

Almost Fedosov and metriplectic structures in the geometry of semisprays

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Abstract

Given a pair (semispray S , almost symplectic form ω) on a tangent bundle, the family of nonlinear connections N such that ω is recurrent with respect to (S, N) with a fixed recurrent factor is determined by using the Obata tensors. In particular, we obtain a characterization for a pair (N, ω) to be recurrent as well as for the triple $(S, \overset{c}{N}, \omega)$ where $\overset{c}{N}$ is the canonical nonlinear connection of the semispray S . In the particular case of vanishing recurrence factor we get the family of almost Fedosov structures associated to a fixed semispray and almost symplectic structure. For a triple (semispray S , almost symplectic form ω , metric g), a characterization for existence of a corresponding almost metriplectic structure is obtained.

Introduction

In two many cited papers, [9, 10], Yung-chow Wong derived several properties of a recurrent tensor field T on a manifold M endowed with a linear connection ∇ . Recall that this means the existence of a 1-form α_T on M such that:

$$\nabla T = \alpha_T \otimes T. \quad (0)$$

For $\alpha_T = 0$ we recover the notion of *parallel* or *covariant constant tensor field*.

The aim of present paper is to extend the notion of recurrence to the geometry of systems of second order differential equations on M or *path geometry* as is usually called. More precisely, given such a system S , it can be considered as a vector field (called *semispray*) on the tangent bundle TM . Then, a type of differential ∇ is naturally associated to S . A main tool in the definition of ∇ is given by a splitting of the iterated tangent bundle $T(TM)$ provided by a distribution N

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on TM supplementary to the vertical distribution. Such an object N is called *nonlinear connection* and a main result in path geometry is that every S yields such a nonlinear connection, $\overset{c}{N}$, indexed by us with c from canonical.

We treat in detail the recurrence of tensor fields T of type $(0, 2)$ in correspondence with the results from [1] (where the metrizable problem is considered) and [5]; therefore, we can say that we search recurrent almost symplectic forms for a given system of second order differential equations, the recurrence factor being fixed. In fact, given a pair (semispray S , almost symplectic structure ω) on TM , the family of nonlinear connections N such that ω is recurrent with respect to (S, N) with a fixed recurrent factor is determined by using the Obata tensors defined by ω . In particular, we obtain a characterization for a pair (N, ω) to be recurrent as well as for the recurrence of the triple $(S, \overset{c}{N}, \omega)$. In the particular case of a vanishing recurrence factor we derive the associated almost Fedosov structures. Recall after [6] that a Fedosov structure is a pair (symplectic form ω , symmetric linear connection ∇) with $\nabla\omega = 0$.

In the last section we put together the results from the (almost) symplectic and metric cases into the (almost) metriplectic structures. Metriplectic systems were introduced in [8] and these systems combine both conservative and dissipative systems; see also [7]. The underline geometrical setting is provided by a pair (almost symplectic structure, metric) with both objects parallel with respect to a linear connection.

1 Nonlinear connections and semisprays on tangent bundles

Let M be a smooth, n -dimensional manifold for which we denote: $C^\infty(M)$ -the algebra of smooth real functions on M , $\mathcal{X}(M)$ -the Lie algebra of vector fields on M , $T_s^r(M)$ -the $C^\infty(M)$ -module of tensor fields of (r, s) -type on M .

A local chart $(U, x = (x^i) = (x^1, \dots, x^n))$ on M lifts to a local chart on the tangent bundle TM given by: $(\pi^{-1}(U), (x, y) = (x^i, y^i))$ where $\pi : TM \rightarrow M$ is the canonical bundle projection. The kernel of the differential of π is an integrable distribution $V(TM)$ with local basis $(\frac{\partial}{\partial y^i})$. An important element of $V(TM)$ is the *Liouville vector field* $\mathbb{C} = y^i \frac{\partial}{\partial y^i}$. $V(TM)$ is called *the vertical distribution* and its elements are *vertical vector fields*.

