Strong Intuitionistic Fuzzy Graphs

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Abstract

We introduce the notion of strong intuitionistic fuzzy graphs and investigate some of their properties. We discuss some propositions of self complementary and self weak complementary strong intuitionistic fuzzy graphs. We introduce the concept of intuitionistic fuzzy line graphs.

1 Introduction

In 1736, Euler first introduced the concept of graph theory. In the history of mathematics, the solution given by Euler of the well known Königsberg bridge problem is considered to be the first theorem of graph theory. This has now become a subject generally regarded as a branch of combinatorics. The theory of graph is an extremely useful tool for solving combinatorial problems in different areas such as geometry, algebra, number theory, topology, operations research, optimization and computer science.

In 1983, Atanassov [6] introduced the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets [31]. Atanassov added a new component (which determines the degree of non-membership) in the definition of fuzzy set. The fuzzy sets give the degree of membership of an element in a given set (and the non-membership degree equals one minus the degree of membership), while intuitionistic fuzzy sets give both a degree of membership and a degree of non-membership which are more-or-less independent from each other, the only requirement is that the sum of these two degrees is not greater than 1. Intuitionistic fuzzy sets have been applied in a wide variety of fields including computer science, engineering, mathematics, medicine, chemistry and economics [5, 13].

In 1975, Rosenfeld [27] introduced the concept of fuzzy graphs. The fuzzy relations between fuzzy sets were also considered by Rosenfeld and he developed the structure of fuzzy graphs, obtaining analogs of several graph theoretical concepts.
Later on, Bhattacharya [8] gave some remarks on fuzzy graphs, and some operations on fuzzy graphs were introduced by Mordeson and Peng [17]. The complement of a fuzzy graph was defined by Mordeson [19] and further studied by Sunitha and Vijayakumar [30]. Mordeson [18] introduced the notion of fuzzy line graph. Bhutani and Rosenfeld introduced the concept of M-strong fuzzy graphs in [11] and studied some of their properties. Akram and Dudek [2] discussed interval-valued fuzzy graphs. The complement of a fuzzy graph. In this article, we introduce the notion of strong intuitionistic fuzzy graphs and investigate some of their properties. We discuss some propositions of self complementary and self weak complementary strong intuitionistic fuzzy graphs.

2 Preliminaries

In this section, we review some definitions that are necessary in the paper.

A graph is an ordered pair $G^* = (V, E)$, where $V$ is the set of vertices of $G^*$ and $E$ is the set of edges of $G^*$. Two vertices $x$ and $y$ in a graph $G^*$ are said to be adjacent in $G^*$ if $(x, y)$ is in an edge of $G^*$ (for simplicity an edge $\{x, y\}$ will be denoted by $xy$). A simple graph is a graph without loops and multiple edges. An isomorphic graph is a graph in which every pair of distinct vertices is connected by an edge. The complete graph on $n$ vertices has $n(n-1)/2$ edges. We will consider only graphs with the finite number of vertices and edges.

An isomorphism of graphs $G_1^*$ and $G_2^*$ is a bijection between the vertex sets of $G_1^*$ and $G_2^*$ such that any two vertices $v_1$ and $v_2$ of $G_1^*$ are adjacent in $G_1^*$ if and only if $f(v_1)$ and $f(v_2)$ are adjacent in $G_2^*$. Isomorphic graphs are denoted by $G_1^* \simeq G_2^*$. By a complementary graph $G^*$ of a simple graph $G$ we mean a graph having the same vertices as $G$ and such that two vertices are adjacent in $G^*$ if and only if they are not adjacent in $G$. A simple graph that is isomorphism to its complement is called self-complementary.

Let $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ be two simple graphs, we can construct several new graphs. The first construction called the Cartesian product of $G_1^*$ and $G_2^*$ gives a graph $G_1^* \times G_2^* = (V, E)$ with $V = V_1 \times V_2$ and

$$E = \{(x, x_2)(x, y_2)|x \in V_1, x_2y_2 \in E_2\} \cup \{(x_1, z)(y_1, z)|x_1y_1 \in E_1, z \in V_2, \}.$$  

The composition of graphs $G_1^*$ and $G_2^*$ is the graph $G_1^*[G_2^*] = (V_1 \times V_2, E^0)$, where

$$E^0 = E \cup \{(x_1, x_2)(y_1, y_2)|x_1y_1 \in E_1, x_2 \neq y_2\}$$

and $E$ is defined as in $G_1^* \times G_2^*$. Note that $G_1^*[G_2^*] \neq G_2^*[G_1^*]$.

The union of graphs $G_1^*$ and $G_2^*$ is defined as $G_1^* \cup G_2^* = (V_1 \cup V_2, E_1 \cup E_2)$. The join of $G_1^*$ and $G_2^*$ is the simple graph $G_1^* + G_2^* = (V_1 \cup V_2, E_1 \cup E_2 \cup E'),$
where $E'$ is the set of all edges joining the nodes of $V_1$ and $V_2$. In this construction it is assumed that $V_1 \cap V_2 = \emptyset$.

**Definition 1.** [31, 32] By a fuzzy subset $\mu$ on a set $X$ is mean a map $\mu : X \to [0,1]$. A map $\nu : X \times X \to [0,1]$ is called a fuzzy relation on $X$ if $\nu(x,y) \leq \min(\mu(x), \mu(y))$ for all $x, y \in X$. A fuzzy relation $\nu$ is symmetric if $\nu(x,y) = \nu(y,x)$ for all $x, y \in X$.

