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MATRIX SUMMABILITY OF STATISTICALLY P-CONVERGENCE SEQUENCES

Richard F. Patterson and Ekrem Savaş

Abstract

Matrix summability is arguable the most important tool used to characterize sequence spaces. In 1993 Kolk presented such a characterization for statistically convergent sequence space using nonnegative regular matrix. The goal of this paper is extended Kolk's results to double sequence spaces via four dimensional matrix transformation. To accomplish this goal we begin by presenting the following multidimensional analog of Kolk's Theorem : Let X be a section-closed double sequence space containing $e^{''}$ and Y an arbitrary sequence space. Then $B \in (st^2_A \cap X, Y)$ if and only if $B \in (c^{''} \cap X, Y)$ and $B^{[K\times K]} \in (X, Y)$ $(\delta_A(K \times K) = 0)$. In addition, to this result we shall also present implication and variation of this theorem.

1 Introduction, notations and preliminary results

Let us begin with the presentation of the following notations and the notion for convergence of double sequences: s set of all ordinary complex sequences; $s^{''}$ set of all double complex sequences; st_A set of A-statistically convergent ordinary sequences; st^2_A set of A-statistically P-convergent double sequences; $st^2_{0,A}$ set of A-statistically P-convergent double null sequences; c ordinary convergence sequences; $c^{''}$ P-convergence double sequences; $c^{''}_0$ P-convergence double null sequences; e single dimensional sequence of all 1's; and $e^{''}$ two dimensional sequence of all 1's.

Definition 1 (Pringsheim, [9]). *A double sequence $x = [x_{k,l}]$ has Pringsheim limit L (denoted by P-$\lim x = L$) provided that given $\epsilon > 0$ there exists $N \in \mathbf{N}$ such that $|x_{k,l} - L| < \epsilon$ whenever $k, l > N$. We shall describe such an x more briefly as " P-convergent".*

Robison and Hamilton in [10] and [5]presented the following definition of regularity of double sequences respectively. The four dimensional matrix A is said to be RH-regular if it maps every bounded P-convergent sequence into a P-convergent

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sequence with the same P-limit. Using this definition Robison and Hamilton both presented the following Silverman-Toeplitz type characterization of RH-regularity, independently.

Theorem 1. *(Robison[10] and Hamilton[5]). The four dimensional matrix A is RH-regular if and only if*

RH_1: $P\text{-}\lim_{m,n} a_{m,n,k,l} = 0$ *for each* k *and* l;
RH_2: $P\text{-}\lim_{m,n} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} = 1$;
RH_3: $P\text{-}\lim_{m,n} \sum_{k=0}^{\infty} |a_{m,n,k,l}| = 0$ *for each* l;
RH_4: $P\text{-}\lim_{m,n} \sum_{l=0}^{\infty} |a_{m,n,k,l}| = 0$ *for each* k;
RH_5: $\sum_{k,l=0,0}^{\infty,\infty} |a_{m,n,k,l}|$ *is P-convergent;*
RH_6: *there exist positive numbers* A *and* B *such that* $\sum_{k,l>B} |a_{m,n,k,l}| < A$.

Let
$$e_{k,l} := \begin{cases} 1, & (k,l)th - \text{position} \\ 0, & \text{otherwise,} \end{cases}$$
and denote the (m,n)-th-section of the double sequence $x = [x_{k,l}]$ is defined as follows:
$$x^{[(m,n)]} = \sum_{k,l=1,1}^{m,n} x_{k,l} e_{k,l}.$$
Let $K \times K$ be an arbitrary double index set $K \times K = \{(k_i, l_j)\}$. The double sequence $x^{[K\times K]} = (y_{k,l})$ where
$$y_{k,l} := \begin{cases} x_{k,l}, & \text{if } (k,l) \in K \times K \\ 0, & \text{otherwise,} \end{cases}$$
We shall call the $K \times K$-section of x. A double sequence space X will be called *section-closed* if $x^{[K\times K]} \in X$ for all $x \in X$ and for every double index set $K \times K$. In addition, the definition for a *subsequence-closed* double sequence space is defined in a similar manner to section-closed using the notion of double subsequence presented in [8]. Let X and Y be two double sequence spaces and $A = (a_{m,n,k,l})$ a four-dimensional infinite matrix. If for each $x = [x_{k,l}] \in X$ the series
$$(Ax)_{m,n} = \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} x_{k,l}$$
P-converges and the resulting double sequence belongs to Y, we say A maps X to Y. We will also denote the set of all four-dimensional matrices that maps X to Y by (X, Y). For a fixed double index set $K \times K = \{(k_i, l_j)\}$ we denote by $A^{[K\times K]}$ the $K \times K$-pairwise-column-section of the four dimensional matrix A. Thus $A^{[K\times K]} = (d_{m,n,k,l})$ where
$$d_{m,n,k,l} := \begin{cases} a_{m,n,k,l}, & (k,l) \in K \times K \\ 0, & \text{otherwise.} \end{cases}$$