The tensor field $J \in T_1^1(TM)$ given by $J = \frac{\partial}{\partial y^i} \otimes dx^i$ is called *the tangent structure*. Two of its properties are: the nilpotence $J^2 = 0$ and $im J (= \ker J) = V(TM)$. A well-known notion in the tangent bundles geometry is:

Definition 1.1([1, p. 336]) A supplementary distribution N to the vertical distribution $V(TM)$:

$$T(TM) = N \oplus V(TM) \tag{1.1}$$

is called *horizontal distribution* or *nonlinear connection*. A vector field belonging to N is called *horizontal*.

A nonlinear connection has a local basis:

$$\frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - N_j^i \frac{\partial}{\partial y^j} \tag{1.2}$$

and the functions $(N_j^i(x, y))$ are called *the coefficients* of N . A basis of $\mathcal{X}(TM)$ adapted to the decomposition (1.1) is $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i})$ called *Berwald basis*. The dual of the Berwald basis is: $(dx^i, \delta y^i = dy^i + N_j^i dx^j)$.

A second remarkable structure on TM is provided by:

Definition 1.2 ([1, p. 336]) $S \in \mathcal{X}(TM)$ is called *semispray* if:

$$J(S) = \mathbb{C}. \tag{1.3}$$

In canonical bundle coordinates:

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i} \tag{1.4}$$

and the functions $(G^i(x, y))$ are *the coefficients* of S . The flow of S is a system of second order differential equations: $\frac{d^2 x^i}{dt^2} = 2G^i(x, \frac{dx}{dt})$ and then the geometry of the pair (M, S) is called *path geometry*.

There is an important relationship between semisprays and nonlinear connections. Firstly, a nonlinear connection $N = (N_j^i)$ yields an unique horizontal semispray denoted $S(N)$ with:

$$G^i = \frac{1}{2} N_j^i y^j. \tag{1.5}$$

In other words:

$$S(N) = y^i \frac{\delta}{\delta x^i}. \tag{1.6}$$

Conversely, a semispray S yields a nonlinear connection $\overset{c}{N}$ given by:

$$\overset{c}{N}_j^i = \frac{\partial G^i}{\partial y^j}. \tag{1.7}$$

Definition 1.3 A semispray S for which the coefficients (G^i) are homogeneous of degree 2 with respect to the variables (y^i) will be called *spray*.

Locally this means, via Euler theorem:

$$2G^i = y^j \frac{\partial G^i}{\partial y^j} \tag{1.8}$$

and then $\overset{c}{N}$ is 1-homogeneous:

$$\overset{c}{N}_j^i = y^a \frac{\partial \overset{c}{N}_j^i}{\partial y^a} \tag{1.9}$$

which yields that $S = S(\overset{c}{N})$ and then S is horizontal with respect to $\overset{c}{N}$.

2 Recurrence and almost Fedosov structures in path geometry

2.1 The general problem of recurrent triples

Let us fix a semispray $S = (G^i)$ and a nonlinear connection $N = (N_j^i)$. Following [2] let us consider:

Definition 2.1 The *dynamical derivative* associate to the pair (S, N) is the map $\overset{SN}{\nabla}: N \rightarrow N$ given by:

$$\overset{SN}{\nabla} X = \overset{SN}{\nabla} \left(X^i \frac{\delta}{\delta x^i} \right) := (S(X^i) + N_j^i X^j) \frac{\delta}{\delta x^i}. \tag{2.1}$$

The dynamical derivative satisfy:

$$\overset{SN}{\nabla} \left(\frac{\delta}{\delta x^i} \right) = N_i^j \frac{\delta}{\delta x^j}, \quad \overset{SN}{\nabla} (X + Y) = \overset{SN}{\nabla} X + \overset{SN}{\nabla} Y, \quad \overset{SN}{\nabla} (fX) = S(f)X + f \overset{SN}{\nabla} X.$$