**Definition 2.** [5] A mapping $A = (\mu_A, \nu_A) : X \to [0,1] \times [0,1]$ is called an intuitionistic fuzzy set in $X$ if $\mu_A(x) + \nu_A(x) \leq 1$ for all $x \in X$, where the mappings $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\nu_A(x)$) of each element $x \in X$ to $A$, respectively.

**Definition 3.** [5] For every two intuitionistic fuzzy sets $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ in $X$, we define

- $(A \cap B)(x) = (\min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x)))$,
- $(A \cup B)(x) = (\max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x)))$.

**Definition 4.** [5] Let $X$ be a nonempty set. Then we call a mapping $A = (\mu_A, \nu_A) : X \times X \to [0,1] \times [0,1]$ an intuitionistic fuzzy relation on $X$ if $\mu_A(x,y) + \nu_A(x,y) \leq 1$ for all $(x, y) \in X$.

**Definition 5.** [5] Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be intuitionistic fuzzy sets on a set $X$. If $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy relation on a set $X$, then $A = (\mu_A, \nu_A)$ is called an intuitionistic fuzzy relation on $B = (\mu_B, \nu_B)$ if $\mu_A(x,y) \leq \min(\mu_B(x), \mu_B(y))$ and $\nu_A(x,y) \geq \max(\nu_B(x), \nu_B(y))$ for all $x, y \in X$. An intuitionistic fuzzy relation $A$ on $X$ is called symmetric if $\mu_A(x,y) = \mu_A(y,x)$ and $\nu_A(x,y) = \nu_A(y,x)$ for all $x, y \in X$.

## 3 Strong intuitionistic fuzzy graphs

Throughout this paper, we denote $G^*$ a crisp graph, and $G$ an intuitionistic fuzzy graph.

**Definition 6.** An intuitionistic fuzzy graph with underlying set $V$ is defined to be a pair $G = (A, B)$ where

(i) the functions $\mu_A : V \to [0,1]$ and $\nu_A : V \to [0,1]$ denote the degree of membership and nonmembership of the element $x \in V$, respectively such that $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for all $x \in V$,

(ii) the functions $\mu_B : E \subseteq V \times V \to [0,1]$ and $\nu_B : E \subseteq V \times V \to [0,1]$ are defined by

$\mu_B((x,y)) \leq \min(\mu_A(x), \mu_A(y))$ and $\nu_B((x,y)) \geq \max(\nu_A(x), \nu_A(y))$

such that $0 \leq \mu_B((x,y)) + \nu_B((x,y)) \leq 1$ for all $(x, y) \in E$. 


We call $A$ the intuitionistic fuzzy vertex set of $V$, $B$ the intuitionistic fuzzy edge set of $G$, respectively. Note that $B$ is a symmetric intuitionistic fuzzy relation on $A$. We use the notation $xy$ for an element of $E$. Thus, $G = (A, B)$ is an intuitionistic graph of $G^* = (V, E)$ if

$$\mu_B(xy) \leq \min(\mu_A(x), \mu_A(y)) \quad \text{and} \quad \nu_B(xy) \geq \max(\nu_A(x), \nu_A(y))$$

for all $xy \in E$.

We now study strong intuitionistic fuzzy graphs.

**Definition 7.** An intuitionistic fuzzy graph $G = (A, B)$ is called strong intuitionistic fuzzy graph if

$$\mu_B(xy) = \min(\mu_A(x), \mu_A(y)) \quad \text{and} \quad \nu_B(xy) = \max(\nu_A(x), \nu_A(y))$$

for all $xy \in E$.

**Example 1.** Consider a graph $G^*$ such that $V = \{x, y, z\}$, $E = \{xy, yz, zx\}$. Let $A$ be an intuitionistic fuzzy subset of $V$ and let $B$ be an intuitionistic fuzzy subset of $E$ defined by

<table>
<thead>
<tr>
<th></th>
<th>$x$</th>
<th>$y$</th>
<th>$z$</th>
<th>$xy$</th>
<th>$yz$</th>
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</tr>
</thead>
<tbody>
<tr>
<td>$\mu_A$</td>
<td>0.2</td>
<td>0.3</td>
<td>0.1</td>
<td>0.2</td>
<td>0.1</td>
<td>0.1</td>
</tr>
<tr>
<td>$\nu_A$</td>
<td>0.4</td>
<td>0.1</td>
<td>0.5</td>
<td>0.4</td>
<td>0.5</td>
<td>0.5</td>
</tr>
</tbody>
</table>

The graph $G$ is represented by the following adjacency matrix

$$A = \begin{bmatrix}
0.2 & 0.4 & 0.1 \\
0.3 & 0.5 & 0.1 \\
0.1 & 0.5 & 0.5
\end{bmatrix}$$

By routine computations, it is easy to see that $G$ is a strong intuitionistic fuzzy graph of $G^*$. 


