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Algebraic Hyper-Structures Associated to Convex Analysis and Applications

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Abstract

In this paper, we generalize some concepts of convex analysis such as convex functions and linear functions on hyper-structures. Based on new definitions we obtain some important results in convex programming. A few suitable examples have been given for better understanding.

1 Introduction and preliminaries

Algebraic hyperstructures are suitable generalizations of classical algebraic structures. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. More exactly, if H is a non-empty set and $\mathcal{P}^*(H)$ is the set of all non-empty subsets of H, then we consider maps of the following type:

$$f_i : H \times H \longrightarrow \mathcal{P}^*(H),$$

where $i \in \{1, 2, \ldots, n\}$ and n is a positive integer. The maps f_i are called (*binary*) *hyperoperations*. For all x, y of H, $f_i(x, y)$ is called the (*binary*) *hyperproduct* of x and y. An algebraic system $(H, f_1, \ldots, f_n)$ is called a (*binary*) *hyperstructure*. Usually, $n = 1$ or $n = 2$.

Under certain conditions, imposed to the maps f_i, we obtain the so-called semihypergroups, hypergroups, hyperrings or hyperfields. Sometimes, external hyperoperations are considered, which are maps of the following type:

$$h : R \times H \longrightarrow \mathcal{P}^*(H),$$

where $R \neq H$. Usually, R is endowed with a ring or a hyperring structure. Several books have been written on this topic, see [1, 2, 6, 9]. Also, see [3, 4, 5, 7, 8, 10]. A recent book on hyperstructures [2] points out on their applications in rough set

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theory, cryptography, codes, automata, probability, geometry, lattices, binary relations, graphs and hypergraphs. Another book [6] is devoted especially to the study of hyperring theory. Several kinds of hyperrings are introduced and analyzed. The volume ends with an outline of applications in chemistry and physics, analyzing several special kinds of hyperstructures: e-hyperstructures and transposition hypergroups. The theory of suitable modified hyperstructures can serve as a mathematical background in the field of quantum communication systems.

Optimization theory is the study of the extremal values of a function: its minima and maxima. Topics in this theory range from conditions for the existence of a unique extremal value to methodsboth analytic and numericfor finding the extremal values and for what values of the independent variables the function attains its extremes. In mathematics, optimization or mathematical programming refers to choosing the best element from some set of available alternatives. Nonlinear programming deals with the problem of optimizing an objective function in the presence of some constraints. In this paper, we consider an optimization problem on algebraic hyperstructures and as special case we obtain the ordinary optimization problem.

2 Convex analysis

In this paper we address a hyper-structures as follows:

$$\star : H \times H \to H \otimes H \subseteq \mathcal{P}^*(H), \tag{1}$$

$$\cdot : F \times H \to H,$$

$$+ : H \times H \to H,$$

where $H \neq \emptyset$, $\star$ is a commutative hyperoperation such that $\star(H \times H) = H \otimes H$, $\cdot$ and $+$ are commutative binary operations and F is a filed. Henceforth, let $F = \mathbb{R}$. Convex functions play an important role in almost all branches of mathematics as well as other areas of science and engineering. Convex functions have many special and important properties. In this paper we introduce a few important topics of convex functions and develop some of their properties. Let define $P^*(X) = \{x \star y \in H \otimes H : x,\ y \in X\}$ for all non-empty subset X in H. At the first, we need the definition of a convex function.

Definition 2.1. Let $f : P^*(X) \to \mathbb{R}$, where X is non-empty convex subset in H. The function f is called a convex function on $P^*(X)$ if

$$f([\lambda x_1 + (1-\lambda)x_2] \star [\lambda y_1 + (1-\lambda)y_2]) \leq \lambda f(x_1 \star y_1) + (1-\lambda) f(x_2 \star y_2)$$

for each $x_1, x_2, y_1, y_2 \in X,\ x_1 \star y_1,\ x_2 \star y_2 \in P^*(X)$ and for all $0 \leq \lambda \leq 1$. The function is called strictly convex on $P^*(X)$ if the inequality is satisfied as a strict inequality for each distinct $x_1 \star y_1,\ x_2 \star y_2 \in P^*(X)$ and $0 < \lambda < 1$. The function f is called concave (strictly concave) on X if $-f$ is convex (strictly convex) on X.

