1}(e)$, $e \in E(X)$, are the *vertex fibres* and the *edge fibres*, respectively. A covering projection is *regular* if there exists a semiregular subgroup $H \leq \mathrm{Aut}(\tilde{X})$, called the *group of covering transformations*, such that the quotient graph $\tilde{X}/H$ is isomorphic to X (that is, if the H-orbits of $\tilde{X}$ coincide with the vertex fibres $q^{-1}(v)$, $v \in V(X)$). In particular, a regular covering q is *p-elementary abelian*, if $H \cong \mathbb{Z}_p^k$, p a prime.

Two covering projections $q\colon \tilde{X} \to X$ and $q'\colon \tilde{X}' \to X$ are *isomorphic* if the following diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{\alpha}} & \tilde{X}' \\ \Big\downarrow q & & \Big\downarrow q' \\ X & \xrightarrow{\alpha} & X \end{array}$$

commutes, that is, if $\alpha q = q\tilde{\alpha}$ for some graph automorphism $\alpha \in \mathrm{Aut}(X)$ and graph isomorphism $\tilde{\alpha}\colon \tilde{X} \to \tilde{X}'$. In particular, q and q' are called *equivalent* if $\alpha = \mathrm{id}_X$. If in the above diagram $\tilde{X}' = \tilde{X}$ and $q' = q$, we say that $\alpha \in \mathrm{Aut}(X)$ *lifts* along the covering projection q, and $\tilde{\alpha}$ is a *lift* of α. The covering projection q is *G-admissible* for a given subgroup $G \leq \mathrm{Aut}(X)$ whenever each $\alpha \in G$ lifts along q. We also say that G *lifts* and the group $\tilde{G} = \{\tilde{\alpha} \mid \alpha \in G\} \leq \mathrm{Aut}(\bar{X})$ is *the lift of G*.

A graph X is called *arc-transitive* (a.k.a. *symmetric*) if the action of its full automorphism group $\mathrm{Aut}(X)$ is transitive on the set of arcs. The following lemma is easy to prove, the second part being a special case of [4].

Lemma 2.1. *Let $q\colon \tilde{X} \to X$ be a G-admissible covering projection for some $G \leq$ $\mathrm{Aut}(X)$ and let $\tilde{G}$ be its lift.*

(a) If $q'\colon \tilde{X} \to X$ is equivalent to q, then q' is G-admissible. If $q'\colon \tilde{X}' \to X$ is isomorphic to q under some $\alpha \in \mathrm{Aut}(X)$, then q' is $\alpha G \alpha^{-1}$-admissible.

(b) If G is arc-transitive on X, then $\tilde{G}$ is arc-transitive on $\tilde{X}$.

If q is G-admissible for some arc-transitive subgroup $G \leq \mathrm{Aut}(X)$, we say that q is an *arc-transitive covering projection*. By Lemma 2.1, arc-transitivity of q implies arc-transitivity of $\tilde{X}$, however, the reverse might not be true in general. Clearly, in order to construct all arc-transitive covering projections it is sufficient to lift the minimal arc-transitive subgroups of $\mathrm{Aut}(X)$. Moreover, by Lemma 2.1 it is enough to consider these subgroups up to conjugacy in $\mathrm{Aut}(X)$ when covering projections are classified up to isomorphism.

By a well-known result of Gross and Tucker [11], any regular covering $q\colon \tilde{X} \to X$ is equivalent to a *covering projection*

$$q_\zeta\colon X \times_\zeta H \to X,\ q_\zeta(u,h) = u,$$

where H is a finite group, called *the voltage group*, with its elements named *voltages*, the function $\zeta\colon A(X) \to H$ satisfying $\zeta(u,v) = (\zeta(v,u))^{-1}$ is the *voltage assignment*, and *the derived covering graph* $X \times_\zeta H$ with vertex set $V(X) \times H$ and edge set $E(X) \times H$ is constructed by joining vertices (u,g) and $(v, g\zeta(u,v))$ by the edge $(\{u,v\}, g)$. It can also be assumed that the voltage assignment ζ is *T-reduced* for some arbitrarily chosen spanning tree T of X, meaning that $\zeta(u,v) = \mathrm{id}_H$ for all arcs $(u,v) \in T$.

Therefore, in order to find all non-equivalent G-admissible regular covers of X for a subgroup $G \leq \mathrm{Aut}(X)$ it suffices, by Lemma 2.1, to find all T-reduced voltage assignments ζ on X such that G lifts along $q_\zeta\colon X \times_\zeta H \to X$ and then reduce these projections up to equivalence.

In order to study elementary abelian covers efficiently we introduce a natural matrix representation of $\mathrm{Aut}(X)$. A choice of a spanning tree T of X together with an ordered set $\{x_1, \ldots, x_r\} \subset A(X)$ containing exactly one arc from each edge of $E(X) \setminus E(T)$ uniquely determines the directed cycles $c_1, \ldots, c_r$; these cycles form an ordered basis of the first (mod p)-homology group $H_1(X, \mathbb{Z}_p) \cong \mathbb{Z}_p^{r\times 1}$, considered as a vector space over $\mathbb{Z}_p$ (note that $r = |E(X)| - |V(X)| + 1$ is the Betti number of X). The induced action of $\alpha \in \mathrm{Aut}(X)$ on base cycles yields an invertible linear transformation on $H_1(X, \mathbb{Z}_p) \cong \mathbb{Z}_p^r$, also denoted by α, and is naturally represented by a matrix $[\alpha] \in \mathbb{Z}_p^{r\times r}$ relative to the chosen basis. Hence, a subgroup $G \leq \mathrm{Aut}(X)$ is linearly represented as $[G] = \{[\alpha] \mid \alpha \in G\} \leq \mathrm{GL}(r, \mathbb{Z}_p)$.

Let X be a connected graph and $\{c_1, \ldots, c_r\}$ an ordered basis for $H_1(X, \mathbb{Z}_p)$. Further, let $\zeta\colon A(X) \to \mathbb{Z}_p^{d\times 1}$ be a voltage assignment on X, and let M_ζ denote the $d \times r$-matrix with

$$\zeta(c_i) = \sum_{x \in c_i} \zeta(x)$$

as columns. In this context, the following lifting criterion was derived by Malnič et al., see Proposition 6.3 and Corollary 6.5 in [15], and Theorem 2.3 in [16].

Theorem 2.2. *With notation above, the following hold:*

(a) The covering graph $X \times_\zeta \mathbb{Z}_p^{d\times 1}$ *is connected if and only if* $\operatorname{rank} M_\zeta = d$.

(b) A subgroup $G \leq \operatorname{Aut}(X)$ *lifts along* $q_\zeta \colon X \times_\zeta \mathbb{Z}_p^{d\times 1} \to X$ *if and only if the rows of* M_ζ *form a basis of a* $[G]$*-invariant* d*-dimensional subspace* $\mathcal{V}_\zeta$ *of* $\mathbb{Z}_p^{1\times r}$.

Remark 2.3. In Theorem 2.2 we have "transposed" the original statement, since vectors are represented as rows and multiplied by matrices on the right throughout this paper.

Finally, adapting Corollary 3.3 of [15] to the context of elementary abelian covers, we obtain the following theorem.

