Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Filomat **26:1** (2012), 177–195 DOI:(will be added later)

Strong Intuitionistic Fuzzy Graphs

Muhammad Akram and Bijan Davvaz

Abstract

We introduce the notion of strong intuitionistic fuzzy graphs and investigate some of their properties. We discuss some propositions of self complementary and self weak complementary strong intuitionistic fuzzy graphs. We introduce the concept of intuitionistic fuzzy line graphs.

1 Introduction

In 1736, Euler first introduced the concept of graph theory. In the history of mathematics, the solution given by Euler of the well known Königsberg bridge problem is considered to be the first theorem of graph theory. This has now become a subject generally regarded as a branch of combinatorics. The theory of graph is an extremely useful tool for solving combinatorial problems in different areas such as geometry, algebra, number theory, topology, operations research, optimization and computer science.

In 1983, Atanassov [6] introduced the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets [31]. Atanassov added a new component(which determines the degree of non-membership) in the definition of fuzzy set. The fuzzy sets give the degree of membership of an element in a given set (and the non-membership degree equals one minus the degree of membership), while intuitionistic fuzzy sets give both a degree of membership and a degree of non-membership which are more-or-less independent from each other, the only requirement is that the sum of these two degrees is not greater than 1. Intuitionistic fuzzy sets have been applied in a wide variety of fields including computer science, engineering, mathematics, medicine, chemistry and economics [5, 13].

In 1975, Rosenfeld [27] introduced the concept of fuzzy graphs. The fuzzy relations between fuzzy sets were also considered by Rosenfeld and he developed the structure of fuzzy graphs, obtaining analogs of several graph theoretical concepts.

2010 *Mathematics Subject Classifications.* 05C99.

Key words and Phrases. Intuitionistic fuzzy sets, Strong intuitionistic fuzzy graphs, Self complementary intuitionistic fuzzy graphs, Intuitionistic fuzzy line graphs. Applications.

Received: Required

Communicated by Miroslav Ćirić

Later on, Bhattacharya [8] gave some remarks on fuzzy graphs, and some operations on fuzzy graphs were introduced by Mordeson and Peng [17]. The complement of a fuzzy graph was defined by Mordeson [19] and further studied by Sunitha and Vijayakumar [30]. Mordeson [18] introduced the notion of fuzzy line graph. Bhutani and Rosenfeld introduced the concept of M-strong fuzzy graphs in [11] and studied some of their properties. Akram and Dudek [2] discussed interval-valued fuzzy graphs. Atanassov [5] introduced the concept of intuitionistic fuzzy relations and intuitionistic fuzzy graphs, and further studied in [26]. In fact, interval-valued fuzzy graphs and intuitionistic fuzzy graphs are two different models that extend theory of fuzzy graph. In this article, we introduce the notion of strong intuitionistic fuzzy graphs and investigate some of their properties. We discuss some propositions of self complementary and self weak complementary strong intuitionistic fuzzy graphs. We study intuitionistic fuzzy line graphs. The definitions and terminologies that we used in this paper are standard.

2 Preliminaries

In this section, we review some definitions that are necessary in the paper.

A *graph* is an ordered pair $G^* = (V, E)$, where V is the set of vertices of G^* and E is the set of edges of G^*. Two vertices x and y in a graph G^* are said to be adjacent in G^* if $\{x, y\}$ is in an edge of G^* (for simplicity an edge $\{x, y\}$ will be denoted by xy). A *simple graph* is a graph without loops and multiple edges. A *complete graph* is a simple graph in which every pair of distinct vertices is connected by an edge. The complete graph on n vertices has $n(n-1)/2$ edges. We will consider only graphs with the finite number of vertices and edges.

An *isomorphism* of graphs G_1^* and G_2^* is a bijection between the vertex sets of G_1^* and G_2^* such that any two vertices v_1 and v_2 of G_1^* are adjacent in G_1^* if and only if $f(v_1)$ and $f(v_2)$ are adjacent in G_2^*. Isomorphic graphs are denoted by $G_1^* \simeq G_2^*$. By a *complementary graph* $\overline{G^*}$ of a simple graph G^* we mean a graph having the same vertices as G^* and such that two vertices are adjacent in $\overline{G^*}$ if and only if they are not adjacent in G^*. A simple graph that is isomorphism to its complement is called *self-complementary*.

Let $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ be two simple graphs, we can construct several new graphs. The first construction called the *Cartesian product* of G_1^* and G_2^* gives a graph $G_1^* \times G_2^* = (V, E)$ with $V = V_1 \times V_2$ and

$$E = \{(x, x_2)(x, y_2) | x \in V_1, x_2y_2 \in E_2\} \cup \{(x_1, z)(y_1, z) | x_1y_1 \in E_1, z \in V_2, \}.$$

The *composition* of graphs G_1^* and G_2^* is the graph $G_1^*[G_2^*] = (V_1 \times V_2, E^0)$, where

$$E^0 = E \cup \{(x_1, x_2)(y_1, y_2) | x_1y_1 \in E_1, x_2 \neq y_2\}$$

and E is defined as in $G_1^* \times G_2^*$. Note that $G_1^*[G_2^*] \neq G_2^*[G_1^*]$.

The *union* of graphs G_1^* and G_2^* is defined as $G_1^* \cup G_2^* = (V_1 \cup V_2, E_1 \cup E_2)$.

The *join* of G_1^* and G_2^* is the simple graph $G_1^* + G_2^* = (V_1 \cup V_2, E_1 \cup E_2 \cup E')$, where E' is the set of all edges joining the nodes of V_1 and V_2. In this construction it is assumed that $V_1 \cap V_2 = \emptyset$.

