Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Filomat **25:4** (2011), 99–108 DOI: 10.2298/FIL1104099E

# ON $I$-ALEXANDROFF AND $I_g$-ALEXANDROFF IDEAL TOPOLOGICAL SPACES

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**Abstract**

In this paper, the notions of $I$-Alexandroff and $I_g$-Alexandroff ideal topological spaces are introduced and studied. Also, characterizations and properties of $I$-Alexandroff and $I_g$-Alexandroff ideal topological spaces are investigated.

## 1 Introduction and preliminaries

Alexandroff spaces were first studied by Alexandroff [2]. It is a topological space in which arbitrary intersection of open sets is open. Equivalently, each singleton has a minimal neighborhood base. Alexandroff spaces have important attentions because of their use in digital topology [10], [14]. In 1998, Arenas et al. [3] introduced and studied generalized Alexandroff topological spaces. Moreover, in 2000, Arenas et al. [4] studied some weak separation axioms related with Alexandroff topological spaces. It is known that any intersection of open sets is $g$-open in a generalized Alexandroff topological spaces [3]. It is shown in [3] that any $T_{\frac{1}{2}}$ $g$-Alexandroff space is locally path-connected, first countable, orthocompact and that in any $T_{\frac{1}{2}}$ $g$-Alexandroff space, the notions of path-connectedness, connectedness and chain-connectedness coincide. Furthermore, Arenas et al. [3] introduced that in digital topology, Khalimsky line [9, 11], various problems are related with generalized Alexandroff spaces. In this paper, the notions of $I$-Alexandroff and $I_g$-Alexandroff ideal topological spaces are introduced and studied. Moreover, characterizations and properties of $I$-Alexandroff and $I_g$-Alexandroff ideal topological spaces are discussed.

In this paper, $(X, \tau)$ or $(Y, \sigma)$ denote a topological space with no separation properties assumed. $Cl(S)$ and $Int(S)$ denote the closure and interior of $S$ in $(X, \tau)$, respectively for a subset $S$ of a topological space $(X, \tau)$. An ideal $I$ on a topological space $(X, \tau)$ is a nonempty collection of subsets of $X$ which satisfies

(1) $S \in I$ and $K \subset S$ implies $K \in I$,

(2) $S \in I$ and $K \in I$ implies $S \cup K \in I$ [12].

2010 *Mathematics Subject Classifications.* 54A05, 54C08, 54D15, 54A10.
*Key words and Phrases.* $I$-Alexandroff ideal space, $I_g$-Alexandroff ideal space, $I_g^*$-closed set.
Received: December 12, 2010; Revised April 4, 2011
Communicated by Ljubiša D.R. Kočinac

For a topological space $(X, \tau)$ with an ideal $I$ on $X$, if $P(X)$ is the set of all subsets of $X$, a set operator $(.)^* : P(X) \to P(X)$, said to be a local function [12] of $S \subset X$ with respect to $\tau$ and $I$ is defined as follows:

$S^*(I, \tau) = \{x \in X : N \cap S \notin I$ for every $N \in \tau(x)\}$ where $\tau(x) = \{N \in \tau : x \in N\}$.

A Kuratowski closure operator $Cl^*(.)$ for a topology $\tau^*(I, \tau)$, said to be the $\star$-topology and finer than $\tau$, is defined by $Cl^*(S) = S \cup S^*(I, \tau)$ [8]. We will briefly write $S^*$ for $S^*(I, \tau)$ and $\tau^*$ or $\tau^*(I)$ for $\tau^*(I, \tau)$. For an ideal $I$ on $X$, $(X, \tau, I)$ is said to be an ideal topological space or briefly an ideal space. For an ideal topological space $(X, \tau, I)$, the collection $\{S \backslash N : S \in \tau$ and $N \in I\}$ is a basis for $\tau^*$ [8].

A subset $S$ of a topological space $(X, \tau)$ is said to be $g$-closed in $(X, \tau)$ [13] if $Cl(S) \subset V$ whenever $S \subset V$ and $V$ is open in $(X, \tau)$. A subset $S$ of a topological space $(X, \tau)$ is called $g$-open in $(X, \tau)$ [13] if $X \backslash S$ is $g$-closed. A subset $S$ of an ideal topological space $(X, \tau, I)$ is said to be $\star$-dense in itself [7] if $S \subset S^*$.