Theorem 2.4. *Let* $M_\zeta, M_{\zeta'} \leq \mathbb{Z}_p^{d\times r}$ *be matrices of rank* d *and let the subspaces* $\mathcal{V}_\zeta, \mathcal{V}_{\zeta'} \leq \mathbb{Z}_p^{1\times r}$ *be spanned by their rows. Then for the covering projections* $q_\zeta, q_{\zeta'}$ *arising from the corresponding voltage assignments* $\zeta, \zeta' \colon A(X) \to \mathbb{Z}_p^{d\times 1}$ *the following hold.*

(a) q_ζ *and* $q_{\zeta'}$ *are equivalent if and only if* $\mathcal{V}_\zeta = \mathcal{V}_{\zeta'}$. *In particular, a choice of basis determines the covers of Theorem 2.2 up to equivalence.*

(b) q_ζ *and* $q_{\zeta'}$ *are isomorphic if and only if* $\mathcal{V}_\zeta = \mathcal{V}_{\zeta'}[\alpha]$ *for some* $\alpha \in \operatorname{Aut}(X)$. *Moreover, if* q_ζ *and* $q_{\zeta'}$ *are both* G*-admissible for a subgroup* $G \leq \operatorname{Aut}(X)$*, then it is enough to check all* α *from some transversal of* G *in* $\operatorname{Aut}(X)$.

Proof. Clearly, (a) is a special case of (b), so proving the later is enough. By Corollary 3.3 in [15], q_ζ and $q_{\zeta'}$ are isomorphic if and only if there exists a group automorphism $\tau \colon \mathbb{Z}_p^{d\times 1}$ and $\alpha \in \operatorname{Aut}(X)$ such that $\tau\zeta = \zeta'\alpha$ for the induced group homomorphisms $\zeta \colon \pi_1(X,u) \to \mathbb{Z}_p^{d\times 1}$ and $\zeta' \colon \pi_1(X, \alpha(u)) \to \mathbb{Z}_p^{d\times 1}$, that is, whenever the diagram on the left commutes:

$$\begin{array}{ccc} \pi(X,u) & \xrightarrow{\alpha} & \pi(X,\alpha(u)) \\ \downarrow \zeta & & \downarrow \zeta' \\ \mathbb{Z}_p^{1\times d} & \xrightarrow{\tau} & \mathbb{Z}_p^{1\times d} \end{array} \qquad \begin{array}{ccc} H_1(X,\mathbb{Z}_p) & \xrightarrow{\alpha} & H_1(X,\mathbb{Z}_p) \\ \downarrow \zeta & & \downarrow \zeta' \\ \mathbb{Z}_p^{1\times d} & \xrightarrow{\tau} & \mathbb{Z}_p^{1\times d} \end{array}$$

Since the voltage group $\mathbb{Z}_p^{1\times d}$ is abelian, the above group homomorphisms $\zeta, \zeta', \alpha, \tau$, induce linear mappings $\zeta, \zeta', \alpha, \tau$ on $H_1(X, \mathbb{Z}_p)$ via the abelianization of fundamental groups, such that diagram on the right commutes whenever the one on the left does. Observe that group homomorphisms ζ, ζ' depend on the choice of base vertex, while the induced linear mappings ζ, ζ' do not. In both cases, ζ, ζ' are surjective since the covering graph is connected.

With regard to our matrix representation (and multiplying vectors by matrices on the right) this is true if and only if the following diagram commutes:

$$\begin{array}{ccc} \mathbb{Z}_p^{1\times r} & \xrightarrow{[\alpha]^t} & \mathbb{Z}_p^{1\times r} \\ \downarrow M_\zeta^t & & \downarrow M_{\zeta'}^t \\ \mathbb{Z}_p^{1\times d} & \xrightarrow{\tau} & \mathbb{Z}_p^{1\times d} \end{array}$$

By an application of the first homomorphism theorem for vector spaces, the later diagram commutes if and only if $\operatorname{Ker} M_\zeta^t$ is mapped to $\operatorname{Ker} M_{\zeta'}^t$ by $[\alpha]^t$. By transposing the last diagram, this holds if and only if $\operatorname{Im} M_\zeta$ is mapped to $\operatorname{Im} M_{\zeta'}$ by $[\alpha]$, that is, $\mathcal{V}_\zeta = \mathcal{V}_{\zeta'}[\alpha]$ for some $\alpha \in \operatorname{Aut}(X)$.

For the final observation, suppose that q_ζ, q'_ζ are both G-admissible, and that $V_\zeta = \mathcal{V}_{\zeta'}[\alpha]$ for some $\alpha \in \operatorname{Aut}(X)$. Then the same is obviously true for any automorphism in the coset αG, so only a single representative of each coset has to be checked. □

3 The classification theorem

Let $X = K_{4,4}$ be a complete bipartite graph on the vertex set $V(X) = \{1, \ldots, 8\}$, with $\{1,2,3,4\}$ and $\{5,6,7,8\}$ as the two sets of bipartition. Fix a spanning tree, for instance,

$$T := \{\{1,5\},\{1,6\},\{1,7\},\{1,8\},\{5,2\},\{5,3\},\{5,4\}\} \subseteq E(X)$$

(see Figure 1). The ordered basis $\{c_1, \ldots, c_9\}$ of the vector space $H_1(X, \mathbb{Z}_p) \cong \mathbb{Z}_p^9$ is uniquely determined by choosing one arc (x, y) from each edge $\{x, y\} \in E(X) \setminus T$, say,

$$\begin{array}{lll} (8,3) \approx c_1 = (8351), & (7,4) \approx c_4 = (7451), & (7,2) \approx c_7 = (7251), \\ (8,4) \approx c_2 = (8451), & (7,3) \approx c_5 = (7351), & (6,3) \approx c_8 = (6351), \\ (6,4) \approx c_3 = (6451), & (6,2) \approx c_6 = (6251), & (8,2) \approx c_9 = (8251). \end{array}$$

8

7

Figure 1: $K_{4,4}$ with chosen

6

Since arc-transitive coverings of X
lifting minimal arc-transitive subgrou
ators. (As already remarked, we cons
jugacy). The lattice of conjugacy cla
was determined using computational so
represented by the following subgroups

1 2 3 5 4

$H_1 = \langle s, r, a\rangle, H_2 = \langle s, r, b\rangle, H_3 = \langle s,$

with the permutations $a, b, c, d, e, r, s,$
the subgroups H_1, H_2, H_3, and H_4 are