Definition 1. [31, 32] *By a fuzzy subset μ on a set X is mean a map $\mu : X \to [0,1]$. A map $\nu : X \times X \to [0,1]$ is called a fuzzy relation on X if $\nu(x,y) \leq \min(\mu(x), \mu(y))$ for all $x, y \in X$. A fuzzy relation ν is symmetric if $\nu(x,y) = \nu(y,x)$ for all $x, y \in X$.*

Definition 2. [5] *A mapping $A = (\mu_A, \nu_A) : X \to [0,1] \times [0,1]$ is called an intuitionistic fuzzy set in X if $\mu_A(x) + \nu_A(x) \leq 1$ for all $x \in X$, where the mappings $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\nu_A(x)$) of each element $x \in X$ to A, respectively.*

Definition 3. [5] *For every two intuitionistic fuzzy sets $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ in X, we define*

- $(A \cap B)(x) = (\min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x)))$,
- $(A \cup B)(x) = (\max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x)))$.

Definition 4. [5] *Let X be a nonempty set. Then we call a mapping $A = (\mu_A, \nu_A) : X \times X \to [0,1] \times [0,1]$ an intuitionistic fuzzy relation on X if $\mu_A(x,y) + \nu_A(x,y) \leq 1$ for all $(x,y) \in X \times X$.*

Definition 5. [5] *Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be intuitionistic fuzzy sets on a set X. If $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy relation on a set X, then $A = (\mu_A, \nu_A)$ is called an intuitionistic fuzzy relation on $B = (\mu_B, \nu_B)$ if $\mu_A(x,y) \leq \min(\mu_B(x), \mu_B(y))$ and $\nu_A(x,y) \geq \max(\nu_B(x), \nu_B(y))$ for all x, $y \in X$. An intuitionistic fuzzy relation A on X is called symmetric if $\mu_A(x,y) = \mu_A(y,x)$ and $\nu_A(x,y) = \nu_A(y,x)$ for all x, $y \in X$.*

3 Strong intuitionistic fuzzy graphs

Throughout this paper, we denote G^* a crisp graph, and G an intuitionistic fuzzy graph.

Definition 6. *An intuitionistic fuzzy graph with underlying set V is defined to be a pair $G = (A, B)$ where*

(i) *the functions $\mu_A : V \to [0,1]$ and $\nu_A : V \to [0,1]$ denote the degree of membership and nonmembership of the element $x \in V$, respectively such that $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for all $x \in V$,*

(ii) *the functions $\mu_B : E \subseteq V \times V \to [0,1]$ and $\nu_B : E \subseteq V \times V \to [0,1]$ are defined by*

$$\mu_B(\{x,y\}) \leq \min(\mu_A(x), \mu_A(y)) \quad \text{and} \quad \nu_B(\{x,y\}) \geq \max(\nu_A(x), \nu_A(y))$$

such that $0 \leq \mu_B(\{x,y\}) + \nu_B(\{x,y\}) \leq 1$ for all $\{x,y\} \in E$.

We call A the intuitionistic fuzzy vertex set of V, B the intuitionistic fuzzy edge set of G, respectively. Note that B is a symmetric intuitionistic fuzzy relation on A. We use the notation xy for an element of E. Thus, $G = (A, B)$ is an intuitionistic graph of $G^* = (V, E)$ if

$$\mu_B(xy) \leq \min(\mu_A(x), \mu_A(y)) \quad \text{and} \quad \nu_B(xy) \geq \max(\nu_A(x), \nu_A(y))$$

for all $xy \in E$.

We now study strong intuitionistic fuzzy graphs.

Definition 7. *An intuitionistic fuzzy graph $G = (A, B)$ is called strong intuitionistic fuzzy graph if*

$$\mu_B(xy) = \min(\mu_A(x), \mu_A(y)) \quad \text{and} \quad \nu_B(xy) = \max(\nu_A(x), \nu_A(y)),$$

for all $xy \in E$.

Example 1. *Consider a graph G^* such that $V = \{x, y, z\}$, $E = \{xy, yz, zx\}$. Let A be an intuitionistic fuzzy subset of V and let B be an intuitionistic fuzzy subset of E defined by*

	x	y	z
μ_A	0.2	0.3	0.1
ν_A	0.4	0.1	0.5

	xy	yz	xz
μ_B	0.2	0.1	0.1
ν_B	0.4	0.5	0.5

y

z

G

Definition 8. *Let $A = (\mu_A, \nu_A)$ and $A' = (\mu'_A, \nu'_A)$ be intuitionistic fuzzy subsets of V_1 and V_2 and let $B = (\mu_B, \nu_B)$ and $B' = (\mu'_B, \nu'_B)$ be intuitionistic fuzzy subsets of E_1 and E_2, respectively. The Cartesian product of two strong intuitionistic fuzzy graphs G_1 and G_2 of the graphs G_1^* and G_2^* is denoted by $G_1 \times G_2 = (A \times A', B \times B')$ and is defined as follows:*

(i) $\begin{cases} (\mu_A \times \mu'_A)(x_1, x_2) = \min(\mu_A(x_1), \mu'_A(x_2)) \\ (\nu_A \times \nu'_A)(x_1, x_2) = \max(\nu_A(x_1), \nu'_A(x_2)) \end{cases}$

for all $(x_1, x_2) \in V$,

(ii) $\begin{cases} (\mu_B \times \mu'_B)((x, x_2)(x, y_2)) = \min(\mu_A(x), \mu'_B(x_2y_2)), \\ (\nu_B \times \nu'_B)((x, x_2)(x, y_2)) = \max(\nu_A(x), \nu'_B(x_2y_2)) \end{cases}$

for all $x \in V_1$, *for all* $x_2y_2 \in E_2$,

(iii) $\begin{cases} (\mu_B \times \mu'_B)((x_1, z)(y_1, z)) = \min(\mu_B(x_1y_1), \mu'_A(z)) \\ (\nu_B \times \nu'_B)((x_1, z)(y_1, z)) = \max(\nu_B(x_1y_1), \nu'_A(z)) \end{cases}$

for all $z \in V_2$, *for all* $x_1y_1 \in E_1$.