**Definition 1.** A subset $S$ of an ideal topological space $(X, \tau, I)$ is said to be

(1) $I_g$-closed [6] in $(X, \tau, I)$ if $S^* \subset N$ whenever $S \subset N$ and $N$ is open in $(X, \tau, I)$.

(2) $I_g$-open [6] in $(X, \tau, I)$ if $X \backslash S$ is $I_g$-closed.

**Theorem 1.** [15] *For a subset $S$ of an ideal topological space $(X, \tau, I)$, $S$ is $I_g$-open if and only if $N \subset Int^*(S)$ whenever $N \subset S$ and $N$ is closed in $X$.*

**Theorem 2.** [15] *For an ideal topological space $(X, \tau, I)$ and $S \subset X$, the following properties are equivalent:*

*(1) $S$ is $I_g$-closed,*

*(2) $Cl^*(S) \subset N$ whenever $S \subset N$ and $N$ is open in $X$.*

**Definition 2.** A topological space $(X, \tau)$ is said to be

(1) Alexandroff [2] if any intersection of open sets is open.

(2) generalized Alexandroff [3] if any intersection of open sets is $g$-open.

## 2 $I$-Alexandroff and $I_g$-Alexandroff ideal spaces

**Definition 3.** An ideal topological space $(X, \tau, I)$ is said to be $I$-Alexandroff if any intersection of open sets is $\star$-open.

**Theorem 3.** *Let $(X, \tau, I)$ be an ideal topological space. The following properties are equivalent:*

*(1) $(X, \tau, I)$ is $I$-Alexandroff,*

*(2) any union of closed sets in $(X, \tau, I)$ is $\star$-closed.*

*Proof.* It follows from the fact that the complement of a $\star$-open set is $\star$-closed. □

**Definition 4.** An ideal topological space $(X, \tau, I)$ is called $I_g$-Alexandroff if any intersection of open sets in $(X, \tau, I)$ is $I_g$-open.

**Theorem 4.** *Let $(X, \tau, I)$ be an ideal topological space. If there exists a point $x \in X$ such that $x$ has only $\star$-neighborhood which is $X$ itself, then $(X, \tau, I)$ is an $I_g$-Alexandroff ideal space.*

*Proof.* Suppose that there exists a point $x \in X$ such that $x$ has only $\star$-neighborhood which is $X$ itself.

Let $\{K_i \colon i \in I\}$ is a family of open sets in $(X, \tau, I)$ for each $i \in I$. We take $K = \bigcap_{i \in I} K_i$. Let $M \subset K$ and $M$ be a closed set.

Suppose that $M = \emptyset$. Then we have $M \subset Int^*(K)$.

Suppose that $M \neq \emptyset$. If $M = X$, then $M \subset K = X$. Hence, $M \subset Int^*(K)$. If $M \neq X$, then $X \backslash M$ is an open set. It follows that $x \notin X \backslash M$ and then $x \in M$. Since $M \subset K$, then $x \in K_i$ for each $i \in I$. Since $x$ has only $\star$-neighborhood which is $X$ itself, then $K_i = X$ for each $i \in I$. Moreover, we have $K = X$ and then $M \subset Int^*(K)$. Hence, $K$ is $I_g$-open.

Thus, $(X, \tau, I)$ is an $I_g$-Alexandroff ideal space. □

**Theorem 5.** *Let $(X, \tau, I)$ be an ideal topological space. If $(X, \tau, I)$ is an $I$-Alexandroff ideal space, then $(X, \tau, I)$ is $I_g$-Alexandroff.*

*Proof.* The proof follows from the fact that any $\star$-open set is $I_g$-open. □

**Remark 1.** The reverse implication of Theorem 5 is not true in general as shown in the following example:

**Example 1.** Suppose that $R$ is the set of real numbers and $\tau = \{(-\frac{1}{n}, \frac{1}{n}) : n \in N \backslash \{0\}\} \cup \{R, \varnothing\}$ where $N$ is the set of naturel numbers. Let $I = \{\varnothing, \{3\}\}$. Then the ideal topological space $(R, \tau, I)$ is $I_g$-Alexandroff by Theorem 4 but $(R, \tau, I)$ is not $I$-Alexandroff. Furthermore, suppose that $J = \{\varnothing\}$. Arenas et al. [3] show that the topological space $(R, \tau)$ is $g$-Alexandroff but $(R, \tau)$ is not Alexandroff. Therefore, the ideal topological space $(R, \tau, J)$ is $I_g$-Alexandroff but $(R, \tau, J)$ is not $I$-Alexandroff.