No.	Automorphism $\alpha \in \mathrm{Aut}(X)$	Matrix $[\alpha] \in \mathbb{Z}_p^{9\times 9}$
1.	$a = (14)(23)$ $m_A = (x-1)(x+1)$ $p_A = (x-1)^3(x+1)^6$	$A = \begin{bmatrix} 0&0&0&0&0&0&0&0&1\\ -1&-1&0&0&0&0&0&0&-1\\ 0&0&-1&0&0&-1&0&-1&0\\ 0&0&0&-1&-1&0&-1&0&0\\ 0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1&0\\ 0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&1&0&0&0\\ 1&0&0&0&0&0&0&0&0 \end{bmatrix}$
2.	$b = (1324)(78)$ $m_B = (x-1)(x+1)(x^2+1)$ $p_B = (x-1)(x+1)^2(x^2+1)^3$	$B = \begin{bmatrix} 0&0&0&-1&-1&0&-1&0&0\\ 0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&1\\ -1&-1&0&0&0&0&0&0&-1\\ 0&0&0&0&0&0&0&1&0\\ 1&0&0&0&0&0&0&0&0\\ 0&0&-1&0&0&-1&0&-1&0\\ 0&0&0&0&1&0&0&0&0 \end{bmatrix}$
3.	$c = (13)(24)(78)$ $m_C = (x-1)(x+1)$ $p_C = (x-1)^4(x+1)^5$	$C = \begin{bmatrix} 0&0&0&-1&-1&0&-1&0&0\\ 0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&1\\ -1&-1&0&0&0&0&0&0&-1\\ 0&0&1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0\\ 0&0&-1&0&0&-1&0&-1&0\\ 0&0&0&1&0&0&0&0&0 \end{bmatrix}$
4.	$d = (5867)$ $m_D = (x+1)(x^2+1)$ $p_D = (x+1)^3(x^2+1)^3$	$D = \begin{bmatrix} -1&0&0&0&-1&0&0&-1&0\\ 0&-1&-1&-1&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&1&0&0&0\\ 1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&-1&-1&0&-1 \end{bmatrix}$
5.	$e = (57)(68)$ $m_E = (x-1)(x+1)$ $p_E = (x-1)^3(x+1)^6$	$E = \begin{bmatrix} 0&0&0&0&0&0&0&1&0\\ 0&0&1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0\\ 0&-1&-1&-1&0&0&0&0&0\\ -1&0&0&0&-1&0&0&-1&0\\ 0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&-1&-1&0&-1\\ 1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0 \end{bmatrix}$
6.	$r = (56)(78)$ $m_R = (x-1)(x+1)$ $p_R = (x-1)^3(x+1)^6$	$R = \begin{bmatrix} 0&0&0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0&0&0\\ 0&-1&-1&-1&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&-1&-1&0&-1\\ 0&0&0&0&0&0&0&0&1\\ -1&0&0&0&-1&0&0&-1&0\\ 0&0&0&0&0&0&1&0&0 \end{bmatrix}$
7.	$s = (15)(26)(38)(47)$ $m_S = (x-1)(x+1)$ $p_S = (x-1)^3(x+1)^6$	$S = \begin{bmatrix} -1&0&0&0&0&0&0&0&0\\ 0&0&0&0&-1&0&0&0&0\\ 0&0&0&0&0&0&-1&0&0\\ 0&0&0&-1&0&0&0&0&0\\ 0&-1&0&0&0&0&0&0&0\\ 0&0&0&0&0&-1&0&0&0\\ 0&0&-1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&-1\\ 0&0&0&0&0&0&0&-1&0 \end{bmatrix}$
8.	$t = (15)(26)(3748)$ $m_T = (x-1)(x+1)(x^2+1)$ $p_T = (x-1)^2(x+1)^3(x^2+1)^2$	$T = \begin{bmatrix} 0&-1&0&0&0&0&0&0&0\\ 0&0&0&-1&0&0&0&0&0\\ 0&0&0&0&0&0&-1&0&0\\ 0&0&0&0&-1&0&0&0&0\\ -1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&-1&0&0&0\\ 0&0&0&0&0&0&0&-1&0\\ 0&0&0&0&0&0&0&0&-1\\ 0&0&-1&0&0&0&0&0&0 \end{bmatrix}$

Table 1: Generators of minimal arc-transitive subgroups of $\mathrm{Aut}(X)$, and their matrix representation together with characteristic and minimal polynomials.

By Theorems 2.2 and 2.4, all arc-transitive elementary abelian covers of X are described up to equivalence in terms of invariant subspaces for a concrete matrix representation of one of the above subgroups. Recall that for $\alpha \in \mathrm{Aut}(X)$ the coefficients α_{ij} of the matrix $[\alpha] \in \mathbb{Z}_p^{9\times 9}$ are defined by

$$\alpha(c_j) = \sum_{i=1}^{n} \alpha_{ij} c_j.$$

For instance, for $a = (14)(23) \in \mathrm{Aut}(X)$ we have

$$a(c_1) = (8254) \approx -c_2 + c_9,$$

so the first column of the matrix $A = [a]$ is equal to $[0, -1, 0, 0, 0, 0, 0, 0, 1]^t$. In the same fashion, the respective matrices for all generators of $H_1, \ldots, H_6$ were computed and gathered in Table 1, along with their characteristic and minimal polynomials.

Our goal now is to find all H_i-invariant subspaces $\mathbb{Z}_p^{9\times 9}$ for all primes p. By Maschke's Theorem [14], if the order of a finite group $G \leq \mathrm{GL}(V, \mathbb{F})$ is not divisible by the field characteristic, then any G-invariant subspace has a G-invariant direct complement, and so all G-invariant subspaces of V are direct sums of some minimal ones – it is therefore enough to find these. Since $|H_i| = 32$ or 64, Maschke's Theorem applies for all $p \neq 2$. For $p = 2$ this might not be true, but in this case the problem is easily solved using computational software such as [1], and is omitted.

Theorem 3.1. *Let $X = K_{4,4}$ with the spanning tree T as above. For $p \neq 2$ a prime, the non-equivalent minimal regular p-elementary abelian covers of X, which are admissible for some H_i, $i = 1, \ldots, 6$, arise from T-reduced voltage assignments $\zeta_n \colon A(X) \to \mathbb{Z}_p^{k\times 1}$, $k \leq r$, given by matrices M_{ζ_n} in Table 2.*

Theorem 3.2. *The isomorphism classes of the covering projections of Theorem 3.1 are the following:*

$$\{\zeta_1, \zeta_2, \zeta_3\}, \{\zeta_4, \zeta_5\}, \{\zeta_6, \zeta_7\}, \{\zeta_8, \zeta_9, \zeta_{10}, \zeta_{11}\},$$
$$\{\zeta_{12}, \zeta_{13}\}, \{\zeta_{14}, \zeta_{15}\}, \{\zeta_{16}\}, \{\zeta_{17}\}.$$

Proof. To see this, observe that for all p we have $\mathcal{V}_{\zeta_1} = \mathcal{V}_{\zeta_2}[(34)(78)] = \mathcal{V}_{\zeta_3}[(24)(67)]$, where

$$[(34)(78)] = \begin{bmatrix} 0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0 \\ 1&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&1 \\ 0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0 \end{bmatrix}, [(24)(67)] = \begin{bmatrix} 1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&1&0 \\ 0&0&0&1&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0 \end{bmatrix}$$

are the matrices corresponding to $(34)(78), (24)(67) \in \mathrm{Aut}(X)$. Hence the covering projections $q_{\zeta_1}, q_{\zeta_2}, q_{\zeta_3}$ are isomorphic by Theorem 2.4. In a similar fashion we have

$$\mathcal{V}_{\zeta_4} = \mathcal{V}_{\zeta_5}[(34)(78)], \quad \mathcal{V}_{\zeta_6} = \mathcal{V}_{\zeta_7}[(34)], \quad \mathcal{V}_{\zeta_{12}} = \mathcal{V}_{\zeta_{13}}[(34)], \quad \mathcal{V}_{\zeta_{14}} = \mathcal{V}_{\zeta_{15}}[(34)(78)],$$
$$\mathcal{V}_{\zeta_8} = \mathcal{V}_{\zeta_9}[(34)] = \mathcal{V}_{\zeta_{10}}[(23)(68)] = \mathcal{V}_{\zeta_{11}}[(24)(67)],$$