Proposition 1. *If G_1 and G_2 are the strong intuitionistic fuzzy graphs, then $G_1 \times G_2$ is a strong intuitionistic fuzzy graph.*

Proof. It is straightforward. □

Proposition 2. *If $G_1 \times G_2$ is strong intuitionistic fuzzy graph, then at least G_1 or G_2 must be strong.*

Proof. Suppose that G_1 and G_2 are not strong intuitionistic fuzzy graphs. Then there exist $x_1y_1 \in E_1$ and $x_2y_2 \in E_2$ such that

$$\mu_{B_1}(x_1y_1) < \min(\mu_{A_1}(x), \mu_{A_1}(y)),\ \mu_{B_2}(x_1y_1) < \min(\mu_{A_2}(x), \mu_{A_2}(y)) \quad (1)$$

$$\nu_{B_1}(x_1y_1) > \max(\nu_{A_1}(x), \nu_{A_1}(y)),\ \nu_{B_2}(x_1y_1) > \max(\nu_{A_2}(x), \nu_{A_2}(y)) \quad (2)$$

Assume that

$$\mu_{B_2}(x_2y_2) \leq \mu_{B_1}(x_1y_1) < \min(\mu_{A_1}(x_1), \mu_{A_1}(y_1)) \leq \mu_{A_1}(x_1) \quad (3)$$

Let

$$E = \{(x, x_2)(x, y_2) | x_1 \in V_1, x_2y_2 \in E_2\} \cup \{(x_1, z)(y_1, z) | z \in V_2, x_1y_1 \in E_1\}.$$

Consider $(x, x_2)(x, y_2) \in E$, we have

$$\begin{aligned} (\mu_{B_1} \times \mu_{B_2})((x, x_2)(x, y_2)) &= \min(\mu_{A_1}(x), \mu_{B_2}(x_2y_2)) \\ &< \min(\mu_{A_1}(x), \mu_{A_2}(x_2), \mu_{A_2}(y_2) \end{aligned}$$

and

$$\begin{aligned} (\mu_{A_1} \times \mu_{A_2})(x_1, x_2) &= \min(\mu_{A_1}(x_1), \mu_{A_2}(x_2)), (\mu_{A_1} \times \mu_{A_2})(x_1, y_2) \\ &= \min(\mu_{A_1}(x_1), \mu_{A_2}(y_2)). \end{aligned}$$

Therefore,

$$\min((\mu_{A_1} \times \mu_{A_2})(x, x_2), (\mu_{A_1} \times \mu_{A_2})(x, y_2)) = \min(\mu_{A_1}(x), \mu_{A_2}(x_2), \mu_{A_2}(y_2)).$$

Hence,

$$(\mu_{B_1} \times \mu_{B_2})((x, x_2)(x, y_2)) < \min((\mu_{A_1} \times \mu_{A_2})(x, x_2), (\mu_{A_1} \times \mu_{A_2})(x, y_2)).$$

Similarly, we can easily show that

$$(\nu_{B_1} \times \nu_{B_2})((x, x_2)(x, y_2)) > \max((\nu_{A_1} \times \nu_{A_2})(x, x_2), (\nu_{A_1} \times \nu_{A_2})(x, y_2)).$$

That is, $G_1 \times G_2$ is not strong intuitionistic fuzzy graph, a contradiction. Hence, this ends the proof. □

Remark 1. *If G_1 is strong and G_2 is not strong, then $G_1 \times G_2$ may or may not be strong.*

Example 2. *We consider the following examples:*

c

a

G_2 *is not strong*

G_1 *is strong*

b

d

(0.4, 0.2)

We sate a nice Proposition without its proof.

Proposition 3. *Let G_1 be a strong intuitionistic fuzzy graph of G_1^*. Then for any intuitionistic fuzzy graph G_2 of G_2^*, $G_1 \times G_2$ is strong if and only if*

$$\mu_A(x_1) \leq \mu_B(x_2y_2), \quad \nu_A(x_1) \geq \nu_B(x_2y_2) \text{ for all } \mathrm{x}_1 \in \mathrm{V}_1 \text{ and } x_2y_2 \in E_2.$$

Definition 9. *Let $A = (\mu_A, \nu_A)$ and $A' = (\mu'_A, \nu'_A)$ be intuitionistic fuzzy subsets of V_1 and V_2 and let $B = (\mu_B, \nu_B)$ and $B' = (\mu'_B, \nu'_B)$ be intuitionistic fuzzy subsets of E_1 and E_2, respectively. The composition of two strong intuitionistic fuzzy graphs G_1 and G_2 of the graphs G_1^* and G_2^* is denoted by $G_1[G_2] = (A \circ A', B \circ B')$ and is defined as follows:*

(i) $\begin{cases} (\mu_A \circ \mu'_A)(x_1, x_2) = \min(\mu_A(x_1), \mu'_A(x_2)), \\ (\nu_A \circ \nu'_A)(x_1, x_2) = \max(\nu_A(x_1), \nu'_A(x_2)), \end{cases}$

for all $(x_1, x_2) \in V$,

(ii) $\begin{cases} (\mu_B \circ \mu'_B)((x, x_2)(x, y_2)) = \min(\mu_A(x), \mu'_B(x_2y_2)), \\ (\nu_B \circ \nu'_B)((x, x_2)(x, y_2)) = \max(\nu_A(x), \nu'_B(x_2y_2)), \end{cases}$

for all $x \in V_1$, for all $x_2y_2 \in E_2$,

(iii) $\begin{cases} (\mu_B \circ \mu'_B)((x_1, z)(y_1, z)) = \min(\mu_B(x_1y_1), \mu'_A(z)), \\ \nu_B \circ \nu'_B)((x_1, z)(y_1, z)) = \max(\nu_B(x_1y_1), \nu'_A(z)) \textit{ for all } z \in V_2, \end{cases}$

for all $x_1y_1 \in E_1$,

(iv) $\begin{cases} (\mu_B \circ \mu'_B)((x_1, x_2)(y_1, y_2)) = \min(\mu'_A(x_2), \mu'_A(y_2), \mu_B(x_1y_1)), \\ (\nu_B \circ \nu'_B)((x_1, x_2)(y_1, y_2)) = \max(\nu'_A(x_2), \nu'_A(y_2), \nu_B(x_1y_1)), \end{cases}$

for all $(x_1, x_2)(y_1, y_2) \in E^0 - E$.

We state the following propositions without their proofs.