**Definition 5.** A subset $S$ of an ideal topological space $(X, \tau, I)$ is called

(1) $I_g^*$-closed in $(X, \tau, I)$ if $Cl(S) \subset N$ whenever $S \subset N$ and $N$ is $\star$-open in $(X, \tau, I)$.

(2) $I_g^*$-open in $(X, \tau, I)$ if $X \backslash S$ is $I_g^*$-closed.

**Remark 2.** Let $(X, \tau, I)$ be an ideal topological space. The following diagram holds for a subset $S$ of $X$:

$$\begin{array}{ccccc} I_g^*\text{-open} & \longrightarrow & g\text{-open} & \longrightarrow & I_g\text{-open} \\ \uparrow & & & \nearrow & \\ \text{open} & \longrightarrow & \star\text{-open} & & \end{array}$$

None of these implications is reversible as shown in the following examples and in [8].

**Example 2.** Let $X = \{a, b, c, d\}$ and $\tau = \{X, \varnothing, \{a\}, \{b, c\}, \{a, b, c\}\}$ and $I = \{\varnothing, \{a\}, \{d\}, \{a, d\}\}$. Then the set $\{c\}$ is $I_g^*$-open but it is nether open nor $\star$-open. The set $\{b, c, d\}$ is $\star$-open but it is not $g$-open. The set $\{a, c\}$ is $I_g$-open but it is not $\star$-open.

**Example 3.** Let $X = \{a, b, c, d\}$ and $\tau = \{X, \varnothing, \{a\}, \{a, b\}, \{c, d\}, \{a, c, d\}\}$ and $I = \{\varnothing, \{a\}, \{d\}, \{a, d\}\}$. Then the set $\{a, b, d\}$ is $g$-open but it is not $I_g^*$-open.

**Theorem 6.** *For a subset $S$ of an ideal topological space $(X, \tau, I)$, $S$ is $I_g^*$-open if and only if $N \subset Int(S)$ whenever $N \subset S$ and $N$ is $\star$-closed in $(X, \tau, I)$.*

*Proof.* Let $S$ be an $I_g^*$-open set in $(X, \tau, I)$. Suppose that $N \subset S$ and $N$ is $\star$-closed in $(X, \tau, I)$. It follows that $X \backslash S \subset X \backslash N$ and $X \backslash N$ is $\star$-open in $(X, \tau, I)$. Since $X \backslash S$ is $I_g^*$-closed, then $Cl(X \backslash S) \subset X \backslash N$. We have $Cl(X \backslash S) = X \backslash Int(S) \subset X \backslash N$. Thus, $N \subset Int(S)$. The converse is similar. □

**Theorem 7.** *Let $(X, \tau, I)$ be an ideal topological space. The following properties are equivalent:*

*(1) $(X, \tau, I)$ is an $I_g$-Alexandroff ideal space,*

*(2) Any intersection of $I_g^*$-open sets in $(X, \tau, I)$ is $I_g$-open.*

*Proof.* (1) ⇒ (2) : Let $(X, \tau, I)$ be an $I_g$-Alexandroff ideal space. Suppose that $\{S_i : i \in I\}$ is a family of $I_g^*$-open sets. We take $S = \underset{i \in I}{\cap} S_i$. Let $K \subset X$ be a closed set and $K \subset S$. We have $K \subset S_i$ for each $i \in I$. Since $S_i$ is $I_g^*$-open set for every $i \in I$, then $K \subset Int(S_i)$ for each $i \in I$. We take $M = \underset{i \in I}{\cap} Int(S_i)$. Since $(X, \tau, I)$ is an $I_g$-Alexandroff ideal space, then $M = \underset{i \in I}{\cap} Int(S_i)$ is $I_g$-open. Since $M = \underset{i \in I}{\cap} Int(S_i)$ is $I_g$-open and $K \subset M$, then $K \subset Int^*(M)$. Hence, $Int^*(M) \subset Int^*(S)$ and thus, $K \subset Int^*(S)$. It follows that $S$ is $I_g$-open.

