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Summation formulae Involving Harmonic Numbers

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Abstract

Several summation formulae for finite and infinite series involving the classical harmonic numbers are presented.

The classical harmonic numbers are defined by

$$H_0 = 0 \quad \text{and} \quad H_m = \sum_{k=1}^{m} \frac{1}{k} \quad \text{for} \quad m \in \mathbb{N}.$$

They have interesting applications in various fields, such as analysis, number theory, combinatorics, and computer science. Several important properties of these numbers can be found, for example, in [10, Sections 6.3 and 6.4].

There exist many elegant identities for finite sums involving harmonic numbers (cf. [4, 5, 9]). For instance, we have

$$\sum_{k=0}^{n} \binom{n}{k}^2 H_k = \binom{2n}{n} \big(2H_n - H_{2n}\big) \tag{1}$$

and

$$\sum_{k=1}^{m} (-1)^k \binom{m}{k} \frac{k(H_{n+k} - H_n)}{\binom{n+k}{k}} = \frac{m(m^2 - m - n^2)}{(m+n)^2(m+n-1)^2}. \tag{2}$$

Identity (1) is due to Paule–Schneider [11] and (2) was proved by Sofo [13].

Of special interest are series involving H_n and related expressions, which can be exemplified by

$$\sum_{n=1}^{\infty} \frac{H_n}{n^3} = \frac{\pi^4}{72} \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{H_n^2}{n^2} = \frac{17\pi^4}{360}.$$

The first formula is a classical result discovered by Euler in 1775. For their proofs and more series involving H_n, refer to [1, 2, 3, 9].

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Continuing the study of harmonic numbers, the aim of this paper is three fold. First, we offer new identities for sums involving H_n. Among others we prove the following counterpart of identitiy (2) due to Sofo

$$\sum_{k=1}^{m}(-1)^k\binom{m}{k}\frac{k(H_{n+k}-H_k)}{\binom{n+k}{k}}=\frac{mn(1-H_{m+n})}{(m+n)(m+n-1)}.$$

Second, for an arbitrary natural numbers n, we find the following infinite series expression (see Theorem 2.3):

$$\sum_{m=1}^{\infty}\frac{n!}{(m)_{n+1}}\Big\{H_{m+n}-H_n\Big\}=\sum_{m=0}^{\infty}\sum_{k=0}^{m}\binom{m}{k}\frac{(-1)^k}{(k+n+1)^2}=\frac{1}{n^2}.$$

Finally, we review an algebraic identity (see Theorem 3.1 and Theorem 3.2) whose special case results in

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\sum_{1\le j_1\le j_2\le\cdots\le j_\ell\le k}\prod_{i=1}^{\ell}\frac{1}{j_i}=\frac{1}{n^\ell}$$

which may be a natural extension of the identity proposed by Díaz-Barrero [7].

In order to assure accuracy of the results, the identities involving harmonic numbers displayed in this paper are verified through *Mathematica*. This can also be done by the computer algebra package *Sigma* developed by Schneider [12], which has been shown to be efficient in dealing with binomial–harmonic sums .

1 Finite Sums Involving Harmonic Numbers

The convolution formula of Chu–Vandermonde is one of the fundamental binomial identities, which may be restated as

$$\sum_{k=0}^{m}(-1)^k\binom{x+k}{k}\binom{m+y}{m-k}=\binom{y-x+m-1}{m}. \tag{3}$$

As has been done by Chu–DeDonno [4, 5] and Paule-Schneider [11], applying derivative operators to the known binomial identities may lead to summation formulae involving harmonic numbers. Following this approach, we shall derive, from the binomial convolution just displayed, several harmonic number identities, resembling those obtained recently by Sofo [13].

It is not hard to check that (3) can be rewritten in the equivalent form

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}\binom{x+k}{k}\Big/\binom{y+k}{k}=\frac{(y-x)_m}{(y+1)_m} \tag{4}$$

where the rising factorial is given by

$$(y)_0=1 \quad\text{and}\quad (y)_m=y(y+1)\cdots(y+m-1) \quad\text{for}\quad m\in\mathbb{N}.$$

Extending slightly the classical harmonic numbers, we define

$$H_0(x) = 0 \quad \text{and} \quad H_m(x) = \sum_{k=1}^{m} \frac{1}{x+k} \quad \text{for} \quad m \in \mathbb{N}.$$

Now computing the derivative with respect to x and then letting $y = n$, we get from (4) the following identity.