Proposition 4. *If G_1 and G_2 are the strong intuitionistic fuzzy graphs, then $G_1[G_2]$ is a strong intuitionistic fuzzy graph.*

Proposition 5. *If $G_1[G_2]$ is strong intuitionistic fuzzy graph, then at least G_1 or G_2 must be strong.*

Definition 10. *Let $A = (\mu_A, \nu_A)$ and $A' = (\mu'_A, \nu'_A)$ be intuitionistic fuzzy subsets of V_1 and V_2 and let $B = (\mu_B, \nu_B)$ and $B' = (\mu'_B, \nu'_B)$ be intuitionistic fuzzy subsets of E_1 and E_2, respectively. The join of two intuitionistic fuzzy graphs G_1 and G_2 of the graphs G_1^* and G_2^* is denoted by $G_1 + G_2 = (A + A', B + B')$ and is defined as follows:*

(i) $\begin{cases} (\mu_A + \mu'_A)(x) = (\mu_A + \mu'_A)(x), \\ (\nu_A + \nu'_A)(x) = (\nu_A + \nu'_A)(x) \end{cases}$

if $x \in V_1 \cup V_2$,

$$\text{(ii)} \begin{cases} (\mu_B + \mu'_B)(xy) = (\mu_B \cup \mu'_B)(xy) = \mu_B(xy) \\ (\nu_B + \nu'_B)(xy) = (\nu_B \cap \nu'_B)(xy) = \nu_B(xy) \end{cases}$$

if $xy \in E_1 \cup E_2$*,*

$$\text{(iii)} \begin{cases} (\mu_B + \mu'_B)(xy) = \min(\mu_A(x), \mu'_A(y)) \\ (\nu_B + \nu'_B)(xy) = \max(\nu_A(x), \nu'_A(y)) \end{cases}$$

if $xy \in E'$*.*

Proposition 6. *If G_1 and G_2 are the strong intuitionistic fuzzy graphs, then $G_1 + G_2$ is a strong intuitionistic fuzzy graph.*

Definition 11. *Let $A = (\mu_A, \nu_A)$ and $A' = (\mu'_A, \nu'_A)$ be intuitionistic fuzzy subsets of V_1 and V_2 and let $B = (\mu_B, \nu_B)$ and $B' = (\mu'_B, \nu'_B)$ be intuitionistic fuzzy subsets of E_1 and E_2, respectively. The union of two strong intuitionistic fuzzy graphs G_1 and G_2 of the graphs G_1^* and G_2^* is denoted by $G_1 \cup G_2 = (A \cup A', B \cup B')$ and is defined as follows:*

$$\text{(A)} \begin{cases} (\mu_A \cup \mu'_A)(x) = \mu_A(x) \quad \text{if} \quad x \in V_1 \cap \overline{V_2}, \\ (\mu_A \cup \mu'_A)(x) = \mu'_A(x) \quad \text{if} \quad x \in V_2 \cap \overline{V_1}, \\ (\mu_A \cup \mu'_A)(x) = \max(\mu_A(x), \mu'_A(x)) \quad \text{if} \quad x \in V_1 \cap V_2. \end{cases}$$

$$\text{(B)} \begin{cases} (\nu_A \cap \nu'_A)(x) = \nu_A(x) \quad \text{if} \quad x \in V_1 \cap \overline{V_2}, \\ (\nu_A \cap \nu'_A)(x) = \nu'_A(x) \quad \text{if} \quad x \in V_2 \cap \overline{V_1}, \\ (\nu_A \cap \nu'_A)(x) = \min(\nu_A(x), \nu'_A(x)) \quad \text{if} \quad x \in V_1 \cap V_2. \end{cases}$$

$$\text{(C)} \begin{cases} (\mu_B \cup \mu'_B)(xy) = \mu_B(xy) \quad \text{if} \quad xy \in E_1 \cap \overline{E_2}, \\ (\mu_B \cup \mu'_B)(xy) = \mu'_B(xy) \quad \text{if} \quad xy \in E_2 \cap \overline{E_1}, \\ (\mu_B \cup \mu'_B)(xy) = \max(\mu_B(xy), \mu'_B(xy)) \quad \text{if} \quad xy \in E_1 \cap E_2. \end{cases}$$

$$\text{(D)} \begin{cases} (\nu_B \cap \nu'_B)(xy) = \nu_B(xy) \quad \text{if} \quad xy \in E_1 \cap \overline{E_2}, \\ (\nu_B \cap \nu'_B)(xy) = \nu'_B(xy) \quad \text{if} \quad xy \in E_2 \cap \overline{E_1}, \\ (\nu_A \cap \nu'_B)(xy) = \min(\nu_B(xy), \nu'_B(xy)) \quad \text{if} \quad xy \in E_1 \cap E_2. \end{cases}$$

Remark 2. *The union of two strong intuitionistic fuzzy graphs may not be a strong intuitionistic fuzzy graph as it can be seen in the following example.*

a

b

G_1 is strong

(iii)

$$\overline{\mu_B}(xy) = \begin{cases} 0 & \textit{if } \ \mu_B(xy) > 0, \\ \min(\mu_A(x), \mu_A(y)) & \textit{if } \ \textit{if } \mu_B(xy) = 0, \end{cases}$$

$$\overline{\nu_B}(xy) = \begin{cases} 0 & \textit{if } \ \nu_B(xy) > 0, \\ \max(\nu_A(x), \nu_A(y)) & \textit{if } \ \textit{if } \nu_B(xy) = 0. \end{cases}$$

Remark 3. *If $G = (A, B)$ is an intuitionistic fuzzy graph of $G^* = (V, E)$. Then from Definition 12, it follows that $\overline{\overline{G}}$ is given by the intuitionistic fuzzy graph $\overline{\overline{G}} = (\overline{\overline{A}}, \overline{\overline{B}})$ on $G^* = (V, E)$ where $\overline{\overline{A}} = A$ and*

$$\overline{\overline{\mu_B}}(xy) = \min(\mu_A(x), \mu_A(y)), \ \overline{\overline{\nu_B}}(xy) = \max(\nu_A(x), \nu_A(y)) \quad for \ all \ xy \in E.$$

Thus $\overline{\overline{\mu_B}} = \mu_B$ and $\overline{\overline{\nu_B}} = \nu_B$ on V where $B = (\mu_B, \nu_B)$ is the strongest intuitionistic fuzzy relation on A. For any intuitionistic fuzzy graph G, $\overline{G}$ is strong intuitionistic fuzzy graph and $G \subseteq \overline{\overline{G}}$.