Natural density was generalized by Freeman and Sember in [2] by replacing C_1 with a nonnegative regular summability matrix $A = (a_{n,k})$. Thus, if K is a subset of N then the A-density of K is given by $\delta_A(K) = \lim_n \sum_{k\in K} a_{n,k}$ if the limit exists. Let $K \times K \subset N \times N$ be a two-dimensional set to positive integers and let $K^{''}(m,n)$ be the numbers of (i,j) in $K \times K$ such that $i \leq m$ and $j \leq n$. The two-dimensional analogues of natural density is defined as follows in [4]: The lower asymptotic density of a set $K \times K \subset N \times N$ is define as

$$\delta(K \times K) = \liminf_{m,n} \frac{K^{''}(m,n)}{mn}.$$

In case the double sequence $[\frac{K^{''}(m,n)}{mn}]$ has a limit in the Pringsheim sense then we say that $K \times K$ has a double natural density as

$$P - \lim_{m,n} \frac{K^{''}(m,n)}{mn} = \delta(K \times K).$$

Quite recently, Mursaleen and Edely [4], defined the statistical analogue for double sequences $x = (x_{k,l})$ as follows: A real double sequences $x = (x_{k,l})$ is said to be P-statistically convergent to L provided that for each $\epsilon > 0$

$$P - \lim_{m,n} \frac{1}{mn}\{\text{number of } (j,k) : j < m \text{ and } k < n, |x_{j,k} - L| \geq \epsilon\} = 0.$$

In this case we write $st_2 - \lim_{m,n} x_{m,n} = L$ and we denote the set of all P-statistical convergent double sequences by st_2.

Let $K \times K \subset N \times N$ be a two-dimensional set of positive integers, then the A-density of $K \times K$ is given by

$$\delta_A(K \times K) = P - \lim_{m,n} \sum_{(k,l)\in K\times K} a_{m,n,k,l},$$

provided that the limit exists. We have the following definition which is defined in [11].

Definition 2. *A double real number sequence x is said to be A-statistically P-convergent to L if, for every positive ϵ*

$$\delta_A(K \times K)(\{(k,l) : |x_{k,l} - L \geq \epsilon\}|) = 0.$$

In this case we write $st^2_A - \lim_{m,n} x_{m,n} = L$ and we denote the set of all A-statistically P-convergent double sequences by st^2_A. We also denote the set of all A-statistically P-convergent double null sequences by $st^2_{0,A}$.

One of the most important paper in summability with respect to statistical analysis was presented by Kolk in 1993. The goal of this paper is to generalize the following result of Kolk to double sequences via convergence in the Pringsheim sense: Let U be a ordinary section-closed sequence space containing e and V an ordinary arbitrary sequence space. Then $B \in (st_A \cap U, V)$ if and only if $B \in (c \cap U, V)$ and $B^{[K]} \in (U, V)$ $(\delta_A(K) = 0)$ where A is the ordinary two dimensional matrix. To accomplish this goal we begin by presenting a multidimensional analog of the following theorem of Agnew [1]: If the matrix $A = (a_{n,k})$ such that

$$\sum_{k=1}^{\infty} |a_{n,k}| < \infty \tag{1}$$

and

$$P - \lim_{n} \max_{\{k:1 \leq k < \infty\}} |a_{n,k}| = 0. \tag{2}$$

Then there exists at least one divergent sequence of 0's and 1's that is A summable.

2 Main results

We begin this section with the following multidimensional generalization of Agnew's theorem.