It is straightforward to extend the action of $\overset{SN}{\nabla}$ to general horizontal tensor fields by requiring to preserve the tensor product and Leibniz rule. Moreover, we will extend $\overset{SN}{\nabla}$ to a special class of tensor fields:

Definition 2.2 A *d-tensor field* (d from distinguished) on TM is a tensor field whose change of components, under a change of canonical coordinates $(x, y) \rightarrow (\tilde{x}, \tilde{y})$ on TM , involves only factors of type $\frac{\partial \tilde{x}}{\partial x}$ and (or) $\frac{\partial \tilde{y}}{\partial y}$.

Examples 2.3 i) $\left(\frac{\delta}{\delta x^i}\right)$ and $\left(\frac{\partial}{\partial y^i}\right)$ are components of d-tensor fields of (1, 0)-type.

ii) (dx^i) and (δy^i) are components of d-tensor fields of (0, 1)-type.

iii) (G^i) are not components of a d-tensor field since a change of coordinates implies:

$$2\tilde{G}^i = 2\frac{\partial \tilde{x}^i}{\partial x^j} G^j - \frac{\partial \tilde{y}^i}{\partial x^j} y^j$$

but it results that given two semisprays $\overset{1}{S}$ and $\overset{2}{S}$ their difference $X = \overset{2}{S} - \overset{1}{S}$ is a vertical (and then d-) vector field.

iv) (N_j^i) are not components of a d-tensor field since a change of coordinates implies:

$$\frac{\partial \tilde{x}^j}{\partial x^k} N_i^k = \tilde{N}_k^j \frac{\partial \tilde{x}^k}{\partial x^i} + \frac{\partial \tilde{y}^j}{\partial x^i}.$$

It follows that given two nonlinear connections $\overset{1}{N}$ and $\overset{2}{N}$ their difference $F = \overset{2}{N} - \overset{1}{N} = \left(F_j^i = \overset{2}{N}_j^i - \overset{1}{N}_j^i \right)$ is a d-tensor field of (1, 1)-type. Here, we thought of the difference $\overset{2}{N} - \overset{1}{N}$ in terms of associated projectors, namely if v, h_1, h_2 are

the projectors given by the decomposition (1.1) then $\frac{2}{N} - \frac{1}{N}$ is corresponding to $h_2 - h_1$ which is a projector together with v .

Definition 2.4 An *almost symplectic structure* ω on TM is a d-tensor field of $(0, 2)$ -type of the local (diagonal) form: $\omega = \omega_{ij} dx^i \wedge \delta y^j$, which is skew-symmetric and non-degenerate.

It results for the components $\omega_{ij} = \omega_{ij}(x, y) = \omega(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j})$ the following properties:

- 1) (skew-symmetry) $\omega_{ij} = -\omega_{ji}$,
- 2) (non-degeneration) $\det(\omega_{ij}) \neq 0$; then there exists the d-tensor field of $(2, 0)$ -type $\omega^{-1} = (\omega^{ij})$.

An important property of ω is that N and $V(TM)$ are Lagrangian distributions.

Definition 2.5 The *dynamical derivative* of the almost symplectic structure ω with respect to the pair (S, N) is $\overset{SN}{\nabla} \omega : N \times N \rightarrow N$ given by:

$$\overset{SN}{\nabla} \omega(X, Y) = S(\omega(X, Y)) - \omega(\overset{SN}{\nabla} X, Y) - \omega(X, \overset{SN}{\nabla} Y). \tag{2.2}$$

The main notion of this section is:

Definition 2.6 i) Let $\alpha \in C^\infty(TM)$. The almost symplectic structure ω is called α -*recurrent* with respect to the pair (S, N) if:

$$\overset{SN}{\nabla} \omega = \alpha \omega. \tag{2.3}$$

Also, the triple (S, N, ω) will be called an α -*recurrent structure*.

ii) An *almost Fedosov structure* on TM is a triple (S, N, ω) which is 0-recurrent i.e. ω is parallel with respect to $\overset{SN}{\nabla}$.