The following propositions are obvious.

Proposition 7. *$G = \overline{\overline{G}}$ if and only if G is a strong intuitionistic fuzzy graph.*

Proposition 8. *Let $G = (A_i, B_i)$ be a strong intuitionistic fuzzy graph of $G_i^* = (V_i, E_i)$ for $i = 1, 2$. Then the following are true:*

(a) *$G_i \subseteq \overline{\overline{G_i}}$,*

(b) *$\overline{G_i} = \overline{(\overline{\overline{G_i}})}$,*

(c) *If $G_1 \subseteq G_2$, then $\overline{\overline{G_1}} \subseteq \overline{\overline{G_2}}$.*

Proposition 9. *$\overline{\overline{G}}$ is the smaller strong intuitionistic fuzzy graph that contains $G^* = (V, E)$.*

Definition 13. *A strong intuitionistic fuzzy graph G is called self complementary if $G \approx \overline{G}$.*

Example 3. *Consider a graph $G^* = (V, E)$ such that $V = \{a, b, c\}$, $E = \{ab, bc\}$. Consider a strong intuitionistic fuzzy graph G*

b

c

G

Proposition 12. *Let G_1 and G_2 be strong intuitionistic fuzzy graphs. Then $G_1 \approx G_2$ if and only if $\overline{G}_1 \approx \overline{G}_2$.*

Proof. Assume that G_1 and G_2 are isomorphic, there exists a bijective map $f : V_1 \to V_2$ satisfying

$$\mu_{A_1}(x) = \mu_{A_2}(f(x)),\ \nu_{A_1}(x) = \nu_{A_2}(f(x)) \text{ for all } x \in V_1,$$

$$\mu_{B_1}(xy) = \mu_{B_2}(f(x)f(y)),\ \nu_{B_1}(xy) = \nu_{B_2}(f(x)f(y)) \text{ for all } xy \in E_1.$$

By definition of complement, we have

$$\overline{\mu}_{B_1}(xy) = \min(\mu_{A_1}(x), \mu_{A_1}(y) = \min(\mu_{A_2}(f(x)), \mu_{A_2}(f(y))) = \overline{\mu}_{B_2}(f(x)f(y)),$$

$$\overline{\nu}_{B_1}(xy) = \max(\nu_{A_1}(x), \nu_{A_1}(y) = \max(\nu_{A_2}(f(x)), \nu_{A_2}(f(y))) = \overline{\nu}_{B_2}(f(x)f(y)),$$

for all $xy \in E_1$. Hence, $\overline{G}_1 \approx \overline{G}_2$.

The proof of the converse part is straightforward. This completes the proof. □

Definition 14. *An intuitionistic fuzzy graph graph $G = (A, B)$ is called complete if*

$$\mu_B(xy) = \min(\mu_A(x), \mu_A(y)) \text{ and } \nu_B(xy) = \min(\nu_A(x), \nu_A(y),$$

for all $xy \in E$.

We use the notion $C_m(G)$ for a complete intuitionistic fuzzy graph where $|V| = m$.

Definition 15. *An intuitionistic fuzzy graph $G = (A, B)$ is called bigraph if and only if there exists intuitionistic fuzzy graphs $G_i = (A_i, B_i)$ for $i = 1, 2$ of $G = (A, B)$ such that $G = (A, B)$ is the join $G_1 + G_2$ where $V_1 \cap V_2 = \emptyset$ and $E_1 \cap E_2 = \emptyset$. An intuitionistic fuzzy bigraph is said to be complete if and only if $\mu_B(xy) > 0$, $\nu_B(xy) > 0$ for all $xy \in E$.*

We use the notion $C_{m,n}(G)$ for a complete bigraph, where $|V_1| = m$ and $|V_2| = n$.

Proposition 13. $C_{m,n}(G) = C_m(\overline{G_1}) + C_n(\overline{G_2})$.

Proof. It is straightforward. □

Definition 16. *Let G_1 and G_2 be the strong intuitionistic fuzzy graphs. A homomorphism $f : G_1 \to G_2$ is a mapping $f : V_1 \to V_2$ which satisfies the following conditions:*

(a) $\mu_{A_1}(x_1) \leq \mu_{A_2}(f(x_1))$, $\nu_{A_1}(x_1) \geq \nu_{A_2}(f(x_1))$,

(b) $\mu_{B_1}(x_1y_1) \leq \mu_{B_2}(f(x_1)f(y_1))$, $\nu_{B_1}(x_1y_1) \geq \nu_{B_2}(f(x_1)f(y_1))$

for all $x_1 \in V_1$, $x_1y_1 \in E_1$.

Definition 17. *Let G_1 and G_2 be strong intuitionistic fuzzy graphs. An isomorphism $f : G_1 \to G_2$ is a bijective mapping $f : V_1 \to V_2$ which satisfies the following conditions:*

(c) $\mu_{A_1}(x_1) = \mu_{A_2}(f(x_1))$, $\nu_{A_1}(x_1) = \nu_{A_2}(f(x_1))$,

(d) $\mu_{B_1}(x_1y_1) = \mu_{B_2}(f(x_1)f(y_1))$, $\nu_{B_1}(x_1y_1) = \nu_{B_2}(f(x_1)f(y_1))$,

for all $x_1 \in V_1$, $x_1y_1 \in E_1$.