Theorem 1.1.

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} \binom{x+k}{k} \frac{H_k(x)}{\binom{n+k}{k}} = \frac{(n-x)_m}{(n+1)_m} \Big\{ H_{n-1}(-x) - H_{m+n-1}(-x) \Big\}.$$

Three special cases of this identity are worthwhile to examine in the sequel. Firstly for $x = 0$, the corresponding identity reads as

Corollary 1.2.

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} \frac{H_k}{\binom{n+k}{k}} = \frac{n}{m+n} \big\{ H_{n-1} - H_{m+n-1} \big\}. \tag{5}$$

Then for $x = 1$ and $n \geq 2$, we have from Theorem 1.1 the following one

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} (1+k) \frac{H_{k+1} - 1}{\binom{n+k}{k}} = \frac{n(n-1)}{(m+n)(m+n-1)} \big\{ H_{n-2} - H_{m+n-2} \big\}$$

which can easily be simplified through (4) into

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} \frac{(1+k)H_k}{\binom{n+k}{k}} = \frac{n(n-1)(H_{n-2} - H_{m+n-2}) - mn}{(m+n)(m+n-1)}. \tag{6}$$

The difference between (5) and (6) leads to the next formula.

Corollary 1.3 ($n \geq 2$)**.**

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} \frac{kH_k}{\binom{n+k}{k}} = \frac{mn(H_{m+n-2} - H_{n-1} - 1)}{(m+n)(m+n-1)}. \tag{7}$$

Finally for $x = 2$ and $n \geq 3$, the identity in Theorem 1.1 becomes

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} (1+k)(2+k) \frac{H_{k+2} - 3/2}{\binom{n+k}{k}} = \frac{2n(n-1)(n-2)(H_{n-3} - H_{m+n-3})}{(m+n)(m+n-1)(m+n-2)}$$

which simplifies again through (4) into

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} \frac{(1+k)(2+k)H_{k+1}}{\binom{n+k}{k}} = \frac{n(n-1)\{(2n-m-4) + 2(n-2)(H_{n-3} - H_{m+n-3})\}}{(m+n)(m+n-1)(m+n-2)}. \tag{8}$$

Observing the relation

$$(1+k)(2+k)H_{k+1} = k^2H_k + (2+3k)H_k + (2+k)$$

and then simplifying the sum

$$\mathrm{Eq}(8) - 2\mathrm{Eq}(6) - \mathrm{Eq}(7) - \mathrm{Eq}(4\colon x=0, y=n) - \mathrm{Eq}(4\colon x=1, y=n)$$

we find the next summation formula.

Corollary 1.4 ($n \geq 3$)**.**

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}\frac{k^2H_k}{\binom{n+k}{k}} = \frac{mn\{(2m-n)+(m-n)(H_{n-1}-H_{m+n-3})\}}{(m+n)(m+n-1)(m+n-2)}.$$

Alternatively, if applying the derivative operator with respect to y at $y=n$, then we get from (4) another identity.

Theorem 1.5.

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}\binom{x+k}{k}\frac{H_{n+k}}{\binom{n+k}{k}} = \frac{(n-x)_m}{(n+1)_m}\Big\{H_{m+n} - H_{m+n-1}(-x) + H_{n-1}(-x)\Big\}.$$

This is again quite a general formula. First, we state the case $x=0$ as

Corollary 1.6 (Sofo [13, Corollary 1])**.**

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}\frac{H_{n+k}}{\binom{n+k}{k}} = \frac{nH_{n-1}}{m+n} + \frac{n}{(m+n)^2}.$$

Then for $x=1$ and $n\geq 2$, the corresponding identity reads as

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}\frac{(k+1)H_{n+k}}{\binom{n+k}{k}} = \frac{n(n-1)H_{n-2}}{(m+n)(m+n-1)} + \frac{n(n-1)(2m+2n-1)}{(m+n)^2(m+n-1)^2}. \tag{9}$$

Their difference leads to the following identity.