n	Inv. subspace $\mathcal{V}_{\zeta_n}$	Voltage matrix M_{ζ_n}	Admissible for
1	$\langle u_2 \rangle$	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$	H_1
2	$\langle u_4 \rangle$	$\begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$	H_1
3	$\langle u_5 \rangle$	$\begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$	$H_1, \ldots, H_6$
4	$\langle u_2 - u_3 - 2u_4 \rangle$	$\begin{bmatrix} 1 & -1 & -2 & -1 & -1 & -2 & -2 & 0 & 0 \end{bmatrix}$	H_3
5	$\langle u_2 + u_3 \rangle$	$\begin{bmatrix} 1 & 1 & 0 & -1 & 1 & 2 & 0 & 2 & 2 \end{bmatrix}$	H_3
6	$\langle u_2 + iu_3 + (i-1)u_4 \rangle$	$\begin{bmatrix} 1 & i & i-1 & -1 & i & 2i & i-1 & 1+i & 1+i \end{bmatrix}$	H_4, $p = 1 \pmod 4$
7	$\langle u_2 - iu_3 - (i+1)u_4 \rangle$	$\begin{bmatrix} 1 & -i & -i-1 & -1 & -i & -2i & -i-1 & 1-i & 1-i \end{bmatrix}$	H_4, $p = 1 \pmod 4$
8	$\langle u_1, u_3 + u_4 \rangle$	$\begin{bmatrix} 0 & 1 & 1 & 0 & -1 & 0 & -1 & -1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 2 & 1 & 1 & 1 \end{bmatrix}$	H_1, H_2
9	$\langle u_2, u_4 \rangle$	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$	H_1, H_2
10	$\langle u_6, u_7 \rangle$	$\begin{bmatrix} 2 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & -1 & 0 & -1 & 1 & -1 \end{bmatrix}$	H_1
11	$\langle u_8, u_9 \rangle$	$\begin{bmatrix} 0 & 1 & 1 & 2 & 1 & 0 & 1 & 1 & 1 \\ 0 & -1 & 1 & 0 & 1 & 0 & -1 & 1 & -1 \end{bmatrix}$	H_1
12	$\langle u_1, u_2 + u_4 \rangle$	$\begin{bmatrix} 0 & 1 & 1 & 0 & -1 & 0 & -1 & -1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 2 & 1 & 1 & 1 \end{bmatrix}$	H_3, H_4
13	$\langle u_2 - u_3 - 2u_4, u_2 + u_3 \rangle$	$\begin{bmatrix} 1 & -1 & -2 & -1 & -1 & -2 & -2 & 0 & 0 \\ 1 & 1 & 0 & -1 & 1 & 2 & 0 & 2 & 2 \end{bmatrix}$	H_3, H_4
14	$\langle u_6 + iu_8, u_7 + iu_9 \rangle$	$\begin{bmatrix} 2 & 1+i & 1+i & 2i & 1+i & 0 & 1+i & 1+i & 1+i \\ 0 & 1-i & 1+i & 0 & -1+i & 0 & -1-i & 1+i & -1-i \end{bmatrix}$	H_4, $p = 1 \pmod 4$
15	$\langle u_6 - iu_8, u_7 - iu_9 \rangle$	$\begin{bmatrix} 2 & 1-i & 1-i & -2i & 1-i & 0 & 1-i & 1-i & 1-i \\ 0 & 1+i & 1-i & 0 & -1-i & 0 & -1+i & 1-i & -1+i \end{bmatrix}$	H_4, $p = 1 \pmod 4$
16	$\langle u_1, u_2, u_3, u_4 \rangle$	$\begin{bmatrix} 0 & 1 & 1 & 0 & -1 & 0 & -1 & -1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$	$H_1, \ldots, H_6$
17	$\langle u_6, u_7, u_8, u_9 \rangle$	$\begin{bmatrix} 2 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & -1 & 0 & -1 & 1 & -1 \\ 0 & 1 & 1 & 2 & 1 & 0 & 1 & 1 & 1 \\ 0 & -1 & 1 & 0 & 1 & 0 & -1 & 1 & -1 \end{bmatrix}$	$H_1, \ldots, H_6$

Table 2: Non-equivalent minimal H_i-admissible covering projections

demonstrating the other isomorphism relations. Note that for all $\alpha \in \mathrm{Aut}(X)$ the matrix coefficients $[\alpha]_{ij}$ take values in the set $\{\pm 1, 0\}$, and the equalities are independent of the value of $p \neq 2$.

In order to determine the isomorphism classes completely we need to check whether $\zeta_3, \zeta_4, \zeta_6$ of rank 1 are pairwise non-isomorphic, and similarly for $\zeta_8, \zeta_{12}, \zeta_{14}$ of rank 2, as well as for ζ_{16} and ζ_{17} of rank 4 .

For instance, since ζ_{16} and ζ_{17} are both admissible for $H' = \langle H_1, \ldots, H_6 \rangle$, we have to check whether $\mathcal{V}_{16}[\alpha] = \mathcal{V}_{17}$ for some α in a transversal of H' in G, say

$$\{\mathrm{id}, (23), (67), (243), (23)(67), (687), (243)(67), (23)(687), (243)(687)\}.$$

Writing down the corresponding matrices one can immediately see that all of them are permutation matrices. Their action on $M_{\zeta_{17}}$ permutes the columns, so the image of $M_{\zeta_{17}}$ has a zero column. As $M_{\zeta_{16}}$ has no such column, the respective subspaces must be different for all p.

Similarly, in order to see that ζ_4 and ζ_6 are nonisomorphic for all p, it is enough to see that for all α in the transversal of $H_3 \cap H_4$ in G, the row vector

$\begin{bmatrix} 1 & -1 & -2 & -1 & -1 & -2 & -2 & 0 & 0 \end{bmatrix} [\alpha]$ has a zero component, while $\begin{bmatrix} 1 & i & i-1 & -1 & i & 2i & i-1 & 1+i & 1+i \end{bmatrix} [\alpha]$ doesn't.

For the remaining cases, it is helpful to use computational tools to create a list of matrices $[\alpha]$ and $M_{\zeta_n}[\alpha]$ for α in the appropriately chosen transversal of some subgroup in G. However, the final comparison of the respective vector spaces has to be done "manually" as described above. Details are left to the reader. □

Example 3.3. It is easy to see that the covering graph $X \times_{\zeta_3} \mathbb{Z}_p$ is isomorphic to the lexicographic product $C_{4p}[2K_1]$ for all primes $p \neq 2$. In fact, ζ_3 assigns voltage 1 to arcs $(7,3), (7,4), (8,3), (8,4)$ and 0 otherwise, see Figure 2.

8

7

6

5

1

2

3

4

Figure 2: $X \times_{\zeta_3} \mathbb{Z}_p \cong C_{4p}[2K_1]$.

Using Magma, the covering graphs for small primes p are easily constructed and matched with the graphs of the (so far incomplete) Census of edge-transitive tetravalent graphs by Potočnik and Wilson [20]. For instance, the graphs

$$X \times_{\zeta_4} \mathbb{Z}_3 \cong R_{12}(2,7) \text{ and } X \times_{\zeta_6} \mathbb{Z}_5 \cong R_{20}(10,11)$$

belong to the class of the Rose Window graphs.

Proof of Theorem 3.1

For convenience we identify the subgroups $H_i \leq \mathrm{Aut}(X)$ with their corresponding matrix subgroups in $\mathbb{Z}_p^{9\times 9}$. The four subgroups

$$H_1 = \langle R, S, A \rangle, \quad H_2 = \langle R, S, B \rangle, \quad H_3 = \langle R, S, C \rangle, \quad H_4 = \langle R, S, D \rangle$$

all contain matrices R and S. Let $H = \langle R, S \rangle$. In order to derive admissible covers for each of the groups H_1, H_2, H_3, H_4, it is convenient to compute certain H-invariant subspaces first.

Elementary calculation shows that the respective minimal polynomials are $m_R =$

$m_S = (x-1)(x+1)$, so both matrices are diagonalizable. Moreover, we have

$$\begin{aligned} \mathrm{Ker}(S-I) \cap \mathrm{Ker}(R+I) &= \langle u_1 \rangle, \\ \mathrm{Ker}(S+I) \cap \mathrm{Ker}(R+I) &= \langle u_2, u_3, u_4 \rangle, \\ \mathrm{Ker}(S+I) \cap \mathrm{Ker}(R-I) &= \langle u_5 \rangle, \\ \mathrm{Ker}(S-I) \cap \mathrm{Ker}(R-I) &= \{0\}, \end{aligned} \tag{1}$$

where

$$\begin{aligned} u_1 &= (0,1,1,0,-1,0,-1,-1,1), \\ u_2 &= (1,0,0,0,0,1,0,1,1), \\ u_3 &= (0,1,0,-1,1,1,0,1,1), \\ u_4 &= (0,0,1,1,0,1,1,0,0), \\ u_5 &= (1,1,0,1,1,0,0,0,0) \end{aligned}$$

(again note that we multiply row vectors with matrices on the right). Thus, the subspace $\langle u_1, \dots, u_5 \rangle$ is H-invariant and has a (possibly non-unique) H-invariant complement by Maschke's Theorem.