Definition 18. *Let* G_1 *and* G_2 *be strong intuitionistic fuzzy graphs. Then, a weak isomorphism* $f : G_1 \to G_2$ *is a bijective mapping* $f : V_1 \to V_2$ *which satisfies the following conditions:*

(e) f *is homomorphism,*

(f) $\mu_{A_1}(x_1) = \mu_{A_2}(f(x_1))$, $\nu_{A_1}(x_1) = \nu_{A_2}(f(x_1))$,

for all $x_1 \in V_1$. *Thus, a weak isomorphism preserves the weights of the nodes but not necessarily the weights of the arcs.*

Definition 19. *Let* G_1 *and* G_2 *be the strong intuitionistic fuzzy graphs. A co-weak isomorphism* $f : G_1 \to G_2$ *is a bijective mapping* $f : V_1 \to V_2$ *which satisfies*

(g) f *is homomorphism,*

(h) $\mu_{B_1}(x_1y_1) = \mu_{B_2}(f(x_1)f(y_1))$, $\nu_{B_1}(x_1y_1) = \nu_{B_2}(f(x_1)f(y_1))$

for all $x_1y_1 \in V_1$. *Thus a co- weak isomorphism preserves the weights of the arcs but not necessarily the weights of the nodes.*

Remark 4.

1. *If* $G_1 = G_2 = G$, *then the homomorphism* f *over itself is called an endomorphism. An isomorphism* f *over* G *is called an automorphism*

2. *Let* $A = (\mu_A, \nu_A)$ *be a strong intuitionistic fuzzy graph with an underlying set* V. *Let Aut(G) be the set of all strong intuitionistic automorphisms of* G. *Let* $e : G \to G$ *be a map defined by* $e(x) = x$ *for all* $x \in V$. *Clearly,* $e \in Aut(G)$.

3. *If* $G_1 = G_2$, *then the weak and co-weak isomorphisms actually become isomorphic.*

4. *If* $f : V_1 \to V_2$ *is a bijective map, then* $f^{-1} : V_2 \to V_1$ *is also a bijective map.*

We state the following Propositions without their proofs.

Proposition 14. *Let* G_1 *and* G_2 *be strong intuitionistic fuzzy graphs. If there is a weak isomorphism between* G_1 *and* G_2, *then there is a weak isomorphism between* $\overline{G}_1$ *and* $\overline{G}_2$.

Proposition 15. *Let* G_1 *and* G_2 *be strong intuitionistic fuzzy graphs. If there is a co-weak isomorphism between* G_1 *and* G_2, *then then there is a homomorphism between* $\overline{G}_1$ *and* $\overline{G}_2$.

4 Intuitionistic fuzzy line graphs

In graph theory, the line graph $L(G^*)$ of a simple graph G^* is another graph $L(G^*)$ that represents the adjacencies between edges of G^*. Given a graph G^*, its line graph $L(G^*)$ is a graph such that:

- each vertex of $L(G^*)$ represents an edge of G^*; and
- two vertices of $L(G^*)$ are adjacent if and only if their corresponding edges share a common endpoint ("are adjacent") in G^*.

Definition 20. [18] *Let $G^* = (V, E)$ be an undirected graph, where $V = \{v_1, v_2, \cdots, v_n\}$. Let $S = \{v_i, x_{i1}, \cdots, x_{iq_i}\}$ where $x_{ij} \in E$ has vertex v_i, i=1, 2, $\cdots, n$, $j = 1, 2, \cdots, q_i$. Let $S = \{S_1, S_2, \cdots, S_n\}$. Let $T = \{S_iS_j | S_i, S_j \in S,\ S_i \cap S_j \neq \emptyset, i \neq j\}$. Then $P(S) = (S, T)$ is an intersection graph and $P(S) = G^*$. The line graph $L(G^*)$ is by definition the intersection graph $P(E)$. That is, $L(G^*) = (Z, W)$ where $Z = \{\{x\} \cup \{u_x, v_x\} | x \in E, u_x, v_x \in V, x = u_xv_x\}$ and $W = \{S_xS_y | S_x \cap S_y \neq \emptyset, x, y \in E, x \neq y\}$, and $S_x = \{x\} \cup \{u_x, v_x\}$, $x \in E$.*

We now discuss intuitionistic fuzzy line graphs.

Definition 21. *Let $A_1 = (\mu_{A_1}, \nu_{A_1})$ and $B_1 = (\mu_{B_1}, \nu_{B_1})$ be intuitionistic fuzzy subsets of V and E, respectively. Let $A_2 = (\mu_{A_2}, \nu_{A_2})$ and $B_2 = (\mu_{B_2}, \nu_{B_2})$ be intuitionistic fuzzy sets of Z and W, respectively. We define an intuitionistic fuzzy line graph $L(G) = (A_2, B_2)$ of the intuitionistic fuzzy graph $G = (A_1, B_1)$ as follows:*

(1) $\mu_{A_2}(S_x) = \mu_{B_1}(x) = \mu_{B_1}(u_xv_x)$,

(2) $\nu_{A_2}(S_x) = \nu_{B_1}(x) = \nu_{B_1}(u_xv_x)$,

(3) $\mu_{B_2}(S_xS_y) = \min(\mu_{B_1}(x), \mu_{B_1}(y))$,

(4) $\nu_{B_2}(S_xS_y) = \max(\nu_{B_1}(x), \nu_{B_1}(y))$,

for all $S_x, S_y \in Z$, $S_xS_y \in W$.

Example 4. *Consider a graph $G^* = (V, E)$ such that $V = \{v_1, v_2, v_3, v_4\}$ and $E = \{x_1 = v_1v_2, x_2 = v_2v_3, x_3 = v_3v_4, x_4 = v_4v_1\}$. Let A_1 be an intuitionistic fuzzy subset of V and let B_1 be an intuitionistic fuzzy subset of E defined by*

	v_1	v_2	v_3	v_4
μ_{A_1}	0.2	0.4	0.4	0.3
ν_{A_1}	0.5	0.3	0.5	0.2

	x_1	x_2	x_3	x_4
μ_{B_1}	0.1	0.3	0.2	0.1
ν_{B_1}	0.6	0.6	0.7	0.7

v_1

v_2

G

v_4

v_3 $(0.1, 0.6)$

$(0.1, 0.7)$

$(0.3, 0.$

$($

Proposition 16. *Every intuitionistic fuzzy line graph is a strong intuitionistic fuzzy graph.*