The aim of this section is to find all nonlinear connections which together with fixed (S, ω) form an α -recurrent structure. In order to answer at this question, a look at example 2.3 iv) gives necessary a study of two operators, called *Obata* in the following, acting on the space of d-tensor fields of $(1, 1)$ -type of local (horizontal) forms $X = X_j^i \frac{\delta}{\delta x^i} \otimes dx^j$:

$$O(\omega)_{kl}^{ij} = \frac{1}{2} (\delta_k^i \delta_l^j - \omega^{ij} \omega_{kl}), \quad \overset{*}{O}(\omega)_{kl}^{ij} = \frac{1}{2} (\delta_k^i \delta_l^j + \omega^{ij} \omega_{kl}). \tag{2.4}$$

The Obata operators are supplementary projectors:

$$O_{bj}^{ia} \overset{*}{O}_{la}^{bk} = \overset{*}{O}_{bj}^{ia} O_{la}^{bk} = 0, \quad O_{bj}^{ia} O_{la}^{bk} = O_{lj}^{ik}, \quad \overset{*}{O}_{bj}^{ia} \overset{*}{O}_{la}^{bk} = \overset{*}{O}_{lj}^{ik} \tag{2.5}$$

(for simplicity we give up to denote ω into these O) and the tensorial equations involving these operators has solutions as follows:

Proposition 2.7 *The system of equations:*

$$\overset{*}{O}(\omega)_{bj}^{ia}(X_a^b) = A_j^i, \quad (O(\omega)_{bj}^{ia}(X_a^b) = A_j^i) \tag{2.6}$$

with X as unknown has a solution if and only if:

$$O(\omega)_{bj}^{ia}(A_a^b) = 0, \quad (\overset{*}{O}(\omega)_{bj}^{ia}(A_a^b) = 0) \tag{2.7}$$

and then, the general solution is:

$$X_j^i = A_j^i + O(\omega)_{bj}^{ia} (Y_a^b), \quad \left(X_j^i = A_j^i + \overset{*}{O}(\omega)_{bj}^{ia} (Y_a^b) \right) \quad (2.8)$$

with Y an arbitrary d -tensor field of $(1, 1)$ -type.

We are ready for one of the main results of paper:

Theorem 2.8 Set S , ω and α .

i) The family $\mathcal{N}(S, \omega, \alpha)$ of all nonlinear connections $N = (N_j^i)$ such that (S, N, ω) is α -recurrent is given by:

$$N_j^i = \frac{1}{2} \overset{c}{N}_j^i - \frac{1}{2} \omega^{ia} \omega_{bj} \overset{c}{N}_a^b + \frac{1}{2} \omega^{ia} S(\omega_{aj}) - \frac{\alpha}{2} \delta_j^i + O(\omega)_{bj}^{ia} (X_a^b). \quad (2.9)$$

ii) The family $\mathcal{N}_F(S, \omega)$ of all nonlinear connections $N = (N_j^i)$ such that (S, N, ω) is an almost Fedosov structure on TM is given by:

$$N_j^i = \frac{1}{2} \overset{c}{N}_j^i - \frac{1}{2} \omega^{ia} \omega_{bj} \overset{c}{N}_a^b + \frac{1}{2} \omega^{ia} S(\omega_{aj}) + O(\omega)_{bj}^{ia} (X_a^b). \quad (2.9F)$$

In these relations $X = (X_a^b)$ is an arbitrary horizontal d -tensor field of $(1, 1)$ -type. It follows that $\mathcal{N}(S, \omega, \alpha)$ is a $C^\infty(TM)$ -affine module over the $C^\infty(TM)$ -module of horizontal d -tensor fields of $(1, 1)$ -type.