Corollary 1.7 (Sofo [13, Corollary 2])**.**

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}\frac{kH_{n+k}}{\binom{n+k}{k}} = -\frac{mn(2m+2n-1)}{(m+n)^2(m+n-1)^2} - \frac{mnH_{n-1}}{(m+n)(m+n-1)}.$$

Finally for $x=2$ and $n\geq 3$, the identity corresponding to Theorem 1.5 yields

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}\frac{(1+k)(2+k)H_{n+k}}{\binom{n+k}{k}} = \frac{2n(n-1)(n-2)(H_{n-3}+H_{m+n}-H_{m+n-3})}{(m+n)(m+n-1)(m+n-2)}. \tag{10}$$

By combining (10) with Corollaries 1.6 and 1.7, we obtain the next identity.

Corollary 1.8 ($n \geq 3$)**.**

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}\frac{k^2 H_{n+k}}{\binom{n+k}{k}} = \frac{mn(m-n)H_{n-1}}{(m+n)(m+n-1)(m+n-2)} + \frac{mn(4m^3+6m^2n-9m^2-6mn+4m-2n^3+3n^2)}{(m+n)^2(m+n-1)^2(m+n-2)^2}.$$

2 Infinite Series Involving Harmonic Numbers

Considering the differences between the expressions given in Corollaries 1.2 and 1.6, Corollaries 1.3 and 1.7 as well as Corollaries 1.4 and 1.8, we find respectively the following three identities.

Proposition 2.1.

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}\frac{H_{n+k}-H_k}{\binom{n+k}{k}} = \frac{nH_{m+n}}{m+n}, \tag{11a}$$

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}\frac{k(H_{n+k}-H_k)}{\binom{n+k}{k}} = \frac{mn(1-H_{m+n})}{(m+n)(m+n-1)}, \tag{11b}$$

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}\frac{k^2(H_{n+k}-H_k)}{\binom{n+k}{k}} = \frac{mn(1+n-2m)+mn(m-n)H_{m+n}}{(m+n)(m+n-1)(m+n-2)}. \tag{11c}$$

They can be considered as counterparts of (2) due to Sofo [13].

In addition, the difference between the expressions given in in Theorems 1.1 and 1.5 reads as

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}\binom{x+k}{k}\frac{H_{n+k}-H_k(x)}{\binom{n+k}{k}} = \frac{(n-x)_m}{(n+1)_m}H_{m+n}. \tag{12}$$

For $i = 0, 1, 2$, the special cases corresponding to $x = i$ and $n = i+1$ give respectively the following three identities.

Proposition 2.2.

$$\sum_{k=0}^{m}\binom{m}{k}\frac{(-1)^k}{(k+1)^2} = \frac{H_{m+1}}{m+1}, \tag{13a}$$

$$\sum_{k=0}^{m}\binom{m}{k}\frac{(-1)^k}{(k+2)^2} = \frac{H_{m+2}-1}{(m+1)(m+2)}, \tag{13b}$$

$$\sum_{k=0}^{m}\binom{m}{k}\frac{(-1)^k}{(k+3)^2} = \frac{2H_{m+3}-3}{(m+1)(m+2)(m+3)}. \tag{13c}$$

Observe that

$$\frac{H_{m+1}}{(m+1)(m+2)}=\frac{1}{(m+2)^2}+\frac{H_{m+1}}{m+1}-\frac{H_{m+2}}{m+2}.$$

Dividing both sides of (13a) by $m+2$ and then summing over $0\le m<\infty$, we derive through the telescoping method the formula

$$\sum_{n=1}^{\infty}\frac{H_n}{n(n+1)}=\sum_{m=0}^{\infty}\frac{1}{2+m}\sum_{k=0}^{m}\binom{m}{k}\frac{(-1)^k}{(k+1)^2}=\frac{\pi^2}{6}\tag{14}$$

which is well–known and can be found, for example, in [1, Equation 3.44].

Analogously, we can deduce from (13b) the identity

$$\sum_{n=1}^{\infty}\frac{H_{n+1}-1}{n(n+1)}=\sum_{m=0}^{\infty}\sum_{k=0}^{m}\binom{m}{k}\frac{(-1)^k}{(k+2)^2}=1.\tag{15}$$

In view of

$$\frac{2H_{m+3}-3}{(m+1)(m+2)(m+3)}=\frac{H_{m+1}}{m+1}-2\frac{H_{m+2}}{m+2}+\frac{H_{m+3}}{m+3}$$

we get from (13c) further

$$\sum_{n=1}^{\infty}\frac{2H_{n+2}-3}{n(n+1)(n+2)}=\sum_{m=0}^{\infty}\sum_{k=0}^{m}\binom{m}{k}\frac{(-1)^k}{(k+3)^2}=\frac{1}{4}.\tag{16}$$