(2) ⇒ (1) : Suppose that any intersection of $I_g^*$-open sets in $(X, \tau, I)$ is $I_g$-open. Since every open set is $I_g^*$-open by Remark 2, it follows from (2) that any intersection of open sets in $(X, \tau, I)$ is $I_g$-open. Thus, $(X, \tau, I)$ is an $I_g$-Alexandroff ideal space. □

**Theorem 8.** *Let $(X, \tau, I)$ be an ideal topological space. The following properties are equivalent:*

*(1) $(X, \tau, I)$ is an $I_g$-Alexandroff ideal space,*

*(2) any union of $I_g^*$-closed sets in $(X, \tau, I)$ is $I_g$-closed.*

*Proof.* It follows from Theorem 7. □

**Theorem 9.** *Let $(X, \tau, I)$ be an ideal topological space. The following properties are equivalent:*

*(1) $(X, \tau, I)$ is an $I_g$-Alexandroff ideal space,*

*(2) any union of closed sets in $(X, \tau, I)$ is $I_g$-closed.*

*Proof.* It follows from the fact that the complement of an $I_g$-open set is $I_g$-closed. □

**Theorem 10.** *Let $(X, \tau, I)$ be an ideal topological space and $S \subset X$. If $(X, \tau, I)$ is an $I$-Alexandroff ideal space, then $S$ is an $I$-Alexandroff ideal space.*

*Proof.* Let $(X, \tau, I)$ be an $I$-Alexandroff ideal space. Suppose that $\{K_i : i \in I\}$ is a family of open sets in $(S, \tau_S)$. We take $K = \underset{i\in I}{\cap} K_i$. It follows that $K_i = S \cap N_i$ where $N_i$ is open in $(X, \tau, I)$ for each $i \in I$. Therefore, we have

$$K = \underset{i\in I}{\cap} K_i = \underset{i\in I}{\cap} (S \cap N_i) = S \cap (\underset{i\in I}{\cap} N_i).$$

Since $(X, \tau, I)$ is an $I$-Alexandroff ideal space, then $\underset{i\in I}{\cap} N_i$ is $\star$-open in $(X, \tau, I)$. It follows that $K = S \cap (\underset{i\in I}{\cap} N_i)$ is $\star$-open in $S$. Thus, $S$ is an $I$-Alexandroff ideal space. □

**Theorem 11.** *Let $(X, \tau, I)$ be an ideal topological space. If $(X, \tau, I)$ is $T_1$ and an $I_g$-Alexandroff ideal space, then $(X, \tau, I)$ is a discrete ideal space with respect to $\tau^*$.*

*Proof.* Let $(X, \tau, I)$ be $T_1$ and an $I_g$-Alexandroff ideal space. Suppose that $x \in X$. Since $(X, \tau, I)$ is a $T_1$ space, then for each $y \neq x$, there exists an open set $S_y$ containing $x$ such that $y \notin S_y$. It follows that $\{x\} = \underset{y\neq x}{\cap} S_y$. Since $(X, \tau, I)$ is a $T_1$-space, then $\{x\}$ is a closed set. Since $(X, \tau, I)$ is an $I_g$-Alexandroff ideal space, then $\{x\}$ is an $I_g$-open set. Since $\{x\} \subset \{x\}$, then we have $\{x\} \subset Int^*(\{x\})$. It follows that $\{x\}$ is $\star$-open in $(X, \tau, I)$. Thus, $(X, \tau, I)$ is a discrete ideal space with respect to $\tau^*$. □

**Theorem 12.** *Let $(X, \tau, I)$ be an ideal topological space. If $(X, \tau, I)$ is a discrete ideal space with respect to $\tau^*$, then $(X, \tau, I)$ is an $I_g$-Alexandroff ideal space.*

*Proof.* Let $(X, \tau, I)$ be a discrete ideal space with respect to $\tau^*$. Suppose that $\{K_i : i \in I\}$ is a family of open sets in $(X, \tau, I)$. It follows that $\underset{i\in I}{\cap} K_i$ is $\star$-open in $(X, \tau, I)$. By Remark 2, $\underset{i\in I}{\cap} K_i$ is $I_g$-open. Hence, $(X, \tau, I)$ is an $I_g$-Alexandroff ideal space. □