All H-invariant complements can now be computed as follows. First, if

$$\{u_1, \dots, u_5, u_6', u_7', u_8', u_9'\}$$

is a basis of the whole space, then all complements of $\langle u_1, \dots, u_5 \rangle$ are uniquely described by

$$\mathcal{U} = \left\langle u_6' + \sum_1^5 a_{6,i} u_i,\ u_7' + \sum_1^5 a_{7,i} u_i,\ u_8' + \sum_1^5 a_{8,i} u_i,\ u_9' + \sum_1^5 a_{9,i} u_i \right\rangle$$

for different choices of $a_{j,i}$. In order for $\mathcal{U}$ to be H-invariant, the condition

$$\mathcal{U} = \mathcal{U}R = \mathcal{U}S$$

translates to a linear system for the coefficients $a_{j,i}$. To keep the computation as simple as possible we choose vectors $u_6', \dots, u_9'$, such that

$$\mathrm{Ker}(S-I) = \langle u_1, u_8', u_9' \rangle \text{ and } \mathrm{Ker}(S+I) = \langle u_2, \dots, u_5, u_6', u_7' \rangle,$$

for instance,

$$\begin{aligned} u_6' &= (1,0,0,0,0,0,0,0,0), \\ u_7' &= (0,0,0,0,0,1,0,0,0), \\ u_8' &= (0,1,0,0,0,-1,0,0,0), \\ u_9' &= (0,0,0,0,0,0,0,1,-1). \end{aligned}$$

Then the equation $\mathcal{U} = \mathcal{U}S$ implies

$$a_{6,1}, a_{7,1}, a_{8,2}, \dots, a_{8,5}, a_{9,2}, \dots, a_{9,5} = 0.$$

The condition $\mathcal{U} = \mathcal{U}R$ is now readily solved. We find that $\mathcal{U}$ is unique and given by $\mathcal{U} = \langle u_6, u_7, u_8, u_9 \rangle$, where

$$
\begin{aligned}
u_6 &= (2, 1, 1, 0, 1, 0, 1, 1, 1),\\
u_7 &= (0, 1, 1, 0, -1, 0, -1, 1, -1),\\
u_8 &= (0, 1, 1, 2, 1, 0, 1, 1, 1),\\
u_9 &= (0, -1, 1, 0, 1, 0, -1, 1, -1).
\end{aligned}
$$

Denote by P the matrix with vectors u_i as rows. Then P is invertible for all primes $p \neq 2$, so an alternative representation of automorphisms from $\mathrm{Aut}(X)$ is provided by the relation $X' = PXP^{-1}$:

$$
A' = \begin{bmatrix}
0 & 0 & -1 & -1 & & & & & \\
0 & 1 & 0 & 0 & & & & & \\
-1 & 0 & 0 & 1 & & & & & \\
0 & 0 & 0 & -1 & & & & & \\
 & & & & -1 & & & & \\
 & & & & & 0 & -1 & 0 & 0\\
 & & & & & -1 & 0 & 0 & 0\\
 & & & & & 0 & 0 & -1 & 0\\
 & & & & & 0 & 0 & 0 & -1
\end{bmatrix}, \qquad
B' = \begin{bmatrix}
0 & 0 & 1 & 1 & & & & & \\
0 & 0 & 0 & -1 & & & & & \\
-1 & -1 & 0 & 0 & & & & & \\
0 & 1 & 0 & 0 & & & & & \\
 & & & & -1 & & & & \\
 & & & & & 0 & 0 & -1 & 0\\
 & & & & & 0 & 0 & 0 & -1\\
 & & & & & 0 & -1 & 0 & 0\\
 & & & & & -1 & 0 & 0 & 0
\end{bmatrix},
$$

$$
C' = \frac{1}{2}\begin{bmatrix}
0 & 2 & 0 & 2 & & & & & \\
1 & 0 & -1 & -1 & & & & & \\
-1 & -2 & -1 & 1 & & & & & \\
1 & 0 & 1 & 1 & & & & & \\
 & & & & -2 & & & & \\
 & & & & & -1 & 1 & -1 & -1\\
 & & & & & 1 & -1 & -1 & -1\\
 & & & & & -1 & -1 & 1 & -1\\
 & & & & & -1 & -1 & -1 & 1
\end{bmatrix}, \quad
D' = \frac{1}{2}\begin{bmatrix}
0 & -2 & 0 & -2 & & & & & \\
1 & 0 & -1 & -1 & & & & & \\
-1 & 2 & -1 & -3 & & & & & \\
1 & 0 & 1 & 1 & & & & & \\
 & & & & -2 & & & & \\
 & & & & & -1 & 1 & -1 & -1\\
 & & & & & 1 & -1 & -1 & -1\\
 & & & & & 1 & 1 & -1 & 1\\
 & & & & & 1 & 1 & 1 & -1
\end{bmatrix}
$$

$$
R' = \begin{bmatrix}
-1 & 0 & 0 & 0 & & & & & \\
0 & -1 & 0 & 0 & & & & & \\
0 & 0 & -1 & 0 & & & & & \\
0 & 0 & 0 & -1 & & & & & \\
 & & & & 1 & & & & \\
 & & & & & 0 & -1 & 0 & 0\\
 & & & & & -1 & 0 & 0 & 0\\
 & & & & & 0 & 0 & 0 & -1\\
 & & & & & 0 & 0 & -1 & 0
\end{bmatrix}, \qquad
S' = \begin{bmatrix}
1 & 0 & 0 & 0 & & & & & \\
0 & -1 & 0 & 0 & & & & & \\
0 & 0 & -1 & 0 & & & & & \\
0 & 0 & 0 & -1 & & & & & \\
 & & & & -1 & & & & \\
 & & & & & -1 & 0 & 0 & 0\\
 & & & & & 0 & 1 & 0 & 0\\
 & & & & & 0 & 0 & -1 & 0\\
 & & & & & 0 & 0 & 0 & 1
\end{bmatrix}
$$

Let

$$
\mathcal{U}_1 = \langle u_1, \ldots, u_4 \rangle, \quad \mathcal{U}_2 = \langle u_5 \rangle, \quad \text{and} \quad \mathcal{U}_3 = \langle u_6, \ldots, u_9 \rangle.
$$

Then $\mathcal{V} = \mathcal{U}_1 \oplus \mathcal{U}_2 \oplus \mathcal{U}_3$, and the block diagonal form of A', B', C', D', R', S' implies that $\mathcal{U}_1$,$\mathcal{U}_2$ and $\mathcal{U}_3$ are H_i-invariant for $i = 1, \ldots, 4$. The following lemma sharpens this observation. (In what follows, notation $(a_1, \ldots, a_9)'$ for the vector $v = \sum_i a_i u_i \in \mathcal{V}$ will be used repeatedly, implying that $vA = (a_1, \ldots, a_9)'A'$, etc.)