Proposition 17. *$L(G) = (A_2, B_2)$ is an intuitionistic fuzzy line graph corresponding to intuitionistic fuzzy graph $G = (A_1, B_1)$.*

Proposition 18. *$L(G) = (A_2, B_2)$ is an intuitionistic fuzzy line graph of some intuitionistic fuzzy graph $G = (A_1, B_1)$ if and only if*

$$\mu_{B_2}(S_xS_y) = \min(\mu_{A_2}(S_x), \mu_{A_2}(S_y)) \quad \text{for all } \mathrm{S_x}, \mathrm{S_y} \in \mathrm{W},$$

$$\nu_{B_2}(S_xS_y) = \max(\nu_{A_2}(S_x), \nu_{A_2}(S_y)) \quad \text{for all } \mathrm{S_x}, \mathrm{S_y} \in \mathrm{W}.$$

Proof. Assume that $\mu_{B_2}(S_xS_y) = \min(\mu_{A_2}(S_x), \mu_{A_2}(S_y))$ for all $S_x,\ S_y \in W$. We define $\mu_{A_1}(x) = \mu_{A_2}(S_x)$ for all $x \in E$. Then

$$\mu_{B_2}(S_xS_y) = \min(\mu_{A_2}(S_x), \mu_{A_2}(S_y)) = \min(\mu_{A_2}(x), \mu_{A_2}(y)),$$

$$\nu_{B_2}(S_xS_y) = \max(\nu_{A_2}(S_x), \nu_{A_2}(S_y)) = \max(\nu_{A_2}(x), \nu_{A_2}(y)).$$

An intuitionistic fuzzy set $A_1 = (\mu_{A_1}, \nu_{A_1})$ that yields that the property

$$\mu_{B_1}(xy) \leq \min(\mu_{A_1}(x), \mu_{A_1}(y)),$$

$$\nu_{B_1}(xy) \geq \max(\nu_{A_1}(x), \nu_{A_1}(y))$$

will suffice.

The converse part is obvious. □

Proposition 19. *If $L(G) = (A_2, B_2)$ is an intuitionistic fuzzy line graph of intuitionistic fuzzy graph $G = (A_1, B_1)$. Then $L(G^*) = (Z, W)$ is the line graph of $G^* = (V, E)$.*

Proof. Since $G = (A_1, B_1)$ is an intuitionistic fuzzy graph and $L(G)$ is an intuitionistic fuzzy line graph,

$$\mu_{A_1}(S_x) = \mu_{B_1}(x), \nu_{A_1}(S_x) = \nu_{B_1}(x) \quad for\ all\ x \in E$$

and so $S_x \in Z \Leftrightarrow x \in E$. Also

$$\mu_{B_2}(S_xS_y) = \min(\mu_{B_2}(x), \mu_{B_2}(y)),$$

$$\nu_{B_2}(S_xS_y) = \max(\nu_{B_2}(x), \nu_{B_2}(y))$$

for all $S_x, S_y \in W$, and so

$$W = \{S_xS_y | s_x \cap S_y \neq \emptyset, x, y \in E, x \neq y\}.$$

This completes the proof. □

Not all graphs are line graphs of some graphs. The following result tell us when an intuitionistic fuzzy graph is an intuitionistic fuzzy line graph of some intuitionistic fuzzy graph.

Proposition 20. *$L(G) = (A_2, B_2)$ is an intuitionistic fuzzy line graph if and only if $L(G^*) = (Z, W)$ is a line graph and*

$$\mu_{B_2}(uv) = \min(\mu_{A_2}(u), \mu_{A_2}(v)) \quad \text{for all uv} \in \text{W},$$

$$\nu_{B_2}(uv) = \max(\nu_{A_2}(u), \nu_{A_2}(v)) \quad \text{for all uv} \in \text{W}.$$

Proof. Follows from Propositions 18 and 19. □

We sate the following results without their proofs.

Theorem 1. *Let $L(G) = (A_2, B_2)$ be the intuitionistic fuzzy line graph corresponding to intuitionistic fuzzy graph $G = (A_1, B_1)$. Suppose that $G^* = (V, E)$ is connected. Then there exists a week isomorphism of $L(G)$ onto G if and only if G^* is a cyclic and for all $v \in V$, $x \in E$, $\mu_{A_1}(v) = \mu_{B_1}(x)$, $\nu_{A_1}(v) = \nu_{B_1}(x)$, i.e., $A_1 = (\mu_{A_1}, \nu_{A_1})$ and $B_1 = (\mu_{B_1}, \nu_{B_1})$ are constant functions on V and E, respectively, taking on the same value.*

Theorem 2. *Let $L(G) = (A_2, B_2)$ be the intuitionistic fuzzy line graph corresponding to intuitionistic fuzzy graph $G = (A_1, B_1)$. Suppose that $G^* = (V, E)$ is connected. If f is a weak isomorphism of G onto $L(G)$, then f is an isomorphism.*

Theorem 3. *Let G and H be intuitionistic fuzzy graphs of G^* and H^*, respectively, such that G^* and H^* are connected. Let $L(G)$ and $L(H)$ be the intuitionistic fuzzy line graphs corresponding to G and H, respectively. Suppose that it is not the case that one of G^* and H^* is complete graph K_3 and other is bipartite complete graph $K_{1,3}$. If $L(G)$ and $L(H)$ are isomorphic, then G and H are line-isomorphic.*

Acknowledgement: The authors are thankful to the referee for his valuable comments and suggestions for improving the paper.

5 Conclusions

An intuitionistic fuzzy set is a generalization of the notion of a fuzzy set. Intuitionistic fuzzy models give more precision, flexibility and compatibility to the system as compared to the classic and fuzzy models. We have introduced the concepts of (i) strong intuitionistic fuzzy graphs, (ii) intuitionistic fuzzy line graphs, and have presented some of their properties in this paper. It is clear that the most of these results can be simply extended to (S, T)-fuzzy graphs, where S and T are given imaginable triangular norms. The obtained results can be applied in various areas of engineering, computer science: artificial intelligence, signal processing, pattern recognition, robotics, computer networks, expert systems, and medical diagnosis. Our future plan to extend our research of fuzzification to (1) Bipolar fuzzy hypergraphs; (2) Intuitionistic fuzzy hypergraphs; (3) Vague hypergraphs; (4) Interval-valued hypergraphs; (5) Soft fuzzy hypergraphs.