Definition 8. Let $A = (\mu_A, \nu_A)$ and $A' = (\mu_A', \nu_A')$ be intuitionistic fuzzy subsets of $V_1$ and $V_2$ and let $B = (\mu_B, \nu_B)$ and $B' = (\mu_B', \nu_B')$ be intuitionistic fuzzy subsets of $E_1$ and $E_2$, respectively. The Cartesian product of two strong intuitionistic fuzzy graphs $G_1$ and $G_2$ of the graphs $G_1'$ and $G_2'$ is denoted by $G_1 \times G_2 = (A \times A', B \times B')$ and is defined as follows:

(i) \[
\begin{align*}
(\mu_A \times \mu_A')(x_1, x_2) &= \min(\mu_A(x_1), \mu_A'(x_2)) \\
(\nu_A \times \nu_A')(x_1, x_2) &= \max(\nu_A(x_1), \nu_A'(x_2))
\end{align*}
\]
for all $(x_1, x_2) \in V$,

(ii) \[
\begin{align*}
(\mu_B \times \mu_B')(x_1, x_2) &= \min(\mu_B(x_1), \mu_B'(x_2,y_2)), \\
(\nu_B \times \nu_B')(x_1, x_2) &= \max(\nu_B(x_1), \nu_B'(x_2,y_2))
\end{align*}
\]
for all $x \in V_1$, for all $x_2y_2 \in E_2$,

(iii) \[
\begin{align*}
(\mu_B \times \mu_B')(x_1, z(y_1, z)) &= \min(\mu_B(x_1, y_1), \mu_B'(z)) \\
(\nu_B \times \nu_B')(x_1, z(y_1, z)) &= \max(\nu_B(x_1, y_1), \nu_B'(z))
\end{align*}
\]
for all $z \in V_2$, for all $x_1y_1 \in E_1$.

Proposition 1. If $G_1$ and $G_2$ are the strong intuitionistic fuzzy graphs, then $G_1 \times G_2$ is a strong intuitionistic fuzzy graph.

Proof. It is straightforward. \qed

Proposition 2. If $G_1 \times G_2$ is strong intuitionistic fuzzy graph, then at least $G_1$ or $G_2$ must be strong.

Proof. Suppose that $G_1$ and $G_2$ are not strong intuitionistic fuzzy graphs. Then there exist $x_1y_1 \in E_1$ and $x_2y_2 \in E_2$ such that

\[
\begin{align*}
\mu_{B_1}(x_1y_1) &< \min(\mu_{A_1}(x), \mu_{A_1}(y)), \quad \mu_{B_2}(x_1y_1) < \min(\mu_{A_2}(x), \mu_{A_2}(y)) \\
\nu_{B_1}(x_1y_1) &> \max(\nu_{A_1}(x), \nu_{A_1}(y)), \quad \nu_{B_2}(x_1y_1) > \max(\nu_{A_2}(x), \nu_{A_2}(y))
\end{align*}
\]
Assume that

\[
\mu_{B_2}(x_2y_2) \leq \mu_{B_1}(x_1y_1) < \min(\mu_{A_1}(x_1), \mu_{A_1}(y_1)) \leq \mu_{A_1}(x_1)
\]
Let

\[E = \{(x, x_2)(y_2) \in V_1 \times x_2y_2 \in E_2 \} \cup \{(x_1, z)(y_1, z) \in V_2 \times x_1y_1 \in E_1 \}.
\]
Consider $(x, x_2)(y, y_2) \in E$, we have

\[
(\mu_{B_1} \times \mu_{B_2})(x, x_2)(y, y_2) = \min(\mu_{A_1}(x), \mu_{A_2}(x_2), \mu_{A_1}(y), \mu_{A_2}(y_2)) < \min(\mu_{A_1}(x), \mu_{A_2}(x_2), \mu_{A_1}(y), \mu_{A_2}(y_2))
\]
and

\[
(\mu_{A_1} \times \mu_{A_2})(x_1, x_2) = \min(\mu_{A_1}(x_1), \mu_{A_2}(x_2)), \quad \mu_{A_1} \times \mu_{A_2}(x_1, y_2) = \min(\mu_{A_1}(x_1), \mu_{A_2}(y_2)).
\]
Therefore,
\[ \min((\mu_{A_1} \times \mu_{A_2})(x, x_2), (\mu_{A_1} \times \mu_{A_2})(x, y_2)) = \min(\mu_{A_1}(x), \mu_{A_2}(x_2), \mu_{A_2}(y_2)). \]
Hence,
\[ (\mu_{B_1} \times \mu_{B_2})(((x, x_2)(x, y_2)) < \min((\mu_{A_1} \times \mu_{A_2})(x, x_2), (\mu_{A_1} \times \mu_{A_2})(x, y_2)). \]
Similarly, we can easily show that
\[ (\nu_{B_1} \times \nu_{B_2})(((x, x_2)(x, y_2)) > \max((\nu_{A_1} \times \nu_{A_2})(x, x_2), (\nu_{A_1} \times \nu_{A_2})(x, y_2)). \]
That is, \( G_1 \times G_2 \) is not strong intuitionistic fuzzy graph, a contradiction. Hence, this ends the proof.

\[ \square \]

**Remark 1.** If \( G_1 \) is strong and \( G_2 \) is not strong, then \( G_1 \times G_2 \) may or may not be strong.

**Example 2.** We consider the following examples:

\[ c \]

\[ a \]

\[ G_2 \text{ is not strong} \]

\[ b \]

\[ G_1 \text{ is strong} \]

\[ d \]
We state a nice Proposition without its proof.