Lemma 3.4. *If some nontrivial H-invariant subspace $\mathcal{W} \leq \mathcal{V}$ is inclusion minimal, then $\mathcal{W} \leq \mathcal{U}_i$ for some i. More precisely, if $\mathcal{W}$ contains some vector*

$$
w = \sum_{i=1}^{9} a_i u_i = (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9)',
$$

then $\mathcal{W}$ contains vectors

$$a_1u_1 = (a_1, 0, 0, 0, 0, 0, 0, 0, 0)' \quad (2)$$
$$a_2u_2 + a_3u_3 + a_4u_4 = (0, a_2, a_3, a_4, 0, 0, 0, 0, 0)' \quad (3)$$
$$a_5u_5 = (0, 0, 0, 0, a_5, 0, 0, 0, 0)' \quad (4)$$
$$a_6u_6 + a_8u_8 = (0, 0, 0, 0, 0, a_6, 0, a_8, 0)' \quad (5)$$
$$a_7u_9 + a_9u_9 = (0, 0, 0, 0, 0, 0, a_7, 0, a_9)'. \quad (6)$$

Proof. Suppose $\mathcal{W} \leq \mathcal{V}$ is H-invariant and $w = (a_1, \ldots, a_9)' \in \mathcal{W}$. Then the vectors

$$w(I + S)/2 = (a_1, 0, 0, 0, 0, 0, a_7, 0, a_9)' \text{ and}$$

$$w(I + S)RSR/2 = (a_1, 0, 0, 0, 0, 0, 0, -a_7, 0, -a_9)'$$

are both in $\mathcal{W}$, and hence also $(a_1, 0, 0, 0, 0, 0, 0, 0, 0)', (0, 0, 0, 0, 0, 0, a_7, 0, a_9)' \in \mathcal{W}$. In a similar fashion we get that $(0, a_2, a_3, a_4, 0, 0, 0, 0, 0)'$, $(0, 0, 0, 0, a_5, 0, 0, 0, 0)'$, $(0, 0, 0, 0, 0, a_6, 0, a_8, 0)' \in \mathcal{W}$. Thus, $w = w_1 + w_2 + w_3$, where

$$w_1 = (a_1, a_2, a_3, a_4, 0, 0, 0, 0, 0)',$$
$$w_2 = (0, 0, 0, 0, a_5, 0, 0, 0, 0)',$$
$$w_3 = (0, 0, 0, 0, 0, a_6, a_7, a_8, a_9)'.$$

Hence $\mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$, where $\mathcal{W}_i = \mathcal{W} \cap \mathcal{U}_i$ are H-invariant. If $\mathcal{W}$ is minimal, then $\mathcal{W} = \mathcal{W}_i$ for some i. □

Now we look at each of the cases $H_1, \ldots, H_4$ separately. The shorthand notation

$$(a_1, a_2, a_3, a_4)_1 = (a_1, a_2, a_3, a_4, 0, \ldots, 0)' \in \mathcal{U}_1$$

and

$$(a_6, a_7, a_8, a_9)_3 = (0, \ldots, 0, a_6, a_7, a_8, a_9)' \in \mathcal{U}_3$$

is used througout the rest of the paper.

Subcase H_1

By Lemma 3.4 it is enough to find the minimal H_1-invariant subspaces of $\mathcal{U}_1$ and $\mathcal{U}_3$. By computation we find that the 1-dimensional H_1-invariant subspaces of $\mathcal{U}_1$ are $\langle u_2 \rangle$ and $\langle u_4 \rangle$. Any minimal 2-dimensional H_1-invariant subspace of $\mathcal{U}_1$ must complement $\langle u_2, u_4 \rangle$, and is thus of the form $\mathcal{W} = \langle (1, a_2, 0, a_4)_1, (0, b_2, 1, b_4)_1 \rangle$. By Lemma 3.4 (2), $u_1 = (1, 0, 0, 0)_1 \in \mathcal{W}$. Thus, $-u_1A = (0, 0, 1, 1)_1 = u_3 + u_4 \in \mathcal{W}$. As $\langle u_1, u_3 + u_4 \rangle$ is H_1-invariant, it is the unique minimal 2-dimensional H_1-invariant subspace of $\mathcal{U}_1$. Suppose $\mathcal{W}$ is 3-dimensional and H_1-invariant. If $\mathcal{W} = \langle u_2, u_3, u_4 \rangle$, then $\mathcal{W}$ is not minimal. If $\mathcal{W} \neq \langle u_2, u_3, u_4 \rangle$ then $\mathcal{W}$ contains some vector $(1, a_2, a_3, a_4)_1$. Again by Lemma 3.4 (2), $u_1 \in \mathcal{W}$, so $\langle u_1, u_3 + u_4 \rangle \leq \mathcal{W}$ and $\mathcal{W}$ is not minimal. We conclude that $\langle u_2 \rangle, \langle u_4 \rangle, \langle u_1, u_3 + u_4 \rangle$ are the minimal H_1-invariant subspaces of $\mathcal{U}_1$.

On the other hand, the subspaces $\langle u_6, u_7 \rangle, \langle u_8, u_9 \rangle \le \mathcal{U}_3$ are obviously H_1-invariant. If another $\mathcal{W} \le \mathcal{U}_3$ is H_1-invariant and $(a_6, a_7, a_8, a_9)_3 \in \mathcal{W}$, then also $(a_6, 0, a_8, 0)_3 \in \mathcal{W}$. Applying the action of A and Lemma 3.4 again, we see that $(a_6, 0, 0, 0)_3, (0, a_6, 0, 0)_3 \in \mathcal{W}$, Hence $a_6 = 0$ or $\langle u_6, u_7 \rangle \le \mathcal{W}$. Similarly, $(0, 0, a_8, 0)_3 \in \mathcal{W}$ implies that $a_8 = 0$ or $\langle u_8, u_9 \rangle \le \mathcal{W}$.

Thus,

$$\mathcal{V} = \langle u_2 \rangle \oplus \langle u_4 \rangle \oplus \langle u_1, u_3 + u_4 \rangle \oplus \langle u_5 \rangle \oplus \langle u_6, u_7 \rangle \oplus \langle u_8, u_9 \rangle$$

is the unique decomposition into minimal H_1-invariant subspaces.

Subcase H_2

Suppose $\mathcal{W} \le \mathcal{U}_1$ is some minimal H_2-invariant subspace. If $(a_1, a_2, a_3, a_4)_1 \in \mathcal{W}$ for vector with $a_1 \neq 0$, then $u_1 \in \mathcal{W}$ by Lemma 3.4. Since $u_1 B = u_3 + u_4$, we have $\mathcal{W} = \langle u_1, u_3 + u_4 \rangle$ by minimality. Otherwise, $a_1 = 0$ for all vectors in $\mathcal{W}$. Then $(0, a_2, a_3, a_4)_1 B' = (-a_3, -a_3 + a_4, 0, -a_2)_1 \in \mathcal{W}$ implies $a_3 = 0$. Since one of a_2, a_4 must be nonzero and $B\colon u_2 \mapsto -u_4 \mapsto -u_2$, we get $\mathcal{W} = \langle u_2, u_4 \rangle$.

For $\mathcal{W} \le \mathcal{U}_3$, observe that $B\colon u_6 \mapsto -u_9 \mapsto u_7 \mapsto -u_8 \mapsto u_6$. Thus, if $\mathcal{W}$ is H_2-invariant and contains one of the vectors u_i, $i = 6, \ldots, 9$, then $\mathcal{W} = \mathcal{U}_3$. Now suppose that $\mathcal{W}$ contains some nonzero vector $(a_6, a_7, a_8, a_9)_3$. Applying Lemma 3.4 we have $(a_6, 0, a_8, 0)_3$, $(0, a_7, 0, a_9)_3 \in \mathcal{W}$. Multiplying by B and applying Lemma 3.4 again we get $(0, a_8, 0, 0)_3, (0, 0, a_6, 0)_3, (0, 0, 0, a_7)_3, (a_9, 0, 0, 0)_3 \in \mathcal{W}$. Hence $u_i \in \mathcal{W}$ for some $i = 6, \ldots, 9$.

Thus,

$$\mathcal{V} = \langle u_2, u_4 \rangle \oplus \langle u_1, u_3 + u_4 \rangle \oplus \langle u_5 \rangle \oplus \langle u_6, u_7, u_8, u_9 \rangle$$

is the unique decomposition into minimal H_2-invariant subspaces.