**Remark 3.** The following example shows that the reverse of Theorem 11 is not true in general:

**Example 4.** Suppose that $R$ is the set of real numbers and $\tau = \{(-\frac{1}{n}, \frac{1}{n}) : n \in N\backslash\{0\}\} \cup \{R, \varnothing\}$ where $N$ is the set of naturel numbers. Let $I = P(X)$ which is the power set of $X$. Then $(X, \tau, I)$ is a discrete ideal space with respect to $\tau^*$ but $(X, \tau, I)$ is not a $T_1$-space.

**Definition 6.** [1] Let $(X, \tau, I)$ be an ideal topological space. $(X, \tau, I)$ is said to be an $F^*$-space if every open subset of $(X, \tau, I)$ is $\star$-closed.

**Theorem 13.** *Let $(X, \tau, I)$ be an ideal topological space. If $(X, \tau, I)$ is a $T_1$ and $F^*$-space, then $(X, \tau, I)$ is a discrete ideal space with respect to $\tau^*$.*

*Proof.* Suppose that $(X, \tau, I)$ is a $T_1$ and $F^*$-space. Since $(X, \tau, I)$ is a $T_1$ space, then $\{x\}$ is a closed set for every $x \in X$. Since $(X, \tau, I)$ is an $F^*$-space, then $\{x\}$ is a $\star$-open set for every $x \in X$. It follows that $(X, \tau, I)$ is a discrete ideal space with respect to $\tau^*$. □

**Theorem 14.** [15] *For an ideal topological space $(X, \tau, I)$, every subset of $X$ is $I_g$-closed if and only if every open set is $\star$-closed.*

**Theorem 15.** *Let $(X, \tau, I)$ be an ideal topological space. The following properties are equivalent:*
*(1) $(X, \tau, I)$ is an $F^*$-space,*
*(2) every subset of $(X, \tau, I)$ is an $I_g$-closed set.*

*Proof.* It follows by Theorem 14. □

**Theorem 16.** *Let $(X, \tau, I)$ be an ideal topological space. If $(X, \tau, I)$ is an $F^*$-space, then $(X, \tau, I)$ is an $I_g$-Alexandroff ideal space.*

*Proof.* Suppose that $(X, \tau, I)$ is an $F^*$-space. By Theorem 15, every subset of $(X, \tau, I)$ is an $I_g$-closed set. It follows that $(X, \tau, I)$ is an $I_g$-Alexandroff space. □

**Definition 7.** [5] A topological space $(X, \tau)$ is said to be an $R_0$-space if $Cl(\{x\}) \subset U$ for each $x \in X$ and each open set $U$ with $x \in U$.

**Theorem 17.** *Let $(X, \tau, I)$ be an ideal topological space. If $(X, \tau, I)$ is an $R_0$ and $I_g$-Alexandroff ideal space, then $(X, \tau, I)$ is an $F^*$-space.*

*Proof.* Let $(X, \tau, I)$ be an $R_0$ and $I_g$-Alexandroff ideal space. Suppose that $S \subset X$ is an open set. Since $(X, \tau, I)$ is an $R_0$ space, then we have $Cl(\{x\}) \subset S$ for every $x \in S$. It follows that

$$S = \underset{x \in S}{\cup} Cl(\{x\}).$$

Since $(X, \tau, I)$ is an $I_g$-Alexandroff ideal space, then $S$ is an $I_g$-closed set. Since $S \subset S$ and $S$ is $I_g$-closed, then we have $Cl^*(S) \subset S$. It follows that $S$ is $\star$-closed. Thus, $(X, \tau, I)$ is an $F^*$-space. □

**Theorem 18.** *Let $(X, \tau, I)$ be an ideal topological space and $M \subset X$. If $(X, \tau, I)$ is an $I_g$-Alexandroff ideal space and $M$ is closed, then $M$ is an $I_g$-Alexandroff ideal space.*