Proof We search (N_j^i) of the form:

$$N_j^i = \overset{c}{N}_j^i + F_j^i \quad (2.10)$$

with (F_j^i) a d -tensor field of $(1, 1)$ -type to be determined. The local expression of equation (2.3) is:

$$S(\omega_{uv}) - \omega_{um} N_v^m - \omega_{mv} N_u^m = \alpha \omega_{uv} \quad (2.11)$$

and inserting (2.10) in (2.11) gives:

$$S(\omega_{uv}) - \omega_{um} \overset{c}{N}_v^m - \omega_{mv} \overset{c}{N}_u^m = \omega_{um} F_v^m + \omega_{mv} F_u^m + \alpha \omega_{uv}.$$

Multiplying the last relation with ω^{ku} we get:

$$\omega^{ku} S(\omega_{uv}) - \overset{c}{N}_v^k - \omega^{ku} \omega_{mv} \overset{c}{N}_u^m - \alpha \delta_v^k = F_v^k + \omega^{ku} \omega_{mv} F_u^m = 2 \overset{*}{O}_{av}^{kb} (F_b^a). \quad (2.12)$$

The condition (2.7) is satisfied:

$$\begin{aligned} & O(\omega)_{av}^{kb} \left(\omega^{am} S(\omega_{mb}) - \overset{c}{N}_b^a - \omega^{am} \omega_{lb} \overset{c}{N}_m^l - \alpha \delta_b^a \right) = \\ & = \omega^{km} S(\omega_{mv}) - \overset{c}{N}_v^k - \omega^{km} \omega_{vl} \overset{c}{N}_m^l - \omega^{km} S(\omega_{mv}) + \omega^{km} \omega_{vl} \overset{c}{N}_m^l + \overset{c}{N}_v^k = 0. \end{aligned}$$

It follows:

$$F_j^i = \frac{1}{2} \omega^{im} S(\omega_{mj}) - \frac{1}{2} \overset{c}{N}_j^i - \frac{1}{2} \omega^{ia} \omega_{jb} \overset{c}{N}_a^b - \frac{\alpha}{2} \delta_j^i + O_{aj}^{ib} (X_b^a)$$

and returning to (2.10) we have the conclusion. \square

In the spray case the equation (2.9) admits a simple form:

Proposition 2.9 Fix a spray S and an almost symplectic structure ω .

i) The family $\mathcal{N}(S, \omega, \alpha)$ is:

$$N_j^i = \frac{1}{2} N_j^i - \frac{1}{2} \omega^{ia} \omega_{bj} N_a^b + \frac{1}{2} \omega^{ia} y^m \frac{\delta \omega_{aj}}{\delta x^m} - \frac{\alpha}{2} \delta_j^i + O(\omega)_{bj}^{ia} (X_a^b). \quad (2.13)$$

ii) The family $\mathcal{N}_F(S, \omega)$ is:

$$N_j^i = \frac{1}{2} N_j^i - \frac{1}{2} \omega^{ia} \omega_{bj} N_a^b + \frac{1}{2} \omega^{ia} y^m \frac{\delta \omega_{aj}}{\delta x^m} + O(\omega)_{bj}^{ia} (X_a^b). \quad (2.13F)$$

2.2 Recurrence of a pair (nonlinear connection, almost symplectic form)

Fix a nonlinear connection $N = (N_j^i)$ and associate to N the semispray $S(N)$.

Definition 2.10 The pair (N, ω) is α -recurrent (or almost Fedosov) if the triple $(S(N), N, \omega)$ is so.