Theorem 2. *If the four-dimensional matrix $A = (a_{m,n,k,l})$ such that*

$$\sum_{k,l=1,1}^{\infty,\infty} |a_{m,n,k,l}| < \infty \tag{3}$$

and

$$P - \lim_{m,n} sup_{\{(k,l):1,1 \leq k,l < \infty,\infty\}} |a_{m,n,k,l}| = 0. \tag{4}$$

Then there exists at least one double P-divergent double sequence of 0*'s and* 1*'s that is A summable.*

Proof. Let $\{\alpha_{i,j}\}$ be a double sequence of positive numbers such that $P - \lim_{i,j} \alpha_{i,j} = 0$ and $P - \lim_{i,j} ij\alpha_{i,j} = 0$. Also let $\{\beta_{i,j}\}$ be a double sequence of positive numbers such that $P - \lim_{i,j} \beta_{i,j} = 0$. Condition (2.2) imply there exist two increasing sequences $\{m_i\}$ and $\{n_j\}$ of positive integers such that for each (p, q) with $p, q = 1, 2, 3, \ldots$ we have

$$|a_{m,n,k,l}| \leq \alpha_{p,q} \text{ for } m > m_p \text{ and } n > n_q$$

for each (k, l) with $k, l = 1, 2, 3, \ldots$. For the fixed double sequence (m_p, n_q) (2.1) imply there exist two index sequences $\{k_i\}$ and $\{l_j\}$ both are going to infinite sufficiently fast. Then for each (p, q) where $p, q = 1, 2, 3, \ldots$ we have

$$\sum_{k,l=k_p+1,l_q+1}^{\infty,\infty} |a_{m,n,k,l}| < \beta_{p,q} \tag{5}$$

with $k_p \leq k < k_{p+1}$ and $l_q \leq l < l_{q+1}$. If we let $\{k_p\}$ and $\{l_q\}$ be such that

$$k_{p+1} > k_p + 1 \text{ and } \ l_{q+1} > l_q + 1 \text{ for each } (p,q); p,q = 1,2,3,\ldots,$$

and define $x_{k,l}$ as follows:

$$x_{k,l} := \begin{cases} 0, & k > k_p \ \& \ 1 \leq l \leq l_q; \\ 0, & l > l_q \ \& \ 1 \leq k \leq k_p; \\ 1, & (k,l) = (k_p, l_q); \\ 0, & \text{otherwise.} \end{cases}$$

Note $[x_{k,l}]$ is P-divergent double sequence of 0's and 1's. The transformation of this double sequence yields the following:

$$\begin{aligned} \left| \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} x_{k,l} \right| &= \left| \sum_{i,j=1,1}^{\infty,\infty} a_{m,n,k_i,l_j} \right| \\ &\leq \sum_{i,j=1,1}^{p,q} \left| a_{m,n,k_i,l_j} \right| + \sum_{i,j=p+1,q+1}^{\infty,\infty} \left| a_{m,n,k_i,l_j} \right| \\ &\leq \sum_{i,j=1,1}^{p,q} \alpha_{p,q} + \sum_{i,j=p+1,q+1}^{\infty,\infty} \left| a_{m,n,k_i,l_j} \right| \\ &< pq\alpha_{p,q} + \beta_{p,q}. \end{aligned}$$

Since $P - \lim_{p,q} pq\alpha_{p,q} = 0$ and $P - \lim_{p,q} \beta_{p,q} = 0$ we have produce a P-divergent sequence of 0's and 1's that is A-summable. This completes the proof. □

In addition, to the last theorem let us consider the following:

Theorem 3. *Let $Y \neq s''$ be a subsequence-closed double sequence space. If the matrix A satisfies the conditions of Theorem 2.1, the following statements about a matrix $B = (b_{m,n,k,l})$ are equivalent:*

1. $B \in (s'', Y)$
2. $B^{[K \times K]} \in (s'', Y)$ *for every double index set $K \times K$ with $\delta_A(K \times K) = 0$*
3. $Be^{k,l} \in Y; \{(k,l) \in N \times N\}$ *and there exists a point (k_0, l_0) such that*

$$b_{m,n,k,l} = 0 \text{ for } k > k_0 \text{ and } l > l_0, (m,n) \in N \times N;$$

Proof. Let begin by showing that (1) imply (2). Let $K \times K$ be a double index set and $x = [x_{k,l}]$ a double sequence. Since $K \times K$-section $y = x^{[K \times K]}$ of x belong to s''. We have $B^{[K \times K]}x \in Y$ because $By \in Y$ and $B^{[K \times K]}_{m,n}x = B_{m,n}y; (m,n) \in N \times N$. Thus (1) implies (2).