*Proof.* Let $(X, \tau, I)$ be an $I_g$-Alexandroff space and $M \subset X$ be a closed set. Suppose that $\{S_i : i \in I\}$ is a family of open sets in $(M, \tau_M)$. We take $S = \underset{i \in I}{\cap} S_i$. It follows that $S_i = M \cap K_i$ where $K_i$ is an open set $(X, \tau, I)$ for each $i \in I$. Let $N \subset M$ be a closed set in $(M, \tau_M)$ and $N \subset S$. It follows that $N$ is a closed set in $(X, \tau, I)$ and $N \subset \underset{i \in I}{\cap} K_i$. Since $(X, \tau, I)$ is an $I_g$-Alexandroff ideal space, then we have $N \subset Int^*(\underset{i \in I}{\cap} K_i)$. Also, we have

$$M \cap Int^*(\underset{i \in I}{\cap} K_i) \subset S.$$

Since $M \cap Int^*(\cap_{i\in I} K_i)$ is a $\star$-open set in $M$, then $N \subset Int^*_M(S)$. It follows that $S$ is $I_g$-open in $M$. Hence, $M$ is an $I_g$-Alexandroff ideal space. □

**Remark 4.** Arenas et al. [3] show that any subset of a generalized Alexandroff space $(X, \tau)$ need not be a generalized Alexandroff space. So, for the ideal $J = \{\varnothing\}$ and hence for any ideal $I$ on $X$, any subset of an $I_g$-Alexandroff ideal space $(X, \tau, I)$ need not be an $I_g$-Alexandroff ideal space.

# 3 The relationships

**Definition 8.** A function $f : (X, \tau, I) \to (Y, \sigma, J)$ is said to be $\star$-closed if $f(A)$ is $\star$-closed in $(Y, \sigma, J)$ for every $\star$-closed subset $A$ of $(X, \tau, I)$.

**Theorem 19.** *Let $f : (X, \tau, I) \to (Y, \sigma, J)$ be a continuous and $\star$-closed surjective function. If $(X, \tau, I)$ is an $I$-Alexandroff ideal space, then $(Y, \sigma, J)$ is an $I$-Alexandroff ideal space.*

*Proof.* Suppose that $f : (X, \tau, I) \to (Y, \sigma, J)$ is a continuous and $\star$-closed function. Let $(X, \tau, I)$ be an $I$-Alexandroff ideal space. Suppose that $\{M_i : i \in I\}$ is a family of closed sets in $(Y, \sigma, J)$. Since $f : (X, \tau, I) \to (Y, \sigma, J)$ is continuous, then $N = \underset{i\in I}{\cup} f^{-1}(M_i)$ is a $\star$-closed set in $(X, \tau, I)$. We take $M = \underset{i\in I}{\cup} M_i$. Since $f : (X, \tau, I) \to (Y, \sigma, J)$ is a $\star$-closed function, then

$$f(N) = f(\underset{i\in I}{\cup} f^{-1}(M_i)) = M$$

is $\star$-closed. It follows that $(Y, \sigma, J)$ is an $I$-Alexandroff ideal space. □

**Theorem 20.** *Let $(X, \tau, I)$ be an ideal topological space and $S \subset X$ be $I_g$-closed. If $f : (X, \tau, I) \to (Y, \sigma, J)$ is a continuous and $\star$-closed function, then $f(S)$ is an $I_g$-closed set in $Y$*

*Proof.* Suppose that $S \subset X$ is a $I_g$-closed set and $f : (X, \tau, I) \to (Y, \sigma, J)$ is a continuous and $\star$-closed function. Let $f(S) \subset K$ where $K$ is open in $(Y, \sigma, J)$. It follows that $S \subset f^{-1}(K)$. Since $f : (X, \tau, I) \to (Y, \sigma, J)$ is a continuous function and $S$ is an $I_g$-closed set, then we have $Cl^*(S) \subset f^{-1}(K)$. Moreover, we have $f(Cl^*(S)) \subset f(f^{-1}(K)) \subset K$. Since $f$ is a $\star$-closed function, then

$$Cl^*(f(S)) \subset Cl^*(f(Cl^*(S))) = f(Cl^*(S)) \subset K.$$

It follows that $Cl^*(f(S)) \subset K$ and hence $f(S)$ is an $I_g$-closed set in $(Y, \sigma, J)$. □

**Theorem 21.** *Let $f : (X, \tau, I) \to (Y, \sigma, J)$ be a continuous and $\star$-closed surjective function. If $(X, \tau, I)$ is an $I_g$-Alexandroff ideal space, then $(Y, \sigma, J)$ is an $I_g$-Alexandroff ideal space.*

