Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Filomat **25:4** (2011), 85–97 DOI: 10.2298/FIL1104085N

CESÁRO TYPE OPERATORS ON SPACES OF ANALYTIC FUNCTIONS

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Abstract

In this article, we consider a two parameter family of generalized Cesáro operators $\mathcal{P}^{b,c}$, $\mathrm{Re}\,(b+1) > \mathrm{Re}\,c > 0$, on classical spaces of analytic functions such as Hardy (H^p), $BMOA$ and a- Bloch space (B^a). We Prove that $\mathcal{P}^{b,c}$, $\mathrm{Re}\,(b+1) > \mathrm{Re}\,c > 0$ is bounded on H^p if and only if $p \in (0,\infty)$ and on B^a if and only if $a \in (1,\infty)$ and unbounded on H^∞, $BMOA$ and B^a, $a \in (0,1]$. Also we prove that α- Cesáro operators $\mathcal{C}^\alpha$ is a bounded operator from the Hardy space H^p to the Bergmann space A^p for $p \in (0,1)$. Thus, we improve some well known results of the literature.

1 Introduction

Let D denote the unit disc in the complex plane $\mathbb{C}$. A function f analytic in the unit disc D is belong to the Hardy space H^p, $0 < p < \infty$, if

$$||f||_{H^p} = \sup_{0<r<1} M_p(r,f) < \infty,$$

where its integral mean $M_p(r,f)$ is given by

$$M_p(r,f) = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \right\}^{1/p}.$$

With this norm H^p is a complete metric linear space or Fréchet space if $0 < p < 1$ and a Banach space if $p \geq 1$. It is a Hilbert space for $p = 2$. We let $M_\infty(r,f)$ denote the maximum of $|f(z)|$ on the circle $|z| = r < 1$. Thus for $p = \infty$ the Banach space H^∞ is the class of bounded analytic functions in D with the norm

$$||f||_{H^\infty} = \sup_{z\in D} |f(z)|.$$

2010 *Mathematics Subject Classifications.* 33C05, 30D55, 47B38.
Key words and Phrases. Gaussian hypergeometric functions, Cesáro operators.
Received: November 3, 2010
Communicated by Miodrag Mateljević

The Bergman space A^p $(0 < p \leq \infty)$ consists of all analytic functions f in D for which

$$||f||_{A^p} = \left(\int_0^1 M_p^p(r,f) r\, dr \right)^{1/p} < \infty.$$

and $A^\infty = H^\infty$. Thus A^p is a Banach space if $p \geq 1$, and Fréchet spaces if $0 < p < 1$. The space $BMOA$ containing H^∞ is the space of functions in H^1, of which the radial limit functions have bounded mean oscillation on the unit circle ∂D, more precisely a functions $f \in H^1$ is said to be in the space

$$||f||_\star = \sup \frac{1}{|I|} \int_I |f(e^{i\theta}) - f_I| \frac{d\theta}{2\pi} < \infty,$$

where the supremum is taken over all subarcs $I \subset \partial D$, $|I| = \int_I \frac{d\theta}{2\pi}$ and

$$f_I = \frac{1}{|I|} \int_I f(e^{i\theta}) \frac{d\theta}{2\pi}.$$

The proper inclusions

$$H^\infty \subset BMOA \subset \bigcap_{p<\infty} H^p$$

and $BMOA \subset B$ are well known. Other information on $BMOA$ can be found in the books [5] or [13]. For $a > 0$, we let a-Bloch space denotes B^a is the space of analytic functions f on D such that

$$||f||_{B^a} = |f(0)| + \sup_{z \in D} (1 - |z|)^a |f'(z)| < \infty.$$

It is known that B^a is a Banach space for each $a > 0$ with the above norm. (see for example [14]). When $a > 1$, the space can be identified with space of analytic functions f with

$$|f(0)| + \sup_{z \in D} (1 - |z|)^{a-1} |f(z)| < \infty,$$

(see [14], Proposition 7). When $0 < a < 1$ and $a = 1$, the space B^a can be identified with analytic Lipschitz space (see [4], Theorem B) and Bloch space respectively.