Now let us show that (2) imply (3). Note that if $K \times K$ consist of only a single point (k,l) then $\delta_A(K \times K) = 0$. This implies

$$Be^{(k,l)} = B^{[K \times K]}e^{(k,l)} \in Y \text{ where } (k,l) \in N \times N.$$

Observe that if

$$b_{m,n,k,l} = 0 \text{ for } k > k_0 \text{ and } l > l_0, (m, n) \in N \times N$$

then B must be pairwise row finite. Suppose B is not pairwise row finite, then there exist indices m_0, n_0, and an infinite index set $K \times K(k_i, l_j)$ such that

$$b_{m_0,n_0,k_i,l_j} \neq 0 \text{ for } (i, j) \in N \times N.$$

By Theorem 2.1 we are granted that $\delta_A(K \times K) = 0$. Let us define a double sequence $x = [x_{k,l}]$ as follows

$$x_{k_i,l_j} = \frac{1}{b_{m_0,n_0,k_i,l_j}} \text{ for } (i, j) \in N \times N.$$

This grant us the following

$$B^{[K\times K]}_{m_0,n_0} x = \sum_{i,j} b_{m_0,n_0,k_i,l_j} x_{k_i,l_j} = \infty.$$

Therefore $B^{[K\times K]}x$ does not exists. Thus B must be pairwise-row finite. Note if

$$b_{m,n,k,l} = 0 \text{ for } k > k_0 \text{ and } l > l_0, (m, n) \in N \times N$$

fails to hold then there exist infinite order pair indices sets $K \times K = \{(k_i.l_j)\}$ and $N \times N = \{(m_i, n_j)\}$ such that

$$b_{m,n,k,l} := \begin{cases} \neq 0 & m = m_i, n = n_j, k = k_i, l = l_j \text{ for } (i, j) \in N \times N \\ 0, & m = m_i, n = n_j, \text{ for } (i, j) \in N \times N \end{cases}$$

We can still assume $\delta_A(K \times K) = 0$ by Theorem 2.1. We can choose $z = [z_{k,l}] \in s^{''} \backslash Y$ and consider $x = [x_{k,l}]$ defined as follows

$$x_{k,l} := \begin{cases} \frac{z_{1,1}}{b_{m_1,n_1,k_1,l_1}}, & k = k_1, l = l_1 \\ \frac{\left(z_{k_i,l_j} - \sum_{i,j=1,1}^{i-1,j-1} b_{m_i,n_j,k_i,l_j} x_{k_i,l_j}\right)}{b_{m_i,n_j,k_i,l_j}}, & i > 1, j > 1. \end{cases}$$

Thus

$$B^{[K\times K]}_{m_i,n_j} x = z_{i,j} \text{ where } (i, j) \in N \times N.$$

Therefore $(B^{[K\times K]}_{m_i,n_j} x) \notin Y$ and since Y is a double subsequence-closed, we $B^{[K\times K]}x \notin Y$. Thus (2) imply (3). Finally for (3) imply (1) we observe that for $x \in s^{''}$,

$$Bx = B\left(\sum_{k,l}^{k_0,l_0} x_{k,l} e^{k,l}\right) = \sum_{k,l}^{k_0,l_0} x_{k,l} B e^{k,l} \in Y.$$

Thus (1) holds. This completes the proof. □

Theorem 4. *Let X be a section-closed double sequence space containing e'' and Y an arbitrary sequence space. Then $B \in (st_A^2 \cap X, Y)$ if and only if $B \in (c'' \cap X, Y)$ and $B^{[K\times K]} \in (X, Y)$ $(\delta_A(K \times K) = 0)$.*