Subcase H_3

Again, suppose that $(a_1, a_2, a_3, a_4)_1 \in \mathcal{W}$ for some minimal H_3-invariant subspace $\mathcal{W} \le \mathcal{U}_1$. If $a_1 \neq 0$, then $\mathcal{W} = \langle u_1, u_2 + u_4 \rangle$. If $a_1 = 0$, then $(0, a_2, a_3, a_4)_1 C' \in \mathcal{W}$ implies $a_3 = a_2 + a_4$ and $(0, a_2 + a_4, a_2, -a_4)_1 \in \mathcal{W}$. Moreover, $(0, a_4, -a_4, -2a_4)_1 \in \mathcal{W}$. If $a_4 \neq 0$, then $\mathcal{W} = \langle (0, 1, -1, -2)_1 \rangle$ by minimality. If $a_4 = 0$ and $a_2 \neq 0$, then $\mathcal{W} = \langle (0, 1, 1, 0)_1 \rangle$.

Next, suppose $\mathcal{W} \le \mathcal{U}_3$ is H_3-invariant and non-trivial. If $(a_6, a_7, a_8, a_9)_3 \in \mathcal{W}$, then also $(a_6, 0, a_8, 0)_3$, $(0, a_6, 0, a_8)_3$, $(0, a_7, 0, a_9)_3$, $(a_7, 0, a_9, 0)_3 \in \mathcal{W}$, so $\dim \mathcal{W} \ge 2$ and equals either 2 or 4. If $\dim \mathcal{W} = 2$, then without loss of generality we may assume that $\mathcal{W} = \langle (1, 0, a, 0)_3, (0, 1, 0, a)_3 \rangle$ for some a. But then $(1, 0, a, 0)_3 C = (-1 - a, 1 - a, a - 1, -1 - a)_3 \notin \mathcal{W}$. Hence $\dim \mathcal{W} = 4$, and $\mathcal{W} = \mathcal{U}_3$.

Thus,

$$\mathcal{V} = \langle u_1, u_2 + u_4 \rangle \oplus \langle u_2 - u_3 - 2u_4 \rangle \oplus \langle u_2 + u_3 \rangle \oplus \langle u_5 \rangle \oplus \langle u_6, u_7, u_8, u_9 \rangle$$

is the unique decomposition into minimal H_3-invariant subspaces.

Subcase H_4

Suppose $\mathcal{W} \leq \mathcal{U}_1$ is a minimal H_4-invariant subspace and $(a_1, a_2, a_3, a_4)_1 \in \mathcal{W}$ is nonzero. If $a_1 \neq 0$, then $\mathcal{W} = \langle u_1 = (1,0,0,0)_1, u_1 D = (0,-2,0,-2)_1 \rangle$, as it is H_4-invariant. Otherwise, suppose $a_1 = 0$ for every vector in $\mathcal{W}$. If $w = (0, a_2, a_3, a_4)_1 \in \mathcal{W}$, then also $2wD = (a_2 - a_3 + a_4, 2a_3, -a_2 - a_3 + a_4, -a_2 - 3a_3 + a_4) \in \mathcal{W}$. Thus, $a_4 = a_3 - a_2$ and $w_1 = (0, a_2, a_3, a_3 - a_2)_1, 2w_1 D = (0, a_3, -a_2, -a_2 - a_3) \in W$. It is easy to check that $\langle w_1, w_1 D \rangle$ is H_4-invariant. Its dimension is 1 if and only if $p = 1 \pmod 4$ and $a_4 = (\pm i - 1) a_2$, where $i^2 = -1$. In this case, $\mathcal{W} = \langle z_1 \rangle$ or $\langle z_2 \rangle$, where $z_{1,2} = u_2 \pm i u_3 + (\pm i - 1) u_4$. Otherwise, $\dim \mathcal{W} = 2$ and hence $\mathcal{W} = \langle (0,1,0,-1)_1, (0,0,1,1)_1 \rangle = \langle (0,1,-1,-2)_1, (0,1,1,0)_1 \rangle$.

Finally, if $\mathcal{W} \leq \mathcal{U}_3$ is H_4-invariant and non-trivial we may assume without loss of generality that $\langle (1,0,a,0)_3, (0,1,0,a)_3 \rangle \leq \mathcal{W}$ for some a. Multiplying by D yields $(a, a, -1, -1)_3 \in \mathcal{W}$. If $\dim \mathcal{W} = 2$, it is easy to see that $a^2 = -1$.

Thus,

$$\mathcal{U}_3 = \langle u_1, u_2 + u_4 \rangle \oplus \langle u_2 - u_3 - 2u_4, u_2 + u_3 \rangle \oplus \langle u_5 \rangle \oplus \langle u_6, u_7, u_8, u_9 \rangle$$

for $p = 3 \pmod 4$, or otherwise

$$\mathcal{U}_3 = \langle u_1, u_2 + u_4 \rangle \oplus \langle z_1 \rangle \oplus \langle z_2 \rangle \oplus \langle u_5 \rangle \oplus \langle u_6 + i u_8, u_7 + i u_9 \rangle \oplus \langle u_6 - i u_8, u_7 - i u_9 \rangle$$

are the unique decompositions of $\mathcal{U}_3$ into minimal H_4-invariant subspaces.

Subcases H_5 and H_6

With P as above we have

$$E' = \begin{bmatrix} 0&0&1&1&&&&& \\ 0&1&0&0&&&&& \\ 1&0&0&1&&&&& \\ 0&0&0&-1&&&&& \\ &&&&-1&&&& \\ &&&&&0&1&0&0 \\ &&&&&1&0&0&0 \\ &&&&&0&0&-1&0 \\ &&&&&0&0&0&-1 \end{bmatrix}, T' = \frac{1}{2}\begin{bmatrix} 0&2&0&-2&&&&& \\ -1&0&-1&-1&&&&& \\ -1&-2&1&-1&&&&& \\ 1&0&-1&-1&&&&& \\ &&&&-2&&&& \\ &&&&&-1&-1&-1&1 \\ &&&&&1&1&-1&1 \\ &&&&&-1&1&-1&-1 \\ &&&&&-1&1&1&1 \end{bmatrix}$$

The block-diagonal form of D', E', T' shows that $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3$ are also H_5 and H_6-invariant. We are going to show that these are the only minimal H_5 or H_6-invariant subspaces. Denote by $T_1 = T'|_{\mathcal{U}_1}$ and $T_3 = T'|_{\mathcal{U}_3}$. It is easy to see that the minimal polynomials are $m_{T_1} = m_{T_2} = (x^2+1)(x+1)(x-1)$ (in fact, T is diagonalizable for $p = 1 \pmod 4$). Computing the respective kernels we get

$$\mathrm{Ker}(T_1^2 + I) = \langle v_1, v_2 \rangle\,, \quad \mathrm{Ker}(T_1 + I) = \langle v_3 \rangle\,, \quad \mathrm{Ker}(T_1 - I) = \langle v_4 \rangle\,,$$

and

$$\mathrm{Ker}(T_2^2 + I) = \langle v_6, v_7 \rangle\,, \quad \mathrm{Ker}(T_2 + I) = \langle v_8 \rangle\,, \quad \mathrm{Ker}(T_2 - I) = \langle v_9 \rangle\,,$$

where v_i are rows of the matrix Q below and $v_5 = u_5$. By similarity relation $X'' = QX'Q^{-1}$ we have