For any complex number a, b, $c \neq -n$, $n = 0, 1, 2, \ldots$, the Gaussian hypergeometric function ${}_2F_1(a, b; c; z)$ is defined by power series expansion

$${}_2F_1(a, b; c; z) = F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n)} \frac{z^n}{n!} \quad (|z| < 1),$$

where (a, n) is the shifted factorial defined by Appel's symbol

$$(a, n) = a(a + 1) \ldots (a + n - 1) = \frac{\Gamma(a + n)}{\Gamma(a)}, \quad n \in \mathbb{N} = \{1, 2, \ldots\}$$

and $(a, 0) = 1$ for $a \neq 0$, (see [1, 11]). Obviously, $F(a, b; c; z)$ is an analytic function in the unit disc D. We refer the reader to [1, 11] for a background on Gaussian hypergeometric functions. In [2], authors have used one of the contiguous relation (differ by 1 in the parameter values) involving hypergeometric functions to introduce and study the generalized Cesáro operator. For $b, c \in \mathbb{C}$, the generalized Cesáro operator $\mathcal{P}^{b,c}$, associates any analytic function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on the unit disc D with power series:

$$\mathcal{P}^{b,c} f(z) := \sum_{n=0}^{\infty} \left(\frac{1}{A_n^{b+1;c}} \sum_{k=0}^{n} b_{n-k} a_k \right) z^n, \tag{1}$$

where

$$A_k^{b;c} = \frac{(b,k)}{(c,k)},$$

$b_0 = 1$, and for $k \geq 1$

$$b_k = \frac{(1+b-c)}{b} A_k^{b;c},$$

Notice that the α Cesáro operator introduce by (see [10]) and Classical Cesáro operator are the special cases of generalized Cesáro operator $\mathcal{P}^{b,c}$.

The boundedness of $\mathcal{P}^{b,c}$ on H^p for $0 < p \leq 1$ is proved in [2] and the case $1 < p < \infty$ is proved in [7, 9] for $\mathrm{Re}\,(b+1) > \mathrm{Re}\,c \geq 1$. Other cases, that is for $\mathrm{Re}\,c \in (0,1)$ remain open till date. In [7] also the boundedness of $\mathcal{P}^{b,c}$ is proved on mixed norm space which include Bergman space. In this article, we study once again the action of the operator $\mathcal{P}^{b,c}$ on the space H^p, further on the spaces $BMOA$ and B^a. We show that $\mathcal{P}^{b,c}$ is a bounded operator on H^p if and only if $p \in (0,\infty)$ for $\mathrm{Re}\,(b+1) > \mathrm{Re}\,c > 0$ by giving an alternate proof for $p > 1$. Thus our result improve the condition on the parameters b, c compare to the result given in [7]. In [2] it is proved that $\mathcal{P}^{b,c}$ is bounded on B^a for all $a \in (1,\infty)$ whenever $\mathrm{Re}\,(b+1) > \mathrm{Re}\,c \geq 1$. In this article we study an equivalent condition for the boundedness of $\mathcal{P}^{b,c}$ on B^a, for $a > 0$ with $\mathrm{Re}\,(b+1) > \mathrm{Re}\,c > 0$. Also we give an alternate proof for unboundedness of $\mathcal{P}^{b,c}$ on $BMOA$ using a different technique. In Section 4, we consider the α- Cesáro operator from H^p to A^p for $p \in (0,1)$ and prove their boundedness.

Throughout the text the letter C will denote the constant depending only on related parameters such as p, b, c and so on, C may differ at different occurrences. The symbol $a \sim b$ means that $\frac{a}{b}$ is bounded from above and below by two positive constants. Also throughout the paper, an operator T is unbounded from X into Y means that, if for some $f \in X$ we have $T(f) \notin Y$.

2 Preliminaries

We start by stating a known result which associates an integral representation to the $\mathcal{P}^{b,c}$ operator.

Lemma 1. ([2]) *For* $b, c \in \mathbb{C}$ *with* $\operatorname{Re}(b+1) > \operatorname{Re} c > 0$*, we have*

$$\begin{aligned} &\mathcal{P}^{b,c} f(z) \\ &= \frac{1}{B}\int_0^1 t^{c-1}(1-t)^{b-c}\frac{f(tz)}{(1-tz)^{b+1-c}}F(c-1, c-b-1; c; tz)\,dt \\ &= \frac{z^{-b}}{B}\int_0^z \zeta^{c-1}(z-\zeta)^{b-c}\frac{f(\zeta)}{(1-\zeta)^{b+1-c}}F(c-1, c-b-1; c; \zeta)\,d\zeta, \end{aligned} \tag{2}$$

where $B = B(c, b+1-c)$ *is the usual beta function.*

The following lemmas will be used in the sequel.