The following lemma gives a necessary condition for checking the convexity of functions.

Lemma 2.2. *Let $f : P^*(X) \to \mathbb{R}$ be a convex function, where X is a non-empty convex subset in H. the level set $X^\star_\alpha = \{x,\ y \in X : f(x \star y) \leq \alpha\}$ is a convex set, where α is an arbitrary real number.*

Proof. Suppose that $x_1,\ y_1,\ x_2,\ y_2 \in X^\star_\alpha$, such that $f(x_1 \star y_1) \leq \alpha$ and $f(x_2 \star y_2) \leq \alpha$. Let $x_\lambda = \lambda x_1 + (1-\lambda)x_2$ and $y_\lambda = \lambda y_1 + (1-\lambda)y_2$ for $0 \leq \lambda \leq 1$. Since X is a convex set, x_λ and y_λ are in X. By convexity of f, we have

$$f([\lambda x_1 + (1-\lambda)x_2] \star [\lambda y_1 + (1-\lambda)y_2]) \leq \lambda f(x_1 \star y_1) + (1-\lambda) f(x_2 \star y_2) \leq \alpha.$$

Hence, $x_\lambda,\ y_\lambda \in X^\star_\alpha$, therefore $X^\star_\alpha$ is a convex set. $\square$

Example 1. Suppose that $H = [0, \pi]$. Set $X = H$ and define

$$\begin{aligned} z_{\min} &= \min\{x, y\}, \\ z_{\max} &= \max\{x, y\}, \\ x \star y &= [z_{\min},\ z_{\max}], \\ f(x \star y) &= \sin(\tfrac{z_{\min}+z_{\max}}{2}), \end{aligned}$$

for all $x, y \in X$. Let $\alpha = \frac{1}{2}$. Clearly,

$$X^\star_{\frac{1}{2}} = \left\{ x_1, y_1 \in X :\ \frac{z_{\min}+z_{\max}}{2} \leq \frac{\pi}{6},\ \textit{or}\ \frac{z_{\min}+z_{\max}}{2} \geq \frac{5\pi}{6} \right\}.$$

Since $\frac{\pi}{6},\ \frac{5\pi}{6} \in X^\star_{\frac{1}{2}}$, and

$$f\left(\left[\lambda \frac{\pi}{6} + (1-\lambda)\frac{5\pi}{6}\right] \star \left[\lambda \frac{\pi}{6} + (1-\lambda)\frac{5\pi}{6}\right]\right) > \frac{1}{2},$$

for all $0 < \lambda < 1$, so $\lambda\frac{\pi}{6} + (1-\lambda)\frac{5\pi}{6}, \lambda\frac{\pi}{6} + (1-\lambda)\frac{5\pi}{6} \notin X^\star_{\frac{1}{2}}$, for all $0 < \lambda < 1$. Therefore, the function f is not convex.

Definition 2.3. Let $f : P^*(X) \to \mathbb{R}$, where X is a non-empty subset in H. The *epigraph* of f denoted by *epi*$^\star f$, is a subset of $H \times H \times \mathbb{R}$ defined by

$$epi^\star f = \{(x, y, z) : x,\ y \in X,\ z \in R,\ f(x \star y) \leq z\}$$

The following theorem gives a necessary and sufficient condition for checking the convexity of f.