**Proposition 3.** Let $G_1$ be a strong intuitionistic fuzzy graph of $G_1^*$. Then for any intuitionistic fuzzy graph $G_2$ of $G_2^*$, $G_1 \times G_2$ is strong if and only if

$$\mu_A(x_1) \leq \mu_B(x_2y_2), \quad \nu_A(x_1) \geq \nu_B(x_2y_2) \quad \text{for all } x_1 \in V_1 \text{ and } x_2y_2 \in E_2.$$

**Definition 9.** Let $A = (\mu_A, \nu_A)$ and $A' = (\mu_A', \nu_A')$ be intuitionistic fuzzy subsets of $V_1$ and $V_2$ and let $B = (\mu_B, \nu_B)$ and $B' = (\mu_B', \nu_B')$ be intuitionistic fuzzy subsets of $E_1$ and $E_2$, respectively. The composition of two strong intuitionistic fuzzy graphs $G_1$ and $G_2$ of the graphs $G_1^*$ and $G_2^*$ is denoted by $G_1[G_2] = (A \circ A', B \circ B')$ and is defined as follows:

(i) \[
(\mu_A \circ \mu_A')(x_1, x_2) = \min(\mu_A(x_1), \mu_A'(x_2)), \quad (\nu_A \circ \nu_A')(x_1, x_2) = \max(\nu_A(x_1), \nu_A'(x_2)),
\]
for all $(x_1, x_2) \in V$, 

(ii) \[
(\mu_B \circ \mu_B')(x, x)(x, y_2) = \min(\mu_A(x), \mu_A'(x_2y_2)), \quad (\nu_B \circ \nu_B')(x, x)(x, y_2) = \max(\nu_A(x), \nu_A'(x_2y_2)),
\]
for all $x \in V_1$, for all $x_2y_2 \in E_2$, 

(iii) \[
(\mu_B \circ \mu_B')(x_1, z)(y_1, z) = \min(\mu_B(x_1y_1), \mu_A'(z)), \quad (\nu_B \circ \nu_B')(x_1, z)(y_1, z) = \max(\nu_B(x_1y_1), \nu_A'(z)) \quad \text{for all } z \in V_2,
\]
for all $x_1y_1 \in E_1$, 

(iv) \[
(\mu_B \circ \mu_B')(x_1, x_2)(y_1, y_2) = \min(\mu_A(x_2), \mu_A'(y_2), \mu_B(x_1y_1)), \quad (\nu_B \circ \nu_B')(x_1, x_2)(y_1, y_2) = \max(\nu_A(x_2), \nu_A'(y_2), \nu_B(x_1y_1)),
\]
for all $(x_1, x_2)(y_1, y_2) \in E^0 - E$.

We state the following propositions without their proofs.

**Proposition 4.** If $G_1$ and $G_2$ are the strong intuitionistic fuzzy graphs, then $G_1[G_2]$ is a strong intuitionistic fuzzy graph.

**Proposition 5.** If $G_1[G_2]$ is strong intuitionistic fuzzy graph, then at least $G_1$ or $G_2$ must be strong.

**Definition 10.** Let $A = (\mu_A, \nu_A)$ and $A' = (\mu_A', \nu_A')$ be intuitionistic fuzzy subsets of $V_1$ and $V_2$ and let $B = (\mu_B, \nu_B)$ and $B' = (\mu_B', \nu_B')$ be intuitionistic fuzzy subsets of $E_1$ and $E_2$, respectively. The join of two intuitionistic fuzzy graphs $G_1$ and $G_2$ of the graphs $G_1^*$ and $G_2^*$ is denoted by $G_1 + G_2 = (A + A', B + B')$ and is defined as follows:

(i) \[
(\mu_A + \mu_A')(x) = (\mu_A + \mu_A')(x), \quad (\nu_A + \nu_A')(x) = (\nu_A + \nu_A')(x),
\]
if $x \in V_1 \cup V_2$, 

We state a nice Proposition without its proof.
Remark 2. The union of two strong intuitionistic fuzzy graphs may not be a strong intuitionistic fuzzy graph as it can be seen in the following example.
$G_1$ is strong
(iii) \[
\mu_B(xy) = \begin{cases} 
0 & \text{if } \mu_B(xy) > 0, \\
\min(\mu_A(x), \mu_A(y)) & \text{if } \mu_B(xy) = 0,
\end{cases}
\]
\[
\nu_B(xy) = \begin{cases} 
0 & \text{if } \nu_B(xy) > 0, \\
\max(\nu_A(x), \nu_A(y)) & \text{if } \nu_B(xy) = 0.
\end{cases}
\]

Remark 3. If \( G = (A, B) \) is an intuitionistic fuzzy graph of \( G^* = (V, E) \). Then from Definition 12, it follows that \( \overline{G} \) is given by the intuitionistic fuzzy graph \( \overline{G} = (\overline{A}, \overline{B}) \) on \( G^* = (V, E) \) where \( \overline{A} = A \) and
\[
\overline{\mu}_B(xy) = \min(\mu_A(x), \mu_A(y)), \quad \overline{\nu}_B(xy) = \max(\nu_A(x), \nu_A(y)) \quad \text{for all } xy \in E.
\]

Thus \( \overline{\mu}_B = \mu_B \) and \( \overline{\nu}_B = \nu_B \) on \( V \) where \( B = (\mu_B, \nu_B) \) is the strongest intuitionistic fuzzy relation on \( A \). For any intuitionistic fuzzy graph \( G \), \( \overline{G} \) is strong intuitionistic fuzzy graph and \( G \subseteq \overline{G} \).

The following propositions are obvious.

Proposition 7. \( G = \overline{G} \) if and only if \( G \) is a strong intuitionistic fuzzy graph.