In general for $x=n-1$, we can show from (12) that there holds the identity

$$\sum_{k=0}^{m}\binom{m}{k}\frac{(-1)^k}{(k+n)^2}=\frac{(n-1)!}{(m+1)_n}\Big\{H_{m+n}-H_{n-1}\Big\}.\tag{17}$$

According to the partial fraction decomposition

$$\frac{(n-1)!}{(m+1)_n}=\sum_{i=1}^{n}\binom{n-1}{i-1}\frac{(-1)^{i-1}}{m+i}$$

the three formulae just exhibited suggest that

$$\frac{(n-1)!}{(m+1)_n}\Big\{H_{m+n}-H_{n-1}\Big\}=\sum_{i=1}^{n}(-1)^{i-1}\binom{n-1}{i-1}\frac{H_{m+i}}{m+i}.\tag{18}$$

Writing

$$H_{m+i}=H_{m+n}-\sum_{j=i+1}^{n}\frac{1}{m+j}$$

we see that the last equality is equivalent to the following one

$$\frac{(n-1)!}{(m+1)_n}H_{n-1}=\sum_{1\le i<j\le n}\binom{n-1}{i-1}\frac{(-1)^{i-1}}{(m+i)(m+j)}\tag{19}$$

which can be verified, without difficulty, by induction on n. Then (17) and (18) lead us to the following surprising formula.

Theorem 2.3.

$$\sum_{m=0}^{\infty}\frac{(n-1)!}{(m+1)_n}\Big\{H_{m+n}-H_{n-1}\Big\}=\sum_{m=0}^{\infty}\sum_{k=0}^{m}\binom{m}{k}\frac{(-1)^k}{(k+n)^2}=\frac{1}{(n-1)^2}.$$

This can be accomplished through the limiting process

$$\begin{aligned}\Xi &= \sum_{m=0}^{\infty}\frac{(n-1)!}{(m+1)_n}\Big\{H_{m+n}-H_{n-1}\Big\}\\ &= \lim_{M\to\infty}\sum_{m=0}^{M}\frac{(n-1)!}{(m+1)_n}\Big\{H_{m+n}-H_{n-1}\Big\}\\ &= \lim_{M\to\infty}\sum_{m=0}^{M}\sum_{k=1}^{n}(-1)^{k-1}\binom{n-1}{k-1}\frac{H_{m+k}}{m+k}\\ &= \lim_{M\to\infty}\sum_{k=1}^{n}(-1)^{k-1}\binom{n-1}{k-1}\sum_{m=0}^{M}\frac{H_{m+k}}{m+k}.\end{aligned}$$

Reformulating the last sum as

$$\sum_{m=0}^{M}\frac{H_{m+k}}{m+k}=\sum_{j=1}^{M}\frac{H_j}{j}-\sum_{j=1}^{k-1}\frac{H_j}{j}+\sum_{j=1}^{k}\frac{H_{j+M}}{j+M}$$

and then applying the binomial identity

$$\sum_{k=\ell}^{n}(-1)^k\binom{n-1}{k-1}=(-1)^{\ell}\binom{n-2}{n-\ell}\quad\text{where}\quad \ell=1,2,\cdots,n \tag{20}$$

we can evaluate the infinite series as follows:

$$\begin{aligned}\Xi &= \sum_{k=1}^{n}(-1)^k\binom{n-1}{k-1}\sum_{j=1}^{k-1}\frac{H_j}{j}=\sum_{1\le i\le j<k\le n}\binom{n-1}{k-1}\frac{(-1)^k}{ij}\\ &= \sum_{1\le i\le j<n}\binom{n-2}{j-1}\frac{(-1)^{j-1}}{ij}=\sum_{1\le i\le j<n}\binom{n-1}{j}\frac{(-1)^{j-1}}{i(n-1)}\\ &= \sum_{1\le i<n}\binom{n-2}{i-1}\frac{(-1)^{i-1}}{i(n-1)}=\sum_{1\le i<n}\binom{n-1}{i}\frac{(-1)^{i-1}}{(n-1)^2}=\frac{1}{(n-1)^2}.\end{aligned}$$ □

It should be pointed out that the infinite double series treated in this subsection cannot be evaluated by series rearrangement because they are not absolutely convergent!