Lemma 2. ([3]) *If* $0 < p < \infty$ *and* $f \in H^p$*, then there exists a constant* $C = C(p)$ *independent of* f *such that*

$$\int_0^{2\pi} \sup_{0 \le r < 1} |f(re^{i\theta})|^p \, d\theta \le C\|f\|_{H^p}^p.$$

Lemma 3. ([8]) *Let* $\alpha > -1$, $0 < p \le \infty$ *and* $0 < q < \infty$*. If*

$$\int_0^1 (1-r)^{\alpha+q} M_p^p(r, D^1 f)\,dr < \infty$$

then

$$\int_0^1 (1-r)^{\alpha} M_p^q(r, f)\,dr < \infty.$$

We shall use an integral representation of $\mathcal{P}^{b,c}$ which is obtained by choosing the path of integration between 0 and z as

$$\gamma(t) = \phi_t(z) = \frac{tz}{1-(1-t)z}, \quad t \in [0, 1].$$

Using this path into the second equality of (2) and denoting the term $F(c-1, c-b-1; c; z)$ by $F(z)$, we find

$$\begin{aligned} &\mathcal{P}^{b,c} f(z) \\ &= \frac{z^{-b}}{B}\int_0^1 \phi_t(z)^{c-1}(z-\phi_t(z))^{b-c}\frac{f(\phi_t(z))}{(1-\phi_t(z))^{b+1-c}}F(\phi_t(z))\,\phi_t'(z)dt \\ &= \frac{1}{B}\int_0^1 \frac{t^{c-1}(1-t)^{b-c}}{(1-(1-t)z)^c} f(\phi_t(z))F(\phi_t(z))\,dt. \end{aligned}$$

We define

$$T_t f(z) = \omega_t^c(z) f(\phi_t(z)) F(\phi_t(z)) \quad \text{for} \quad t \in (0, 1],$$

where $\omega_t(z) = \dfrac{1}{1-(1-t)z}$, $z \in D$ is the family of weight functions. It follows that

$$\mathcal{P}^{b,c} f(z) = \frac{1}{B}\int_0^1 T_t f(z)\lambda(t)\,dt, \tag{3}$$

where $\lambda(t) = t^{c-1}(1-t)^{b-c}$.

3 Generalized Cesáro operators on H^P, $BMOA$, B^a spaces

In this section we describe the boundedness of $\mathcal{P}^{b,c}$ on the spaces H^P, $BMOA$ and B^a. Though we provide the proofs only for reals b and c with $b+1 > c > 0$, these can easily be modified to see that all the results remain valid for $b, c \in \mathbb{C}$ and $\mathrm{Re}\,(b+1) > \mathrm{Re}\,c > 0$. Time and again we use techniques developed in [6].

Theorem 4. *Let* $0 < p \leq \infty$ *and* $b, c \in \mathbb{C}$ *be numbers satisfying* $\mathrm{Re}\,(b+1) > \mathrm{Re}\,c > 0$. *Then* $\mathcal{P}^{b,c}$ *is a bounded operator from* H^p *to* H^p *if and only if* $0 < p < \infty$. *Moreover, for* $1 < p < \infty$ *we have*

$$||\mathcal{P}^{b,c}||_{H^p} \leq \begin{cases} C\dfrac{B(b+1,\frac{1}{p})}{B(c,b-c+1+\frac{1}{p})}\max\left(\dfrac{1}{|cp-1|^{1/p}}, \dfrac{1}{2^{c-1/p}}\right) & \text{if } p \neq \frac{1}{c}, \\ C\dfrac{\Gamma(c)}{B(c,b+1-c)} & \text{if } p = \frac{1}{c}. \end{cases}$$

Proof. Case (a), Let $0 < p \leq 1$, for this case (see [2], Theorem 3.2).

Case (b), Let $1 < p < \infty$. In this case our aim is to show that

$$||\mathcal{P}^{b,c}||_{H^p} \leq C||f||_{H^p}$$

for some constant $C > 0$ depending on b, c and p. Suppose $f \in H^p$, as H^p norm is same as L^p norm of the boundary function for $1 < p < \infty$, (3) gives

$$\begin{aligned} &||\mathcal{P}^{b,c}(f)||_{H^p} \\ &= \lim_{r\to 1} M_p(r, \mathcal{P}^{b,c}(f)) \\ &= \left\{\tfrac{1}{2\pi}\int_0^{2\pi} |\mathcal{P}^{b,c}f(e^{i\theta})|^p\, d\theta\right\}^{\frac{1}{p}} \\ &\leq \tfrac{1}{B}\left\{\tfrac{1}{2\pi}\int_0^{2\pi}\left(\int_0^1 |\omega_t(e^{i\theta})|^c |f(\phi_t(e^{i\theta}))||F(\phi_t(e^{i\theta}))|\,\lambda(t)\,dt\right)^p d\theta\right\}^{\frac{1}{p}}. \end{aligned}$$