Proposition 8. Let \( G = (A_i, B_i) \) be a strong intuitionistic fuzzy graph of \( G_i^* = (V_i, E_i) \) for \( i = 1, 2 \). Then the following are true:

(a) \( G_i \subseteq \overline{G}_i \),

(b) \( \overline{G}_i = (\overline{G}_i) \),

(c) If \( G_1 \subseteq G_2 \), then \( \overline{G}_1 \subseteq \overline{G}_2 \).

Proposition 9. \( \overline{G} \) is the smaller strong intuitionistic fuzzy graph that contains \( G^* = (V, E) \).

Definition 13. A strong intuitionistic fuzzy graph \( G \) is called self complementary if \( G \approx \overline{G} \).

Example 3. Consider a graph \( G^* = (V, E) \) such that \( V = \{a, b, c\}, E = \{ab, bc\} \). Consider a strong intuitionistic fuzzy graph \( G \)
Clearly, $G = \overline{G}$. Hence, $G$ is self complementary.

**Proposition 10.**
Let $G$ be a self complementary strong intuitionistic fuzzy graph. Then

\[
\sum_{x \neq y} \mu_B(xy) = \sum_{x \neq y} \min(\mu_A(x), \mu_A(y)) \quad \sum_{x \neq y} \nu_B(xy) = \sum_{x \neq y} \max(\nu_A(x), \nu_A(y)).
\]

**Proposition 11.**
Let $G$ be a strong intuitionistic fuzzy graph. If

\[
\mu_B(xy) = \min(\mu_A(x), \mu_A(y)) \quad \nu_B(xy) = \max(\nu_A(x), \nu_A(y))
\]

for all $x, y \in V$, then $G$ is self complementary.

**Proof.**
Let $G$ be a strong intuitionistic fuzzy graph such that

\[
\mu_B(xy) = \min(\mu_A(x), \mu_A(y)) \quad \nu_B(xy) = \max(\nu_A(x), \nu_A(y))
\]

for all $x, y \in V$. Then $G \approx \overline{G}$ under the identity map $I: V \rightarrow V$. Hence, $G$ is self complementary.
Proposition 12. Let $G_1$ and $G_2$ be strong intuitionistic fuzzy graphs. Then $G_1 \approx G_2$ if and only if $\overline{G}_1 \approx \overline{G}_2$.

Proof. Assume that $G_1$ and $G_2$ are isomorphic, there exists a bijective map $f : V_1 \rightarrow V_2$ satisfying

\[
\mu_{A_1}(x) = \mu_{A_2}(f(x)), \quad \nu_{A_1}(x) = \nu_{A_2}(f(x)) \quad \text{for all } x \in V_1,
\]

\[
\mu_{B_1}(xy) = \mu_{B_2}(f(x)f(y)), \quad \nu_{B_1}(xy) = \nu_{B_2}(f(x)f(y)) \quad \text{for all } xy \in E_1.
\]

By definition of complement, we have

\[
\overline{\mu}_{B_1}(xy) = \min(\mu_{A_1}(x), \mu_{A_1}(y)) = \min(\mu_{A_2}(f(x)), \mu_{A_2}(f(y))) = \overline{\mu}_{B_2}(f(x)f(y)),
\]

\[
\overline{\nu}_{B_1}(xy) = \max(\nu_{A_1}(x), \nu_{A_1}(y)) = \max(\nu_{A_2}(f(x)), \nu_{A_2}(f(y))) = \overline{\nu}_{B_2}(f(x)f(y))
\]

for all $xy \in E_1$. Hence, $\overline{G}_1 \approx \overline{G}_2$.

The proof of the converse part is straightforward. This completes the proof. \qed

Definition 14. An intuitionistic fuzzy graph $G = (A, B)$ is called complete if

\[
\mu_B(xy) = \min(\mu_A(x), \mu_A(y)) \quad \text{and} \quad \nu_B(xy) = \min(\nu_A(x), \nu_A(y)),
\]

for all $xy \in E$.

We use the notion $C_m(G)$ for a complete intuitionistic fuzzy graph where $|V| = m$.

Definition 15. An intuitionistic fuzzy graph $G = (A, B)$ is called bigraph if and only if there exists intuitionistic fuzzy graphs $G_i = (A_i, B_i)$ for $i = 1, 2$ of $G = (A, B)$ such that $G = G_1 + G_2$ where $V_1 \cap V_2 = \emptyset$ and $E_1 \cap E_2 = \emptyset$.

An intuitionistic fuzzy bigraph is said to be complete if and only if $\mu_B(xy) > 0$, $\nu_B(xy) > 0$ for all $xy \in E$.

We use the notion $C_{m,n}(G)$ for a complete bigraph, where $|V_1| = m$ and $|V_2| = n$.

Proposition 13. $C_{m,n}(G) = C_m(\overline{G}_1) + C_n(\overline{G}_2)$.

Proof. It is straightforward. \qed

Definition 16. Let $G_1$ and $G_2$ be the strong intuitionistic fuzzy graphs. A homomorphism $f : G_1 \rightarrow G_2$ is a mapping $f : V_1 \rightarrow V_2$ which satisfies the following conditions:

(a) $\mu_{A_1}(x_1) \leq \mu_{A_2}(f(x_1))$, $\nu_{A_1}(x_1) \geq \nu_{A_2}(f(x_1))$,

(b) $\mu_{B_1}(x_1 y_1) \leq \mu_{B_2}(f(x_1)f(y_1))$, $\nu_{B_1}(x_1 y_1) \geq \nu_{B_2}(f(x_1)f(y_1))$

for all $x_1 \in V_1$, $x_1 y_1 \in E_1$.