Since the canonical nonlinear connection for $S(N)$ is:

$$N_j^i = \frac{1}{2} \left(N_j^i + \frac{\partial N_u^i}{\partial y^j} y^u \right)$$

it follows:

Theorem 2.11 i) The pair (N, ω) is α -recurrent if and only if:

$$N_j^i + \omega^{ia} \omega_{bj} N_a^b + \omega^{im} \frac{\partial \omega_{mj}}{\partial y^b} N_a^b y^a = \omega^{im} \frac{\partial \omega_{mj}}{\partial x^a} y^a - \alpha \delta_j^i \quad (2.14)$$

for all $i, j \in \{1, \dots, n\}$.

ii) The pair (N, ω) is almost Fedosov if and only if:

$$N_j^i + \omega^{ia} \omega_{bj} N_a^b + \omega^{im} \frac{\partial \omega_{mj}}{\partial y^b} N_a^b y^a = \omega^{im} \frac{\partial \omega_{mj}}{\partial x^a} y^a \quad (2.14F)$$

for all $i, j \in \{1, \dots, n\}$.

Proof From (2.9) it results that (N, ω) is α -recurrent if and only if:

$$O^*(\omega)_{uj}^{iv} \left(N_v^u + \omega^{ua} \omega_{vb} N_a^b + \omega^{um} \frac{\partial \omega_{mv}}{\partial y^b} N_a^b y^a + \alpha \delta_v^u \right) = O^*(\omega)_{uj}^{iv} \left(\omega^{um} \frac{\partial \omega_{mv}}{\partial x^a} y^a \right)$$

and a straightforward computation yields the conclusion. \square

Example 2.12 Basic almost symplectic structures Let us consider $\omega_M = \omega_{ij}(x) dx^i \wedge dx^j$ an almost symplectic structure on M . Then we associate an almost symplectic structure in our framework as in Definition 2.4 and then the

relation (2.14) becomes:

$$N_j^i + \omega^{ia}\omega_{jb}N_a^b = \omega^{im}\frac{\partial\omega_{mj}}{\partial x^a}y^a - \alpha\delta_j^i. \quad (2.15)$$

Recall that a symmetric linear connection on M with coefficients $(\Gamma_{jk}^i(x))$ yields the nonlinear connection with the coefficients:

$$N_j^i = \Gamma_{ja}^i y^a. \quad (2.16)$$

Then the associated semispray $S(N)$ is a spray:

$$G^i = \frac{1}{2}\Gamma_{jk}^i y^j y^k. \quad (2.17)$$

Inserting (2.16) in (2.15) we get:

$$\left(\omega^{im}\frac{\partial\omega_{mj}}{\partial x^a} - \Gamma_{ja}^i - \omega^{iu}\omega_{jv}\Gamma_{ua}^v\right)y^a = \alpha\delta_j^i. \quad (2.18)$$

But multiplying the last equation with ω_{ik} we arrive at:

$$\left(\frac{\partial\omega_{jk}}{\partial x^a} - \omega_{ki}\Gamma_{ja}^i - \omega_{ji}\Gamma_{ka}^i\right)y^a = \alpha\omega_{jk} \quad (2.19)$$

which is the usual Christoffel process for the almost symplectic case replaced in the recurrent framework of TM . So, we verified the condition (2.14) in the basic almost symplectic setting.

Let us point out the rôle of homogeneity of the spray (2.17). We remark from (2.19) that α must be a 1-homogeneous on y , i.e. $\alpha(x, y) = \alpha_a(x)y^a$, and then we have:

$$\frac{\partial\omega_{jk}}{\partial x^a} - \omega_{ki}\Gamma_{ja}^i - \omega_{ji}\Gamma_{ka}^i = \alpha_a\omega_{jk} \quad (2.20)$$

for all $a, j, k \in \{1, \dots, n\}$. By considering the 1-form $\alpha_\omega = \alpha_a(x)dx^a$ we recover the starting formula (0) from Introduction for $T = \omega_M$.

A very important remark is that α_ω must satisfy a necessary condition in order to be a recurrence form for an almost symplectic structure. Namely, cycling (2.20) and then summing up, we get:

$$\frac{\partial\omega_{jk}}{\partial x^a} + \frac{\partial\omega_{ka}}{\partial x^j} + \frac{\partial\omega_{aj}}{\partial x^k} = \alpha_a\omega_{jk} + \alpha_j\omega_{ka} + \alpha_k\omega_{aj}. \quad (2.20C)$$

If ω_M is a symplectic structure, namely $d\omega_M = 0$, then the left-hand side of above relation vanishes and then α must be zero which yield a Fedosov structure.