Using Minkowski's inequality and the boundedness of hypergeometric function $F(c-1, c-b-1; c; z)$ on $|z| \leq 1$ for $b+1 > c$ in above calculations, we find

$$\begin{aligned} ||\mathcal{P}^{b,c}(f)||_{H^p} &\leq \frac{1}{B}\int_0^1 \left\{\frac{1}{2\pi}\int_0^{2\pi} |\omega_t(e^{i\theta})|^{cp}|f(\phi_t(e^{i\theta}))F(\phi_t(e^{i\theta}))|^p\,d\theta\right\}^{\frac{1}{p}} \lambda(t)\,dt \\ &\leq \frac{C}{B}\int_0^1 \left\{\frac{1}{2\pi}\int_0^{2\pi} |\omega_t(e^{i\theta})|^{cp}|f(\phi_t(e^{i\theta}))|^p\,d\theta\right\}^{\frac{1}{p}} \lambda(t)\,dt. \qquad (4) \end{aligned}$$

Fix $t \in (0, 1]$ and define

$$A(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{|1-(1-t)e^{i\theta}|^{cp}}\left|f\left(\frac{te^{i\theta}}{1-(1-t)e^{i\theta}}\right)\right|^p d\theta.$$

Let $e^{i\phi} = \dfrac{te^{i\theta}}{1-(1-t)e^{i\theta}} \dfrac{|1-(1-t)e^{i\theta}|}{t}$. A simple calculation shows that

$$\left|\frac{d\phi}{d\theta}\right| \geq \frac{C}{t}, \quad 0 < t \leq 1. \tag{5}$$

Suppose $Mf(e^{i\phi}) = \sup_{0\leq r<1} |f(re^{i\phi})|$ is the radial maximal function. For $\frac{1}{2} \leq t \leq 1$, inequality (5) gives,

$$\begin{aligned} A(t) &\leq C\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{|1-(1-t)e^{i\theta}|^{cp}} \left|Mf(e^{i\phi})\right|^p \frac{d\theta}{d\phi} d\phi \\ &\leq C\frac{1}{2\pi}\int_{-\pi}^{\pi} \left|Mf(e^{i\phi})\right|^p d\phi \\ &\leq C||f||_{H^p}^p \quad \text{(Lemma 2)}. \end{aligned} \tag{6}$$

For $0 < t < \frac{1}{2}$, let $A(t) = A_1(t) + A_2(t)$, where

$$A_1(t) = \frac{1}{2\pi}\int_{0<|\theta|\leq t} \frac{1}{|1-(1-t)e^{i\theta}|^{cp}} \left|f\left(\frac{te^{i\theta}}{1-(1-t)e^{i\theta}}\right)\right|^p d\theta$$

and

$$A_2(t) = \frac{1}{2\pi}\int_{t<|\theta|\leq \pi} \frac{1}{|1-(1-t)e^{i\theta}|^{cp}} \left|f\left(\frac{te^{i\theta}}{1-(1-t)e^{i\theta}}\right)\right|^p d\theta.$$

Using (5) in $A_1(t)$ and since $t < |1-(1-t)e^{i\theta}|$, we have

$$\begin{aligned} A_1(t) &\leq C\frac{1}{2\pi}\int_{0<|\theta|\leq t} \frac{1}{t^{cp}} \left|Mf(e^{i\phi})\right|^p \frac{d\theta}{d\phi} d\phi \\ &\leq C\frac{1}{2\pi}\frac{1}{t^{cp-1}}\int_{0<|\theta|\leq t} \left|Mf(e^{i\phi})\right|^p d\phi \\ &\leq C\frac{1}{t^{cp-1}}||f||_{H^p}^p \quad \text{(Lemma 2)}. \end{aligned}$$

$f\left(\frac{te^{i\theta}}{1-(1-t)e^{i\theta}}\right) \leq C||f||_{H^p}$ as $\left|\frac{te^{i\theta}}{1-(1-t)e^{i\theta}}\right| < 1$ and when $t < |\theta| < \pi$, imply $|\theta| \leq |1-(1-t)e^{i\theta}|$. Hence for $A_2(t)$, we have

$$\begin{aligned} A_2(t) &\leq C\frac{1}{2\pi}\int_{t<|\theta|\leq \pi} \frac{1}{\theta^{cp}}||f||_{H^p}^p \, d\theta \\ &\leq C||f||_{H^p}^p \begin{cases} \frac{1}{|cp-1|}\frac{1}{t^{cp-1}} & \text{if } p \neq \frac{1}{c}, \\ \log 1/t & \text{if } p = \frac{1}{c}. \end{cases} \end{aligned}$$

Therefore, for $0 < t < \frac{1}{2}$,

$$A(t) \leq \begin{cases} C||f||_{H^p}^p \frac{1}{|cp-1|}\frac{1}{t^{cp-1}} & \text{if } p \neq \frac{1}{c}, \\ C(1+\log 1/t)||f||_{H^p}^p & \text{if } p = \frac{1}{c}. \end{cases} \tag{7}$$