Definition 17. Let $G_1$ and $G_2$ be strong intuitionistic fuzzy graphs. An isomorphism $f : G_1 \rightarrow G_2$ is a bijective mapping $f : V_1 \rightarrow V_2$ which satisfies the following conditions:
(c) $\mu_{A_1}(x_1) = \mu_{A_2}(f(x_1)), \nu_{A_1}(x_1) = \nu_{A_2}(f(x_1))$,

(d) $\mu_{B_1}(x_1y_1) = \mu_{B_2}(f(x_1)f(y_1)), \nu_{B_1}(x_1y_1) = \nu_{B_2}(f(x_1)f(y_1))$,

for all $x_1 \in V_1, x_1y_1 \in E_1$.

**Definition 18.** Let $G_1$ and $G_2$ be strong intuitionistic fuzzy graphs. Then, a weak isomorphism $f : G_1 \rightarrow G_2$ is a bijective mapping $f : V_1 \rightarrow V_2$ which satisfies the following conditions:

(e) $f$ is homomorphism,

(f) $\mu_{A_1}(x_1) = \mu_{A_2}(f(x_1)), \nu_{A_1}(x_1) = \nu_{A_2}(f(x_1))$,

for all $x_1 \in V_1$. Thus, a weak isomorphism preserves the weights of the nodes but not necessarily the weights of the arcs.

**Definition 19.** Let $G_1$ and $G_2$ be the strong intuitionistic fuzzy graphs. A co-weak isomorphism $f : G_1 \rightarrow G_2$ is a bijective mapping $f : V_1 \rightarrow V_2$ which satisfies

(g) $f$ is homomorphism,

(h) $\mu_{B_1}(x_1y_1) = \mu_{B_2}(f(x_1)f(y_1)), \nu_{B_1}(x_1y_1) = \nu_{B_2}(f(x_1)f(y_1))$

for all $x_1y_1 \in V_1$. Thus a co-weak isomorphism preserves the weights of the arcs but not necessarily the weights of the nodes.

**Remark 4.** 1. If $G_1 = G_2 = G$, then the homomorphism $f$ over itself is called an endomorphism. An isomorphism $f$ over $G$ is called an automorphism

2. Let $A = (\mu_A, \nu_A)$ be a strong intuitionistic fuzzy graph with an underlying set $V$. Let $\text{Aut}(G)$ be the set of all strong intuitionistic automorphisms of $G$. Let $e : G \rightarrow G$ be a map defined by $e(x) = x$ for all $x \in V$. Clearly, $e \in \text{Aut}(G)$.

3. If $G_1 = G_2$, then the weak and co-weak isomorphisms actually become isomorphic.

4. If $f : V_1 \rightarrow V_2$ is a bijective map, then $f^{-1} : V_2 \rightarrow V_1$ is also a bijective map.

We state the following Propositions without their proofs.

**Proposition 14.** Let $G_1$ and $G_2$ be strong intuitionistic fuzzy graphs. If there is a weak isomorphism between $G_1$ and $G_2$, then there is a weak isomorphism between $\overline{G}_1$ and $\overline{G}_2$.

**Proposition 15.** Let $G_1$ and $G_2$ be strong intuitionistic fuzzy graphs. If there is a co-weak isomorphism between $G_1$ and $G_2$, then there is a homomorphism between $\overline{G}_1$ and $\overline{G}_2$. 
4 Intuitionistic fuzzy line graphs

In graph theory, the line graph $L(G^*)$ of a simple graph $G^*$ is another graph $L(G^*)$ that represents the adjacencies between edges of $G^*$. Given a graph $G^*$, its line graph $L(G^*)$ is a graph such that:

- each vertex of $L(G^*)$ represents an edge of $G^*$; and
- two vertices of $L(G^*)$ are adjacent if and only if their corresponding edges share a common endpoint ("are adjacent") in $G^*$.

**Definition 20.** [18] Let $G^* = (V, E)$ be an undirected graph, where $V = \{v_1, v_2, \ldots, v_n\}$. Let $S = \{x_1, x_2, \ldots, x_q\}$ where $x_{ij} \in E$ has vertex $v_i$, $i=1, 2, \ldots, n$, $j = 1, 2, \ldots, q$. Let $T = \{S_1, S_2, \ldots, S_m\}$. Let $T = \{S_1, S_2, S_3, S_4\}$. Let $\mu = \{S_i, S_j|S_i \in S, S_i \cap S_j \neq \emptyset, i \neq j\}$. Then $P(S) = (S, T)$ is an intersection graph and $P(S) = G^*$. The line graph $L(G^*)$ is by definition the intersection graph $P(E)$. That is, $L(G^*) = (Z, W)$ where $Z = \{x \cup \{u, v\}|x \in E, u, v \in E, x = u, v \in V\}$ and $W = \{S_1, S_2, S_3, S_4\}$.

We now discuss intuitionistic fuzzy line graphs.