*Proof.* Suppose that $f : (X, \tau, I) \to (Y, \sigma, J)$ is a continuous and $\star$-closed surjective function. Let $(X, \tau, I)$ be an $I_g$-Alexandroff ideal space and $\{M_i : i \in I\}$ be a family of closed sets in $(Y, \sigma, J)$. Since $f : (X, \tau, I) \to (Y, \sigma, J)$ is a continuous function, then $K = \underset{i\in I}{\cup} f^{-1}(M_i)$ is an $I_g$-closed set in $(X, \tau, I)$. We take $M = \underset{i\in I}{\cup} M_i$. It follows from Theorem 20 that, $f(K) = f(\underset{i\in I}{\cup} f^{-1}(M_i)) = M$ is an $I_g$-closed set. Thus, $(Y, \sigma, J)$ is an $I_g$-Alexandroff ideal space. □

**Theorem 22.** [15] *Let $(X, \tau, I)$ be a $T_1$ ideal topological space and $A \subset X$. If $A$ is an $I_g$-closed set in $(X, \tau, I)$, then $A$ is $\star$-closed.*

**Theorem 23.** *For a $T_1$ ideal topological space $(X, \tau, I)$, the following properties are equivalent:*
*(1) $(X, \tau, I)$ is an $I$-Alexandroff ideal space,*
*(2) $(X, \tau, I)$ is an $I_g$-Alexandroff ideal space.*

*Proof.* Since any $I_g$-closed set is $\star$-closed in a $T_1$ ideal topological space $(X, \tau, I)$, then by Remark 2 and Theorem 22, $(X, \tau, I)$ is an $I$-Alexandroff ideal space if and only if $(X, \tau, I)$ is an $I_g$-Alexandroff ideal space. □

**Theorem 24.** [15] *Let $(X, \tau, I)$ be an ideal topological space and $A \subset X$. If $A$ is $\star$-dense in itself and $I_g$-closed in $(X, \tau, I)$, then $A$ is $g$-closed.*

**Theorem 25.** *Let $(X, \tau, I)$ be an ideal topological space. Suppose that every subset of $(X, \tau, I)$ is $\star$-dense in itself. Then the following properties are equivalent:*
*(1) $(X, \tau, I)$ is an $I_g$-Alexandroff ideal space,*
*(2) $(X, \tau, I)$ is a generalized Alexandroff space.*

*Proof.* Since every subset is $\star$-dense in itself, then by Remark 2 and Theorem 24, $(X, \tau, I)$ is an $I_g$-Alexandroff ideal space if and only if $(X, \tau, I)$ is a generalized Alexandroff space. □

**Theorem 26.** [6] *Let $(X, \tau, I)$ be an ideal topological space where $I = \{\oslash\}$ and $A \subset X$. Then $A$ is $I_g$-closed if and only if $A$ is $g$-closed.*

**Theorem 27.** *For an ideal topological space $(X, \tau, I)$ where $I = \{\oslash\}$, the following properties are equivalent:*
*(1) $(X, \tau, I)$ is an $I_g$-Alexandroff ideal space,*
*(2) $(X, \tau, I)$ is a generalized Alexandroff space.*

*Proof.* Let $(X, \tau, I)$ be an ideal topological space where $I = \{\oslash\}$ and $S \subset X$. Since $S$ is an $I_g$-closed set if and only if $S$ is a $g$-closed set by Theorem 26, then $(X, \tau, I)$ is an $I_g$-Alexandroff ideal space if and only if $(X, \tau, I)$ is a generalized Alexandroff space. □

**Theorem 28.** *Let $(X, \tau, I)$ be an ideal topological space and $I = \{\oslash\}$. Then the following properties are equivalent:*
*(1) $(X, \tau, I)$ is an Alexandroff space,*
*(2) $(X, \tau, I)$ is an $I$-Alexandroff ideal space.*

*Proof.* Since $I = \{\oslash\}$, then we have $\tau = \tau^*$. It follows that $(X, \tau, I)$ is an Alexandroff space if and only if $(X, \tau, I)$ is an $I$-Alexandroff ideal space. □

**Acknowledgement.** I would like to express my sincere gratitude to the referees.

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