Finally, combining inequalities (6), (7), together with (4), we obtain for $p = \frac{1}{c}$,

$$||\mathcal{P}^{b,c}||_{H^p} \leq ||f||_{H^p}\left(\int_0^{1/2}(1+\log 1/t)^c\lambda(t)\,dt + \int_{1/2}^1 \lambda(t)\,dt\right),$$

since $b+1 > c > 0$, convergence of the following integral

$$\int_0^{1/2}(1+\log 1/t)^c t^{c-1}(1-t)^{b-c}\,dt$$

imply

$$||\mathcal{P}^{b,c}||_{H^p} \leq C\frac{\Gamma(c)}{B(c,b+1-c)}||f||_{H^p},$$

and for $p \neq \frac{1}{c}$

$$\begin{aligned} ||\mathcal{P}^{b,c}||_{H^p} &\leq C||f||_{H^p}\left(\int_0^{1/2}\frac{1}{|cp-1|^{1/p}}t^{1/p-c}\lambda(t)\,dt + \int_{1/2}^1\lambda(t)\,dt\right) \\ &\leq C\max\left(\frac{1}{|cp-1|^{1/p}}, \frac{1}{2^{c-1/p}}\right)||f||_{H^p}\int_0^1 t^{1/p-1}(1-t)^{b-c}\,dt. \end{aligned}$$

If $p = \infty$, it is rather simple to find $\mathcal{P}^{b,c}$ is unbounded on H^∞ *i.e.* there exists $f_1 \in H^\infty$ such that $\mathcal{P}^{b,c}(f_1) \notin H^\infty$. Taking $f_1 \equiv 1 \in H^\infty$, from (1) we have

$$\begin{aligned} \mathcal{P}^{b,c}f_1(z) &= \sum_{n=0}^{\infty}\frac{1+b-c}{b+n}z^n \\ &= \frac{1+b-c}{b}F(b,1;b+1;z), \end{aligned}$$

which gives $\mathcal{P}^{b,c}(f_1) \notin H^\infty$, completes the proof. ∎

For our next result we consider the general version of $\mathcal{P}^{b,c}$ which is defined as below. If μ be a finite positive Borel measure on $(0,1]$, then

$$\mathcal{P}^{b,c}_\mu f(z) = \int_0^1 T_t f(z)\,d\mu(t), \quad \text{for each } z \in D.$$

Theorem 5. *Let μ be a finite positive Borel measure on $(0,1]$ and $b, c \in \mathbb{C}$ with $\mathrm{Re}\,(b+1) > \mathrm{Re}\,c > 0$. Then $\mathcal{P}^{b,c}_\mu$ is a bounded operator on $BMOA$ if and only if $\int_0^1 \frac{d\mu(t)}{t^c} < \infty$.*

Proof. Assume that $\int_0^1 \frac{d\mu(t)}{t^c} < \infty$. It is known [5] that $f \in BMOA$ if and only if there exist f_1 and f_2 analytic in the D with $\mathrm{Re}\,(f_1)$, $\mathrm{Re}\,(f_2)$ in L^∞, $f = f_1 + if_2 + f(0)$, $f_1(0) = f_2(0) = 0$ and $||f||_{BMOA} \sim ||\mathrm{Re}\,(f_1)||_{L^\infty} + ||\mathrm{Re}\,(f_2)||_{L^\infty} + |f(0)|$. Now, let $f \in BMOA$ posses such a decomposition. Define $g(z) = f(z)F(z)$. Since $F(z)$ is bounded on $|z| \leq 1$ for $\mathrm{Re}\,(b+1) > \mathrm{Re}\,c$, gives $g(z) \in BMOA$. Let $g = g_1 + ig_2$ with

g_j analytic in D such that $g_j(0)=0$ and $||g||_{BMOA} \sim ||\mathrm{Re}\,(g_1)||_{L^\infty} + ||\mathrm{Re}\,(g_2)||_{L^\infty}$. Since $|\frac{1}{1-(1-t)z}| \leq \frac{1}{t}$, We have

$$|\mathcal{P}_\mu^{b,c} f(z)| \leq C \int_0^1 \left| g\left(\frac{tz}{1-(1-t)z}\right)\right| \frac{d\mu(t)}{t^c}$$

is well defined for each $z \in D$. Obviously, $\mathrm{Re}\,(\mathcal{P}_\mu^{b,c}(f)) = \mathcal{P}_\mu^{b,c}(\mathrm{Re}\,(f))$. Hence for each j,

$$\begin{aligned} ||\mathcal{P}_\mu^{b,c}(f_j)||_{BMOA} &\leq C\{|\mathcal{P}_\mu^{b,c} f(0)| + ||\mathcal{P}^{b,c}(\mathrm{Re}\,(f_j))||_{L^\infty}\} \\ &\leq C\{|f(0)| + \int_0^1 \frac{d\mu(t)}{t^c} ||g||_{L^\infty}\}. \end{aligned}$$