2.3 Recurrence of a pair (semispray, almost symplectic form)

Let us fix a semispray $S = (G^i)$ and the almost symplectic form ω .

Definition 2.13 The pair (S, ω) is called α -recurrent (or almost Fedosov) if the triple $(S, \overset{c}{N}, \omega)$ is so.

Inserting $\overset{c}{N}$ in the left-hand-side of (2.9) we get:

Theorem 2.14 i) The pair (S, ω) is α -recurrent if and only if:

$$\frac{\partial G^i}{\partial y^j} + \omega^{ia} \omega_{jb} \frac{\partial G^b}{\partial y^a} - \omega^{ia} S(\omega_{aj}) = -\alpha \delta_j^i \tag{2.21}$$

for all $i, j \in \{1, \dots, n\}$.

ii) The pair (S, ω) is almost Fedosov if and only if:

$$\frac{\partial G^i}{\partial y^j} + \omega^{ia} \omega_{jb} \frac{\partial G^b}{\partial y^a} - \omega^{ia} S(\omega_{aj}) = 0 \tag{2.21F}$$

for all $i, j \in \{1, \dots, n\}$.

iii) The spray S makes α -recurrent the almost symplectic form ω if and only if:

$$\frac{\partial G^i}{\partial y^j} + \omega^{ia} \omega_{jb} \frac{\partial G^b}{\partial y^a} - \omega^{ia} y^m \frac{\delta \omega_{aj}}{\delta x^m} = -\alpha \delta_j^i \tag{2.22}$$

for all $i, j \in \{1, \dots, n\}$.

iv) The spray S makes almost Fedosov the almost symplectic form ω if and only if:

$$\frac{\partial G^i}{\partial y^j} + \omega^{ia} \omega_{jb} \frac{\partial G^b}{\partial y^a} - \omega^{ia} y^m \frac{\delta \omega_{aj}}{\delta x^m} = 0 \tag{2.22F}$$

for all $i, j \in \{1, \dots, n\}$.

Proof The left-hand-side of (2.9) becomes:

$$\overset{*}{O}(\omega)_{uj}^{iv} \left(\overset{c}{N}_v^u + \omega^{ua} \omega_{vb} \overset{c}{N}_a^b - \omega^{um} S(\omega_{mv}) \right) = -\alpha \delta_j^i \tag{2.23}$$

and the computations give (2.21). \square

3 Almost metriplectic structures in path geometry

Suppose that, in addition to the pair (semispray S , nonlinear connection N), we are given on TM a pair (almost symplectic structure ω , metric g) where we use the following:

Definition 3.1 A metric g on TM is a d-tensor field of (0, 2)-type of local Sasaki form: $g_{ij} dx^i \otimes dx^j + g_{ij} \delta y^i \otimes \delta y^j$, which is symmetric and non-degenerate.

It results for the components $g_{ij} = g(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j})$ the following properties:

- 1) (symmetry) $g_{ij} = g_{ji}$,
- 2) (non-degeneration) $\det(g_{ij}) \neq 0$; then there exists the d-tensor field of (2, 0)-type $g^{-1} = (g^{ij})$.

The name is justified from the fact that g is a Riemannian metric on TM for which N and $V(TM)$ are mutually orthogonal distributions.

Let the corresponding Obata operators:

$$O(g)^{ij}_{kl} = \frac{1}{2} \left(\delta_k^i \delta_l^j - g^{ij} g_{kl} \right), \quad \overset{*}{O}(g)^{ij}_{kl} = \frac{1}{2} \left(\delta_k^i \delta_l^j + g^{ij} g_{kl} \right). \quad (3.1)$$

Definition 3.2 The data (S, N, ω, g) is an *almost metriplectic structure* if:

$$\overset{SN}{\nabla} \omega = \overset{SN}{\nabla} g = 0 \quad (3.2)$$

where $\overset{SN}{\nabla} g$ is exactly as in formula (2.2) with ω replaced by g .