3 Díaz-Barrero's Monthly Problem 11064

Díaz-Barrero [7] proposed the following monthly problem

$$\sum_{k=1}^{n}(-1)^{k+1}\binom{n}{k}\sum_{1\le i\le j\le k}\frac{1}{ij} = \frac{1}{n^2}. \tag{21}$$

Subsequently, Díaz-Barrero *et al* [8] extended it with an extra variable x. In fact, there exists another generalization, which may be reproduced as the following algebraic identity

Theorem 3.1 (Chu–Yan [6, Theorem 9])**.**

$$\sum_{k=0}^{n}(-1)^{k}\binom{n}{k}\binom{x+k}{k}^{-1}\sum_{0\le j_1\le j_2\le\cdots\le j_\ell\le k}\prod_{i=1}^{\ell}\frac{1}{x+j_i} = \frac{x}{(x+n)^{\ell+1}}.$$

Compared with Theorems 2 and 3 in [8], this theorem not only is more general with multiple ℓ-fold sum, but also has more compact expression. In particular when $x\to 1$ and $n\to n-1$, we deduce easily from this theorem the following identity, which may be considered as a natural extension of (21).

Corollary 3.2.

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\sum_{1\le j_1\le j_2\le\cdots\le j_\ell\le k}\prod_{i=1}^{\ell}\frac{1}{j_i} = \frac{1}{n^\ell}.$$

The proof of Theorem 3.1 included in [6] employed binomial inversions. In fact, we can prove it directly. For simplicity, denote by Ω the multiple sum displayed in Theorem 3.1. According to the binomial relation

$$\binom{n}{k}\binom{x+k}{k}^{-1} = \binom{x+n}{n-k}\binom{x+n}{n}^{-1}$$

the Ω-sum can be reformulated, by interchanging the summation order, as follows:

$$\Omega\binom{x+n}{n} = \sum_{0\le j_1\le j_2\le\cdots\le j_\ell\le n}\prod_{i=1}^{\ell}\frac{1}{x+j_i}\sum_{k=j_\ell}^{n}(-1)^k\binom{x+n}{n-k}.$$

Similar to (20), recall the following well–known binomial identity

$$\sum_{k=m}^{n}(-1)^k\binom{x+n}{n-k} = (-1)^m\binom{x+n-1}{n-m} = (-1)^m\binom{x+n}{n-m}\frac{x+m}{x+n}. \tag{22}$$

Then we can evaluate the innermost sum with respect to k in closed form

$$\sum_{k=j_\ell}^{n}(-1)^k\binom{x+n}{n-k} = (-1)^{j_\ell}\binom{x+n-1}{n-j_\ell} = (-1)^{j_\ell}\binom{x+n}{n-j_\ell}\frac{x+j_\ell}{x+n}$$

which permits us to reduce the Ω-sum accordingly to the following one

$$\Omega\binom{x+n}{n} = \frac{1}{x+n} \sum_{0\leq j_1\leq j_2\leq\cdots\leq j_{\ell-1}\leq n} \prod_{i=1}^{\ell-1} \frac{1}{x+j_i} \sum_{j_\ell=j_{\ell-1}}^{n} (-1)^{j_\ell}\binom{x+n}{n-j_\ell}.$$

Evaluating the last innermost sum with respect to j_ℓ again by (22)

$$\sum_{j_\ell=j_{\ell-1}}^{n} (-1)^{j_\ell}\binom{x+n}{n-j_\ell} = (-1)^{j_{\ell-1}}\binom{x+n}{n-j_{\ell-1}}\frac{x+j_{\ell-1}}{x+n}$$

we see that the Ω-sum becomes consequently

$$\Omega\binom{x+n}{n} = \frac{1}{(x+n)^2} \sum_{0\leq j_1\leq j_2\leq\cdots\leq j_{\ell-2}\leq n} \prod_{i=1}^{\ell-2} \frac{1}{x+j_i} \sum_{j_{\ell-1}=j_{\ell-2}}^{n} (-1)^{j_{\ell-1}}\binom{x+n}{n-j_{\ell-1}}.$$

Iterating this process ℓ-times leads us finally to the equality

$$\Omega\binom{x+n}{n} = \frac{1}{(x+n)^\ell}\binom{x+n-1}{n} = \frac{1}{(x+n)^\ell}\binom{x+n}{n}\frac{x}{x+n}.$$

Canceling the common binomial coefficient from this equation, we get the closed expression stated in Theorem 3.1. □

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