The above inequality gives

$$\begin{aligned} ||\mathcal{P}_\mu^{b,c}(f)||_{BMOA} &\leq C|f(0)| + \int_0^1 \frac{d\mu(t)}{t^c} ||g||_{BMOA} \\ &\leq C\int_0^1 \frac{d\mu(t)}{t^c} ||f||_{BMOA}. \end{aligned}$$

Conversely, Assume that $\mathcal{P}_\mu^{b,c}$ is bounded on $BMOA$. Let $f(z) = \log \dfrac{1}{1-z}$ and $x \in [0,1)$. Then,

$$\mathcal{P}_\mu^{b,c} f(x) \leq C \log \frac{1}{1-x} ||\mathcal{P}_\mu^{b,c}||_{BMOA}. \tag{8}$$

Since $b+1>c$, $F(c-1, c-b-1; c; x)$ is bounded, continuous and positive on $[0,1]$, we can quickly obtain

$$\begin{aligned} \mathcal{P}_\mu^{b,c} f(x) &= \int_0^1 \frac{1}{(1-(1-t)x)^c} \log \frac{1-(1-t)x}{1-x} F\left(\frac{tx}{1-(1-t)x}\right) d\mu(t) \\ &\geq \frac{1}{2}\log \frac{1}{1-x} \int_{\sqrt{(1-x)}}^1 \frac{1}{(1-(1-t)x)^c} F\left(\frac{tx}{1-(1-t)x}\right) d\mu(t) \\ &\geq C \log \frac{1}{1-x} \int_{\sqrt{(1-x)}}^1 \frac{1}{t^c}\, d\mu(t). \end{aligned} \tag{9}$$

Combining (8) and (9), we find

$$\int_{\sqrt{(1-x)}}^1 \frac{1}{t^c}\, d\mu(t) \leq C||\mathcal{P}_\mu^{b,c}||_{BMOA}.$$

The desired conclusion follows if we take $x \to 1-$ in the above estimate. ■

In particular, if we take $\mu(t) = \lambda(t)dt$, then we have following result for $\mathcal{P}^{b,c}$.

Corollary 6. *Let* $b, c \in \mathbb{C}$ *be such that* $\mathrm{Re}\,(b+1) > \mathrm{Re}\,c > 0$. *Then* $\mathcal{P}^{b,c}$ *is unbounded operator from* $BMOA$ *to* $BMOA$ *i.e. there exists* $f \in BMOA$ *but* $\mathcal{P}^{b,c}(f) \notin BMOA$.

Proof. Let $f_2(z) = \log \frac{1}{1-z}$. We see that $f_2 \in BMOA$ (see [2]). Our aim is to show that $\mathcal{P}^{b,c}(f_2) \notin BMOA$. Since $BMOA \subset B^1$, to proof our aim it is suffices to show that $\mathcal{P}^{b,c}(f_2) \notin B^1$. From (3), we find

$$(\mathcal{P}^{b,c})' f_2(x) = \frac{1}{B} I',$$

where

$$I = \int_0^1 \frac{1}{(1-(1-t)x)^c} \log \frac{1-(1-t)x}{1-x} F\left(\frac{tx}{1-(1-t)x}\right) \lambda(t)\, dt$$

and

Since $t < 1-(1-t)x$ for $0 < t < 1$, $F(c-1, c-b-1; c; x)$ is bounded, continuous and positive on $[0,1]$ for $b+1 > c$ and using logarithm function is increasing, we have

$$\begin{aligned} I &\geq C \int_0^1 F\left(\frac{tx}{1-(1-t)x}\right) \log \frac{t}{1-x} t^{-1}(1-t)^{b-c}\, dt \\ &\geq -C \int_0^1 \log \frac{t}{1-x} t^{-1}(1-t)^{b-c}\, dt. \end{aligned}$$

Above inequality gives

$$(1-x)I' \geq -C \int_0^1 \log t\; t^{-1}(1-t)^{b-c}\, dt.$$

The integral $\int_0^1 \log t\; t^{-1}(1-t)^{b-c}\, dt$ is not finite, therefore it is clear that $(1-x)(\mathcal{P}^{b,c})' f_2(x)$ is also not finite, imply $\mathcal{P}^{b,c}(f_2) \notin B^1$ which completes the proof. ∎

Theorem 7. *Let* $a \in (0,\infty)$ *and* $b, c \in \mathbb{C}$. *Then for* $\mathrm{Re}\,(b+1) > \mathrm{Re}\,c > 0$, $\mathcal{P}^{b,c}$ *is a bounded operator on* B^a, *if and only if* $a \in (1,\infty)$.