**Definition 21.** Let $A_1 = (\mu_{A_1}, \nu_{A_1})$ and $B_1 = (\mu_{B_1}, \nu_{B_1})$ be intuitionistic fuzzy subsets of $V$ and $E$, respectively. Let $A_2 = (\mu_{A_2}, \nu_{A_2})$ and $B_2 = (\mu_{B_2}, \nu_{B_2})$ be intuitionistic fuzzy sets of $Z$ and $W$, respectively. We define an intuitionistic fuzzy line graph $L(G) = (A_2, B_2)$ of the intuitionistic fuzzy graph $G = (A_1, B_1)$ as follows:

1. $\mu_{A_2}(S_x) = \mu_{B_1}(x) = \mu_{B_1}(u, v_x)$,
2. $\nu_{A_2}(S_x) = \nu_{B_1}(x) = \nu_{B_1}(u, v_x)$,
3. $\mu_{B_2}(S_x, S_y) = \min(\mu_{B_1}(x), \mu_{B_1}(y))$,
4. $\nu_{B_2}(S_x, S_y) = \max(\nu_{B_1}(x), \nu_{B_1}(y))$,

for all $S_x, S_y \in Z$, $S_x, S_y \in W$.

**Example 4.** Consider a graph $G^* = (V, E)$ such that $V = \{v_1, v_2, v_3, v_4\}$ and $E = \{x_1 = v_1, v_2, x_2 = v_2, v_3, x_3 = v_3, v_4, x_4 = v_4, v_1\}$. Let $A_1$ be an intuitionistic fuzzy subset of $V$ and let $B_1$ be an intuitionistic fuzzy subset of $E$ defined by

<table>
<thead>
<tr>
<th>$v_1$</th>
<th>$v_2$</th>
<th>$v_3$</th>
<th>$v_4$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_{A_1}$</td>
<td>0.2</td>
<td>0.3</td>
<td>0.4</td>
<td>0.3</td>
<td>0.1</td>
<td>0.3</td>
<td>0.2</td>
</tr>
<tr>
<td>$\nu_{A_1}$</td>
<td>0.5</td>
<td>0.6</td>
<td>0.5</td>
<td>0.2</td>
<td>0.6</td>
<td>0.6</td>
<td>0.7</td>
</tr>
<tr>
<td>$\mu_{B_1}$</td>
<td>0.1</td>
<td>0.3</td>
<td>0.2</td>
<td>0.1</td>
<td>0.1</td>
<td>0.3</td>
<td>0.7</td>
</tr>
<tr>
<td>$\nu_{B_1}$</td>
<td>0.6</td>
<td>0.6</td>
<td>0.7</td>
<td>0.7</td>
<td>0.6</td>
<td>0.6</td>
<td>0.7</td>
</tr>
</tbody>
</table>
By routine computations, it is easy to see that $G$ is an intuitionistic fuzzy graph.

Consider a line graph $L(G) = (Z, W)$ such that

$Z = \{S_x^1, S_x^2, S_x^3, S_x^4\}$

and

$W = \{S_x^1 S_x^2, S_x^2 S_x^3, S_x^3 S_x^4, S_x^4 S_x^1\}$.

Let $A_2 = (\mu A_2, \nu A_2)$ and $B_2 = (\mu B_2, \nu B_2)$ be intuitionistic fuzzy sets of $Z$ and $W$, respectively. Then, by routine computations, we have

$\mu A_2(S_x^1) = 0.1$, $\mu A_2(S_x^2) = 0.3$, $\mu A_2(S_x^3) = 0.2$, $\mu A_2(S_x^4) = 0.1$,

$\nu A_2(S_x^1) = 0.6$, $\nu A_2(S_x^2) = 0.6$, $\nu A_2(S_x^3) = 0.7$, $\nu A_2(S_x^4) = 0.7$.

$\mu B_2(S_x^1 S_x^2) = 0.1$, $\mu B_2(S_x^2 S_x^3) = 0.2$, $\mu B_2(S_x^3 S_x^4) = 0.1$, $\mu B_2(S_x^4 S_x^1) = 0.1$,

$\nu B_2(S_x^1 S_x^2) = 0.6$, $\nu B_2(S_x^2 S_x^3) = 0.7$, $\nu B_2(S_x^3 S_x^4) = 0.7$, $\nu B_2(S_x^4 S_x^1) = 0.7$.

$L(G)$ is an intuitionistic fuzzy line graph.

The following propositions are obvious.
Proposition 16. Every intuitionistic fuzzy line graph is a strong intuitionistic fuzzy graph.

Proposition 17. \( L(G) = (A_2, B_2) \) is an intuitionistic fuzzy line graph corresponding to intuitionistic fuzzy graph \( G = (A_1, B_1) \).

Proposition 18. \( L(G) = (A_2, B_2) \) is an intuitionistic fuzzy line graph of some intuitionistic fuzzy graph \( G = (A_1, B_1) \) if and only if

\[
\mu_{B_2}(S_x S_y) = \min(\mu_{A_2}(S_x), \mu_{A_2}(S_y)) \quad \text{for all } S_x, S_y \in W,
\]

\[
\nu_{B_2}(S_x S_y) = \max(\nu_{A_2}(S_x), \nu_{A_2}(S_y)) \quad \text{for all } S_x, S_y \in W.
\]

Proof. Assume that \( \mu_{B_2}(S_x S_y) = \min(\mu_{A_2}(S_x), \mu_{A_2}(S_y)) \) for all \( S_x, S_y \in W \). We define \( \mu_{A_1}(x) = \mu_{A_2}(S_x) \) for all \( x \in E \). Then

\[
\mu_{B_2}(S_x S_y) = \min(\mu_{A_2}(S_x), \mu_{A_2}(S_y)) = \min(\mu_{A_1}(x), \mu_{A_1}(y)),
\]

\[
\nu_{B_2}(S_x S_y) = \max(\nu_{A_2}(S_x), \nu_{A_2}(S_y)) = \max(\nu_{A_1}(x), \nu_{A_1}(y)).
\]

An intuitionistic fuzzy set \( A_1 = (\mu_{A_1}, \nu_{A_1}) \) that yields that the property

\[
\mu_{B_1}(xy) \leq \min(\mu_{A_1}(x), \mu_{A_1}(y)),
\]

\[
\nu_{B_1}(xy) \geq \max(\nu_{A_1}(x), \nu_{A_1}(y))
\]

will suffice.