References

[1] M. Akram, *Intuitionistic (S,T)-fuzzy Lie ideals of Lie algebras*, Quasigroups and Related Systems 15 (2007, 201-218.

[2] M. Akram and W.A. Dudek, *Interval-valued fuzzy graphs*, Computers and Mathematics with Applications 61 (2011), 289-299.

[3] M. Akram and W.A. Dudek, *Intuitionistic fuzzy left k-ideals of semirings*, Soft Comput. 12(2008), 881-890.

[4] A. Alaoui, *On fuzzification of some concepts of graphs*, Fuzzy Sets and Systems 101(1999), 363-389.

[5] K.T. Atanassov, *Intuitionistic fuzzy sets: Theory and applications, Studies in fuzziness and soft computing*, Heidelberg, New York, Physica-Verl., 1999.

[6] K. T. Atanassov, *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems 20(1986), 87-96.

[7] K.T. Atanassov, G. Pasi, R. Yager, V. Atanassova, *Intuitionistic fuzzy graph interpretations of multi-person multi-criteria decision making*, EUSFLAT Conf. 2003, 177-182.

[8] P. Bhattacharya, *Some remarks on fuzzy graphs*, Pattern Recognition Letter **6**(1987), 297-302.

[9] K.R. Bhutani, *On automorphism of fuzzy graphs*, Pattern Recognition Letter **9**(1989), 159-162.

[10] K.R. Bhutani and A. Rosenfeld, *Strong arcs in fuzzy graphs*, Information Sciences **152** (2003), 319322.

[11] K.R. Bhutani and A. Battou, On M-strong fuzzy graphs, Information Sciences **155** (2003), 103109.

[12] R. Biswas, *Intuitionistic fuzzy subgroups*, Mathematical Forum **10**(1989), 37-46.

[13] S.K. De, R. Biswas and A.R. Roy, *An application of intuitionistic fuzzy sets in medical diagnosis*, Fuzzy Sets and Systems **117** (2001), 209-213.

[14] K.P. Huber and M.R. Berthold,*Application of fuzzy graphs for metamodeling*, Proceedings of the 2002 IEEE Conference, 640-644.

[15] S. Mathew and M.S. Sunitha, *Types of arcs in a fuzzy graph*, Information Sciences, **179**(2009), 1760-1768.

[16] S. Mathew and M.S. Sunitha, *Node connectivity and arc connectivity of a fuzzy graph*, Information Sciences, **180**(2010), 519-531.

[17] J.N. Mordeson and C.S. Peng, *Operations on fuzzy graphs*, Information Sciences **79** (1994), 159-170.

[18] J.N. Mordeson, *Fuzzy line graphs*, Pattern Recognition Letter **14**(1993), 381-384.

[19] J.N. Mordeson and P.S. Nair, *Fuzzy graphs and fuzzy hypergraphs*, Physica Verlag, Heidelberg 1998; Second Edition 2001.

[20] J.N. Mordeson and P.S. Nair, *Cycles and cocyles of fuzzy graphs*, Information Sciences **90** (1996), 39-49.

[21] F. Harary, *Graph Theory*, 3rd Edition, Addison-Wesley, Reading, MA, October 1972.

[22] A. Kiss, *An application of fuzzy graphs in database theory*, Pure Mathematics and Applications, **1** (1991), 337-342.

[23] M.G. Karunambigai, P. Rangasamy, K.T. Atanassov and N. Palaniappan, *An intuitionistic fuzzy graph method for finding the shortest paths in networks*, O. Castillo et al. (Eds.): Theor. Adv. and Appl. of Fuzzy Logic, ASC 42, 2007, 3-10.

[24] R. Parvathi and M.G. Karunambigai, *Intuitionistic fuzzy graphs*, Journal of computational Intelligence: Theory and Applications, 2006, 139-150.

[25] G. Pasi, R. Yager, K.T. Atanassov, *Intuitionistic fuzzy graph interpretations of multi-person multi-criteria decision making: generalized net approach*, Intelligent Systems, 2004. Proceedings. 2004 2nd International IEEE Conference, 434-439, vol. 2.

[26] R. Parvathi, M.G. Karunambigai and K.T. Atanassov, Operations on intuitionistic fuzzy graphs, Fuzzy Systems, 2009. FUZZ-IEEE 2009. IEEE International Conference, 1396-1401.

[27] A. Rosenfeld, *Fuzzy graphs*, Fuzzy Sets and their Applications (L.A. Zadeh, K.S. Fu, M. Shimura, Eds.), Academic Press, New York, 1975, 77-95.

[28] A. Shannon, K.T. Atanassov, *A first step to a theory of the intuitionistic fuzzy graphs*, Proceeding of FUBEST (D. Lakov, Ed.), Sofia, 1994, 59-61.

[29] A. Shannon, K.T. Atanassov, *Intuitionistic fuzzy graphs from $\alpha-$, $\beta-$, and (α, b)- levels*, Notes on Intuitionistic Fuzzy Sets **1** (1995), 32-35.

[30] M.S. Sunitha and A. Vijayakumar, *Complement of a fuzzy graph*, Indian Journal of Pure and Applied Mathematics **33**(9),1451-1464.

[31] L.A. Zadeh, *Fuzzy sets*, Information and Control **8** (1965), 338-353.

[32] L.A. Zadeh, *Similarity relations and fuzzy orderings*, Information Sciences **3**(1971), 177-200.

[33] L. A. Zadeh, *The concept of a lingusistic variable and its application to approximate reasoning*, Information Sciences **8**(1975), 199-249.

Muhammad Akram:
Punjab University College of Information Technology, University of the Punjab, Old Campus, P. O. Box 54000, Lahore, Pakistan.
E-mail: m.akram@pucit.edu.pk

Bijan Davvaz:
Department of Mathematics, Yazd University Yazd, Iran.
E-mail: davvaz@yazduni.ac.ir