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}, \qquad T'' = \begin{bmatrix} 0 & -1 & 0 & 0 & & & & & \\ 1 & 0 & 0 & 0 & & & & & \\ 0 & 0 & -1 & 0 & & & & & \\ 0 & 0 & 0 & 1 & & & & & \\ & & & & -1 & & & & \\ & & & & & 0 & -1 & 0 & 0 \\ & & & & & 1 & 0 & 0 & 0 \\ & & & & & 0 & 0 & -1 & 0 \\ & & & & & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$D'' = \frac{1}{2}\begin{bmatrix} 0 & 0 & -1 & -1 & & & & & \\ 0 & 0 & 1 & -1 & & & & & \\ 2 & -2 & 0 & 0 & & & & & \\ 2 & 2 & 0 & 0 & & & & & \\ & & & & -2 & & & & \\ & & & & & -1 & -1 & -1 & -1 \\ & & & & & -1 & -1 & 1 & 1 \\ & & & & & 1 & -1 & -1 & 1 \\ & & & & & 1 & -1 & 1 & -1 \end{bmatrix}, \quad 2E'' = \frac{1}{2}\begin{bmatrix} 0 & 0 & -1 & 1 & & & & & \\ 0 & 0 & -1 & -1 & & & & & \\ -2 & -2 & 0 & 0 & & & & & \\ 2 & -2 & 0 & 0 & & & & & \\ & & & & -2 & & & & \\ & & & & & -1 & 1 & 1 & -1 \\ & & & & & 1 & -1 & 1 & -1 \\ & & & & & 1 & 1 & -1 & -1 \\ & & & & & -1 & -1 & -1 & -1 \end{bmatrix}.$$

Any nontrivial T_1-invariant subspace of $\mathcal{W} \leq \mathcal{U}_1$ must contain at least one of the vectors $v_3, v_4, av_1 + bv_2$ $(a^2 + b^2 \neq 0)$. Suppose that $\mathcal{W}$ is H_5-invariant. If $v_3 = (0,0,1,0)''_1 \in \mathcal{W}$, then apply D'' and T'' consecutively to see that $\mathcal{W} = \mathcal{U}_1$. Similarly, $v_4 \in \mathcal{W}$ implies $\mathcal{W} = \mathcal{U}_1$. If $av_1 + bv_2 = (a,b,0,0)''_1 \in \mathcal{W}$, then apply D'' and T'' to see that $av_3, bv_4 \in \mathcal{W}$, thus one of $v_3, v_4 \in \mathcal{W}$, and $\mathcal{W} = \mathcal{U}_3$ follows.

In exactly the same fashion one can easily check that $\mathcal{U}_3$ is minimal H_5-invariant subspace, and that $\mathcal{U}_1, \mathcal{U}_3$ are both minimal H_6-invariant subspaces.

Finally, we show that there are no other minimal H_5 or H_6-invariant subspaces. As

$$\mathcal{V} = \operatorname{Ker}(T^2 + I) \oplus \operatorname{Ker}(T + I) \oplus \operatorname{Ker}(T - I) = \langle v_1, v_2, v_6, v_7 \rangle \oplus \langle v_3, v_5, v_8 \rangle \oplus \langle v_4, v_9 \rangle,$$

any nontrivial T-invariant subspace must contain a nonzero vector from one of these direct summands. Observe that

$$D''^2 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}.$$

Suppose $\mathcal{W}$ is some minimal H_5-invariant subspace and $(a_1, a_2, 0, 0, 0, a_6, a_7, 0, 0)'' \in \mathcal{W}$ is nonzero. Apply D''^2 to see that $(0,0,0,0,0, a_6 + a_7, a_6 + a_7, 0, 0)'' \in \mathcal{W}$. If $a_6 + a_7 \neq 0$, then $v_6 + v_7 \in \mathcal{W}$, hence $\mathcal{W} = \mathcal{U}_3$ by minimality. If $a_6 + a_7 = 0$, apply D''^2 and T'' several times to see that $(0,0,0,0,0, a_6, a_6, 0, 0)'' \in \mathcal{W}$. By minimality, either $\mathcal{W} = \mathcal{U}_1$ or $\mathcal{W} = \mathcal{U}_3$ follows. The remaining cases can be dealt with in a similar fashion. Thus, $\mathcal{V} = \mathcal{U}_1 \oplus \mathcal{U}_2 \oplus \mathcal{U}_3$ is the unique minimal H_i-invariant decomposition for $i = 5, 6$.

References

[1] W. Bosma, C. Cannon, and C. Playoust, The MAGMA algebra system I: The user language, *J. Symbolic Comput.* **24** (1997), 235–265.

[2] N. L. Biggs, Algebraic graph theory, Cambridge University press, Cambridge, 1974.

[3] R. Diestel, 'Graph Theory', Springer-Verlag, Berlin, 2006.

[4] D. Ž. Djoković, Automorphisms of graphs and coverings, J. Combin. Theory Ser. B, 16 (1974) 243–247.

[5] S. F. Du, J. H. Kwak, M. Y. Xu, Linear criteria for lifting automorphisms of elementary abelian regular coverings, *Linear Algebra Appl.* **373** (2003) 101–119.

[6] S. F. Du, J. H. Kwak, M. Y. Xu, 2-Arc-transitive regular covers of complete graphs having the covering transformation group $\mathbb{Z}_3^p$, *J. Combin. Theory Ser. B* **93** (2005) 73–93.

[7] S. F. Du, D. Marušič, A. O. Waller, On 2-arc-transitive covers of complete graphs, *J. Combin. Theory Ser. B* **74** (1998) 276–290.

[8] A. Malnič, D. Marušič, Š. Miklavič, P. Potočnik, Semisymmetric elementary abelian covers of the MbiusKantor graph, *Discrete Math.* 307 (2007) 2156–2175.

[9] Y. Q. Feng, J. H. Kwak, Cubic symmetric graphs of order a small number times a prime or a prime square, *J. Combin. Theory Ser. B* **97**, no. 4 (2007), 627–646.

[10] Y. Q. Feng, J. H. Kwak, K. Wang, Classifying cubic symmetric graphs of order 8p or 8p 2 , *European J. Combin.* **26**, (2005), 1033–1052.

[11] J. L. Gross and T. W. Tucker, "Topological Graph Theory", Wiley - Interscience, New York, 1987.

[12] B. Kuzman, Arc-transitive elementary abelian covers of the complete graph K_5, submitted.

[13] J. H. Kwak, J. M. Oh, Arc transitive elementary abelian covers of the octahedron graph, *Linear algebra and its applications*, **429**, 2008, 2180–2198.

[14] S. Lang, Algebra, Graduate Texts in Mathematics, 211 (Revised third ed.), New York: Springer-Verlag, 2002.

[15] A. Malnič, D. Marušič, P. Potočnik, Elementary abelian covers of graphs, *J. Algebraic Combin.* **20** (2004), 71–97.

[16] A. Malnič, P. Potočnik, Invariant subspaces, duality, and covers of the Petersen graph, *European J. Combin.* **27** (2006), 971–989.

[17] A. Malnič, R. Nedela and M. Škoviera, Lifting graph automorphisms by voltage assignments, *European J. Combin.* **21** (2000), 927–947.

[18] J. M. Oh, Arc-transitive elementary abelian covers of the Pappus graph, *Discrete Mathematics* (2009), doi:10.1016/j.disc.2009.07.010.

[19] P. Potočnik, S. Wilson, A Census of edge-transitive tetravalent graphs, `http://jan.ucc.nau.edu/~swilson/C4Site/index.html`

[20] P. Potočnik, S. Wilson, Tetravalent edge-transitive graphs of girth at most 4, *J. Combin. Theory Ser. B* **97**, no. 2 (2007), 217–236.

Boštjan Kuzman
Faculty of Education, University of Ljubljana, Kardeljeva pl. 16, 1000 Ljubljana, Slovenia
E-mail: `bostjan.kuzman@pef.uni-lj.si`