Let us fix (S, ω, g) . The set $\mathcal{N}(S, g)$ of nonlinear connections making parallel the metric g is given with a formula similar to (2.9F), [1, p. 339]:

$$N_j^i = \frac{1}{2} \overset{c}{N}_j^i - \frac{1}{2} g^{ia} g_{bj} \overset{c}{N}_a^b + \frac{1}{2} g^{ia} S(g_{aj}) + O(g)_{bj}^{ia} (Y_a^b) \quad (3.3)$$

through a similar proof; see also [4] or [5]. We derive then:

Proposition 3.3 *Let S, ω and g be given. There exists a nonlinear connection N such that (S, N, ω, g) is an almost metriplectic structure on TM if and only if there are two horizontal d -tensor fields of $(1, 1)$ -type, X and Y , such that:*

$$\omega^{ia} S(\omega_{aj}) - \omega^{ia} \omega_{jb} \overset{c}{N}_a^b + O(\omega)_{bj}^{ia} (2X_a^b) = g^{ia} S(g_{aj}) - g^{ia} g_{jb} \overset{c}{N}_a^b + O(g)_{bj}^{ia} (2Y_a^b). \quad (3.4)$$

Then N is given by (2.9) or (3.3).

Example 3.4 Almost Hermitian structures Let (g_M, J) be an almost Hermitian structure on M i.e. J is an almost complex structure, $J^2 = -1_{TM}$, compatible with $g_M = (g_{ij}(x))$, [3, p. 90]:

$$g(J \cdot, J \cdot) = g(\cdot, \cdot). \quad (3.5)$$

Then:

$$\omega_M(\cdot, \cdot) = g_M(J \cdot, \cdot) \quad (3.6)$$

is an almost symplectic structure on M . Let (J_j^i) be the components of J , this means:

$$J_a^i J_j^a = -\delta_j^i \quad (3.7)$$

and then $\omega_M = (\omega_{ij}(x))$ with:

$$\omega_{ij} = g \left(J \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j} \right) = g \left(J_i^a \frac{\partial}{\partial y^a}, \frac{\partial}{\partial y^j} \right) = g_{aj} J_i^a \quad (3.8)$$

with inverse:

$$\omega^{ij} = -g^{ia} J_a^j. \quad (3.9)$$

The condition (3.4) becomes:

$$g^{ir} g_{sb} J_r^a J_j^s \overset{c}{N}_a^b - g^{ib} g_{cj} J_b^a S(J_a^c) + O(\omega)_{bj}^{ia} (2X_a^b) = 2g^{ia} S(g_{aj}) - g^{ia} g_{jb} \overset{c}{N}_a^b + O(g)_{bj}^{ia} (2Y_a^b) \quad (3.10)$$

which can be written:

$$g^{ir} g_{sb} (\delta_r^a \delta_j^s + J_r^a J_j^s) \overset{c}{N}_a^b + O(\omega)_{bj}^{ia} (2X_a^b) = 2g^{ia} S(g_{aj}) + g^{ib} g_{cj} J_b^a S(J_a^c) + O(g)_{bj}^{ia} (2Y_a^b). \quad (3.11)$$

Another form of this relation is:

$$g^{ir} g_{sb} \overset{*}{O}(J)_{rj}^{as} \left(\overset{c}{N}_a^b \right) + O(\omega)_{bj}^{ia} (2X_a^b) = 2g^{ia} S(g_{aj}) + g^{ib} g_{cj} J_b^a S(J_a^c) + O(g)_{bj}^{ia} (2Y_a^b). \quad (3.12)$$

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