Proof. Let $B \in (st_A^2 \cap X, Y)$. Since $c'' \subset st_A^2$ we have $B \in (c'' \cap X, Y)$. Let $K \times K$ be a subset of $N \times N$ with $\delta_A(K \times K) = 0$ and let $x \in X$. Then the $K \times K$-section y of x P-converges A-statistically to 0. Also y belong to X. Thus $y \in st_A^2 \cap X$ and therefore $By \in Y$. In addition since $B_{m,n}^{[K\times K]}x = B_{m,n}y$ where $(n, m) \in N \times N$ we have $B_{m,n}^{[K\times K]}x \in Y$. This implies $B^{[K\times K]} \in (X, Y)$ for every double index set $K\times K$ with $\delta_A(K\times K) = 0$. Now let us consider the converse, let $x \in st_A^2 \cap X$ where $st_A^2 - \lim x_{k,l} = x_0$. We need only to show that $x \in Y$. Without loss of generality we can assume that $x_0 = 0$. If $x \in c''$ then $Bx \in Y$ thus $B \in (c'' \cap X, Y)$. Now suppose that $x \in st_A^2 \backslash c''$ then Theorem 2.2 implies there exists an infinite double index set $K \times K$ with $\delta_A(K \times K) = 0$ such that $P - \lim_{k,l} z_{k,l} = 0$ where $z = z_{k,l}$ is the $(N \times N)\backslash(K \times K)$-section of x. Since $z \in X$ then $Bz \in Y$. Thus $B^{[K\times K]}x \in Y$. Now since $Bx = Bz + B^{[K\times K]}x$ we have $Bx \in Y$. □

If we $st_{0,A}^2$ instead of st_A^2 we obtain the following theorem.

Theorem 5. *Let X be a section-closed double sequence space containing e and Y an arbitrary sequence space. Then $B \in (st_{0,A}^2 \cap X, Y)$ if and only if $B \in (c''_0 \cap X, Y)$ and $B^{[K\times K]} \in (X, Y)$ $(\delta_A(K \times K) = 0)$.*

Theorem 6. *Let $Y \neq s''$ be a subsequence-closed double sequence space. If A is uniformly RH-regular matrix then $(st_A^2, Y) = (st_{0,A}^2, Y) = (s'', Y)$.*

Proof. Since $(s'', Y) \subset (st_A^2, Y) \subset (st_{0,A}^2, Y)$ we need only to prove $(st_{0,A}^2, Y) \subset (s'', Y)$. If $B \in (st_{0,A}^2, Y)$ then by Theorem 2.4 $B^{[K\times K]} \in (s'', Y)$ note $s'' = X$ for every double index set K with $(\delta_A(K \times K) = 0)$. Thus Theorem 2.2 grants us $B \in (s'', Y)$.

□

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References

[1] Agnew, R. P., *A simple sufficient condition that a method of summability be stronger than convergence.* Bull. Amer. Math. Soc. 52, (1946). 128-132.

[2] Freedman, A. R. and Sember, J. J., *Densities and summability*, Pacific J. Math. 95(2), (1981), 293-305.

[3] Kolk, E., *Matrix summability of statistically convergent sequences*, Analysis, 13, (1993), 77-83.

[4] Mursaleen and Edely, O. H. *Statistical convergence of double sequences*, J. Math. Anal. Appl., 288(1), (2003), 223-231.

[5] Hamilton, H. J., *Transformations of Multiple Sequences*, Duke Math. Jour., 2, (1936), 29 - 60.

[6] Patterson, R. F. *Uniformly Summable Double Sequences*, Sci. Math. Hungar. 44(1), (2007), 147-158

[7] Patterson, R. F., *Four dimensioal matrix Characterization of the Pringsheim core*, Southeast Asian Bull. Math. 27, (2004), 99-906.

[8] Patterson, R. F.*Analogues of some Fundamental Theorems of Summability Theory*, Internat. J. Math. & Math. Sci., 23(1), 2000, 1-9.

[9] Pringsheim, A., *Zur theorie der zweifach unendlichen zahlenfolgen*, Mathematische Annalen, 53, (1900), 289 - 321.

[10] Robison, G. M., *Divergent Double Sequences and Series*, Amer. Math. Soc. trans., 28, (1926), 50 - 73.

[11] Savas, E. and Patterson, R. F. *On some Double Sequence Spaces Defined by A Modulus* , Math. Slovaca, 61(2), 245-256, DOI: 10.2478/s12175-011-0009-2. .

Richard F. Patterson:
Department of Mathematics and Statistics, University of North Florida building 11 Jacksonville, Florida, 32224
E-mail: rpatters@unf.edu

Ekrem Savaş:
Istanbul Ticaret University , Department of Mathematics,Üsküdar-Istanbul-TURKEY
E-mail: ekremsavas@yahoo.com, esavas@iticu.edu.tr