The converse part is obvious.

Proposition 19. If \( L(G) = (A_2, B_2) \) is an intuitionistic fuzzy line graph of intuitionistic fuzzy graph \( G = (A_1, B_1) \). Then \( L(G^*) = (Z, W) \) is the line graph of \( G^* = (V, E) \).

Proof. Since \( G = (A_1, B_1) \) is an intuitionistic fuzzy graph and \( L(G) \) is an intuitionistic fuzzy line graph,

\[
\mu_{A_1}(S_x) = \mu_{B_1}(x), \nu_{A_1}(S_x) = \nu_{B_1}(x) \quad \text{for all } x \in E
\]

and so \( S_x \in Z \iff x \in E \). Also

\[
\mu_{B_2}(S_x S_y) = \min(\mu_{B_2}(x), \mu_{B_2}(y)),
\]

\[
\nu_{B_2}(S_x S_y) = \max(\nu_{B_2}(x), \nu_{B_2}(y))
\]

for all \( S_x, S_y \in W \), and so

\[
W = \{S_x S_y | s_x \cap S_y \neq \emptyset, x, y \in E, x \neq y \}.
\]

This completes the proof.
Not all graphs are line graphs of some graphs. The following result tell us when an intuitionistic fuzzy graph is an intuitionistic fuzzy line graph of some intuitionistic fuzzy graph.

**Proposition 20.** \( L(G) = (A_2, B_2) \) is an intuitionistic fuzzy line graph if and only if \( L(G^*) = (Z, W) \) is a line graph and
\[
\mu_{B_2}(uv) = \min(\mu_{A_2}(u), \mu_{A_2}(v)) \quad \text{for all } uv \in W,
\]
\[
\nu_{B_2}(uv) = \max(\nu_{A_2}(u), \nu_{A_2}(v)) \quad \text{for all } uv \in W.
\]

**Proof.** Follows from Propositions 18 and 19.

We state the following results without their proofs.

**Theorem 1.** Let \( L(G) = (A_2, B_2) \) be the intuitionistic fuzzy line graph corresponding to intuitionistic fuzzy graph \( G = (A_1, B_1) \). Suppose that \( G^* = (V, E) \) is connected. Then there exists a weak isomorphism of \( L(G) \) onto \( G \) if and only if \( G^* \) is a cyclic and for all \( v \in V, x \in E \),
\[
\mu_{A_1}(v) = \mu_{B_1}(x), \quad \nu_{A_1}(v) = \nu_{B_1}(x),
\]
i.e., \( A_1 = (\mu_{A_1}, \nu_{A_1}) \) and \( B_1 = (\mu_{B_1}, \nu_{B_1}) \) are constant functions on \( V \) and \( E \), respectively, taking on the same value.

**Theorem 2.** Let \( L(G) = (A_2, B_2) \) be the intuitionistic fuzzy line graph corresponding to intuitionistic fuzzy graph \( G = (A_1, B_1) \). Suppose that \( G^* = (V, E) \) is connected. If \( f \) is a weak isomorphism of \( G \) onto \( L(G) \), then \( f \) is an isomorphism.

**Theorem 3.** Let \( G \) and \( H \) be intuitionistic fuzzy graphs of \( G^* \) and \( H^* \), respectively, such that \( G^* \) and \( H^* \) are connected. Let \( L(G) \) and \( L(H) \) be the intuitionistic fuzzy line graphs corresponding to \( G \) and \( H \), respectively. Suppose that it is not the case that one of \( G^* \) and \( H^* \) is complete graph \( K_3 \) and other is bipartite complete graph \( K_{1,3} \). If \( L(G) \) and \( L(H) \) are isomorphic, then \( G \) and \( H \) are line-isomorphic.

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## 5 Conclusions

An intuitionistic fuzzy set is a generalization of the notion of a fuzzy set. Intuitionistic fuzzy models give more precision, flexibility and compatibility to the system as compared to the classic and fuzzy models. We have introduced the concepts of (i) strong intuitionistic fuzzy graphs, (ii) intuitionistic fuzzy line graphs, and have presented some of their properties in this paper. It is clear that the most of these results can be simply extended to \((S, T)\)-fuzzy graphs, where \( S \) and \( T \) are given imaginable triangular norms. The obtained results can be applied in various areas of engineering, computer science: artificial intelligence, signal processing, pattern recognition, robotics, computer networks, expert systems, and medical diagnosis. Our future plan to extend our research of fuzzification to (1) Bipolar fuzzy hypergraphs; (2) Intuitionistic fuzzy hypergraphs; (3) Vague hypergraphs; (4) Interval-valued hypergraphs; (5) Soft fuzzy hypergraphs.
References


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