Proof. Let $f \in B^a, a \in (1,\infty)$. It is easy to see that $|f(\phi_t(z))| \leq C||f||_{B^a}(1-|\phi_t(z)|)^{1-a}$. A simple calculation gives

$$|f(\phi_t(z))| \leq C||f||_{B^a} \left(\frac{|1-(1-t)z|}{(1-|z|)}\right)^{a-1}.$$

Using this in the integral representation of $\mathcal{P}^{b,c}$ given in (3) and since $F(z)$ is bounded on $|z| \leq 1$, we find

$$\begin{aligned} (1-|z|)^{a-1}|\mathcal{P}^{b,c} f(z)| &\leq \frac{(1-|z|)^{a-1}}{B} \int_0^1 \frac{|f(\phi_t(z))F(\phi_t(z))|}{|1-(1-t)z|^c} \lambda(t)\, dt \\ &\leq C \int_0^1 \frac{||f||_{B^a}}{|1-(1-t)z|^{c-a+1}} \lambda(t)\, dt \\ &\leq C||f||_{B^a} \int_0^1 \frac{\lambda(t)}{|1-(1-t)z|^{c-a+1}}\, dt. \qquad (10) \end{aligned}$$

$$\frac{1}{|1-(1-t)z|^{c-a+1}} \leq \begin{cases} \frac{1}{t^{c-a+1}} & \text{if } c-a+1 \geq 0, \\ \frac{1}{2^{c-a+1}} & \text{if } c-a+1 < 0. \end{cases}$$

As $a > 1$, above expression together with inequality (10), we have

$$(1-|z|)^{a-1}|\mathcal{P}^{b,c}f(z)| \leq C||f||_{B^a}.$$

Since $|\mathcal{P}^{b,c}f(0)| = |f(0)|$, last inequality gives $\mathcal{P}^{b,c}(f) \in B^a$. Conversely, unboundedness of $\mathcal{P}^{b,c}$ on B^1 is contained in the proof of last corollary (see also [2], Theorem 3.9(i)). Let $a \in (0,1)$ and $f_3 \equiv 1$. Then it is obivious that $f_1 \in B^a$. From the definition (1), we have

$$\mathcal{P}^{b,c}f_3(z) = \frac{1+b-c}{b}F(b,1;b+1;z).$$

In view of the well-known Gauss identity [1, 11]

$$F(a,b;c;z) = (1-z)^{c-a-b}F(c-a,c-b,c;z), \tag{11}$$

and using derivative formula of hypergeometric function, we obtain

$$(\mathcal{P}^{b,c})'f_3(z) = \frac{1+b-c}{b+1}(1-z)^{-1}F(1,b;b+2;z).$$

Now

$$\begin{aligned} (1-|z|)^a|(\mathcal{P}^{b,c})'f_3(z)| &= \frac{1+b-c}{b+1}(1-|z|)^a|(1-z)^{-1}F(1,b;b+2;z)| \\ &\geq \frac{1+b-c}{b+1}(1-|z|)^{a-1}|F(1,b;b+2;z)|. \end{aligned}$$

Since $F(1,b;b+2;z)$ is bounded on $|z| \leq 1$ and $a \in (0,1)$, imply $\mathcal{P}^{b,c}(f_3) \notin B^a$, which completes the proof. ∎

4 α-Cesáro operator

In this section we discuss the boundedness of the α- Cesáro operator $\mathcal{C}^\alpha$ from the Hardy space H^p to the Bergman space A^p for $p \in (0,1)$ by rewriting $\mathcal{C}^\alpha$ as Hadamard or convolution product. Although the result follows from ([12], Theorem 3.1(i)) here we give an alternate and simple proof.

The Hadamard product $f * g$ is defined by $(f*g)(z) = \sum_{n=0}^\infty a_n b_n z^n$, whenever $f(z) = \sum_{n=0}^\infty a_n z^n$ and $g(z) = \sum_{n=0}^\infty b_n z^n$. If f and g are analytic in the unit disc D, then

$$(f*g)(\rho z) = \frac{1}{2\pi}\int_0^{2\pi} f(\rho e^{it})g(ze^{-it})\,dt, \quad 0 < \rho < 1.$$

For $\alpha \in \mathbb{C}$ with Re $\alpha > -1$, the α- Cesáro operator (see [10] or [12]) is defined by

$$\mathcal{C}^\alpha f(z) := \sum_{n=0}^{\infty} \left(\frac{(1,n)}{(\alpha+2,n)} \sum_{k=0}^{n} \frac{(\alpha+1,n-k)}{(1,n-k)} a_k \right) z^n$$

We can rewrite the above formula of $\mathcal{C}^\alpha$ with the following expression.

$$\mathcal{C}^\alpha f(z) = G_1(z) * G_2(z),$$

where $G_1(z) = \dfrac{f(z)}{(1-z)^{\alpha+1}}$ and $G_2(z) = F(1,1;\alpha+2;z)$. Therefore,

$$\begin{aligned} \mathcal{C}^\alpha f(\rho z) &= \frac{1}{2\pi} \int_0^{2\pi} F(\rho e^{it}) G(z e^{-it})\, dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{f(\rho e^{it})}{(1-\rho e^{it})^{\alpha+1}} F(1,1;\alpha+2; z e^{-it})\, dt. \end{aligned} \tag{12}$$

Differentiation with respect to z in (12), gives

$$(\mathcal{C}^\alpha f(\rho z))' = \frac{1}{2\pi(\alpha+1)} \int_0^{2\pi} \frac{f(\rho e^{it})}{(1-\rho e^{it})^{\alpha+1}} F(2,2;\alpha+3; z e^{-it}) e^{-it}\, dt.$$

Using the Gauss identity (11) and since $F(\alpha+1,\alpha+1;\alpha+3;z)$ is bounded on $|z| \leq 1$ if $\operatorname{Re}\alpha < 1$, we have

$$\begin{aligned} |\rho(\mathcal{C}^\alpha)' f(\rho z)| &\leq \frac{1}{2\pi(\alpha+1)} \int_0^{2\pi} \left| \frac{f(\rho e^{it}) F(2,2;\alpha+3; z e^{-it}) e^{-it}}{(1-\rho e^{it})^{\alpha+1}} \right| dt \\ &\leq C \int_0^{2\pi} \left| \frac{f(\rho e^{it})}{(1-\rho e^{it})^{\alpha+1}} \right| |1 - z e^{-it}|^{\alpha-1}\, dt \\ &\leq C \int_0^{2\pi} \frac{(1-|z|)^{\alpha-1}}{(1-\rho)^{\alpha+1}} |f(\rho e^{it})|\, dt. \end{aligned} \tag{13}$$

We will use above inequality to prove our next theorem.

Theorem 8. *Let $\alpha \in \mathbb{C}$ with $-1 < \operatorname{Re}\alpha < 1$ and $0 < q \leq p < 1$. If $f \in H^p$, then $\mathcal{C}^\alpha(f) \in A^q$.*

Proof. (i) Let $0 < q < p < 1$. If $f \in H^p$, then ([3], Theorem 5.9) gives

$$M_1(r,f) \leq C \frac{1}{(1-r)^{1/p-1}} \quad . \tag{14}$$

Assume that $\operatorname{Re}\alpha < 1$, taking $|z| = r = \rho$ and $z = re^{i\theta}$ in the last inequality of (13), a simple calculations show that

$$\begin{aligned} \int_0^1 \rho(1-\rho)^q M_q^q(\rho^2, (\mathcal{C}^\alpha)' f)\, d\rho &\leq \int_0^1 (1-\rho)^{-q} M_1^q(\rho, f)\, d\rho \\ &\leq \int_0^1 (1-\rho)^{\frac{-q}{p}}\, d\rho \quad \text{(by (14))} \\ &< \infty \quad (\text{ as } q < p). \end{aligned}$$

Since

$$\int_0^1 (1-\rho)^q M_q^q(\rho, f)\, d\rho \le C \int_0^1 (1-\rho)^q \rho M_q^q(\rho^2, f)\, d\rho$$

then

$$\int_0^1 (1-\rho)^q M_q^q(\rho, (\mathcal{C}^\alpha)' f)\, d\rho < \infty.$$

Taking $\alpha = 0$ and $p = q$ in Lemma 3, the above inequality gives

$$\int_0^1 M_q^q(r, \mathcal{C}^\alpha f)\, dr < \infty.$$

That is $\mathcal{C}^\alpha(f) \in A^q$.

(ii) If $0 < q = p < 1$ and $f \in H^p$, then

$$\int_0^1 (1-r)^{-p} M_1^p(r, f)\, dr < \infty,$$

by Hardy-Littlewood theorem ([3], Theorem 5.11). Similar to above case, we have

$$\int_0^1 r(1-r)^p M_p^p(r^2, (\mathcal{C}^\alpha)' f)\, dr < \int_0^1 (1-\rho)^{-p} M_1^p(r, f)\, dr < \infty.$$

Desired result follows as before. This complete the proof. ∎

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