Theorem 2.4. *Let $f : P^*(X) \to \mathbb{R}$, where X is a non-empty convex subset in H. f is a convex function if and only if epi$^\star f$ is a convex set.*

Proof. Let $epi^{\star} f$ be a convex set, and $x_1,\ y_1,\ x_2,\ y_2 \in X$. Suppose that $(x_1, y_1, f(x_1 \star y_1))$ and $(x_2, y_2, f(x_2 \star y_2))$ in $epi^{\star} f$. we have:

$$(\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2, \lambda f(x_1 \star y_1) + (1-\lambda) f(x_2 \star y_2)) \in epi^{\star} f,$$

for all $0 \le \lambda \le 1$. Therefore, by definition, we have

$$f([\lambda x_1 + (1-\lambda)x_2] \star [\lambda y_1 + (1-\lambda)y_2]) \le \lambda f(x_1 \star y_1) + (1-\lambda) f(x_2 \star y_2),$$

for all $0 \le \lambda \le 1$. Conversely, assume that f is convex, (x_1, y_1, z_1) and (x_2, y_2, z_2) in $epi^{\star} f$. We have

$$f([\lambda x_1 + (1-\lambda)x_2] \star [\lambda y_1 + (1-\lambda)y_2]) \le \lambda f(x_1 \star y_1) + (1-\lambda) f(x_2 \star y_2) \le \lambda z_1 + (1-\lambda) z_2.$$

Since $\lambda x_1 + (1-\lambda)x_2,\ \lambda y_1 + (1-\lambda)y_2 \in X$, so $(\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2, \lambda z_1 + (1-\lambda)z_2) \in epi^{\star} f$, and hence $epi^{\star} f$ is a convex set. $\square$

Example 2. Let $H = \mathbb{R}^+ \times \mathbb{R}^+$. Suppose that

$$\begin{aligned} &z_{\min} = \min\{x_1, x_2, y_1, y_2\}, \\ &z_{\max} = \max\{x_1, x_2, y_1, y_2\}, \\ &(x_1, y_1) \star (x_2, y_2) = [z_{\min},\ z_{\max}] \times [z_{\min},\ z_{\max}] \subseteq \mathbb{R}^+ \times \mathbb{R}^+ \end{aligned}$$

and $f : H \otimes H \to \mathbb{R}$ is defined by $f((x_1, y_1) \star (x_2, y_2)) = z_{\max} - z_{\min}$, for all $(x_1, y_1),\ (x_2, y_2) \in X$, where X is any non-empty convex subset in H. Let $((\bar{x}_1, \bar{y}_1), (\bar{x}_2, \bar{y}_2), \bar{z})$ and $((\hat{x}_1, \hat{y}_1), (\hat{x}_2, \hat{y}_2), \hat{z})$ in $epi^{\star} f$ and $\bar{x}_\lambda = \lambda \bar{x}_1 + (1-\lambda)\bar{x}_2$, $\bar{y}_\lambda = \lambda \bar{y}_1 + (1-\lambda)\bar{y}_2$, $\hat{x}_\lambda = \lambda \hat{x}_1 + (1-\lambda)\hat{x}_2$, $\hat{y}_\lambda = \lambda \hat{y}_1 + (1-\lambda)\hat{y}_2$. One can show that

$$\max\{\bar{x}_\lambda, \bar{y}_\lambda, \hat{x}_\lambda, \hat{y}_\lambda\} - \min\{\bar{x}_\lambda, \bar{y}_\lambda, \hat{x}_\lambda, \hat{y}_\lambda\} \le \lambda \bar{z} + (1-\lambda)\hat{z}.$$

That is, $((\bar{x}_\lambda, \bar{y}_\lambda),\ (\hat{x}_\lambda, \hat{y}_\lambda),\ \lambda \bar{z} + (1-\lambda)\hat{z}) \in epi^{\star} f$. Therefore, $epi^{\star} f$ is a convex set and by using Theorem 2.4, we conclude that the function f is also a convex function.

3 Applications

In this section, we consider some applications of convex analysis on hyper-structures. At first, we formulate Optimization Model (OM) and then present some examples in this respect. Consider an optimization problem on hyper-structures as follows:

$$\min_{x, y \in X} f(x \star y), \tag{2}$$

where $f : H \otimes H \to \mathbb{R}$ is a real function and X is any non-empty subset in H.

3.1 Examples

The three following examples show that, how we can use (OM) in practice. First of all, we show that the OM is a generalization of usual optimization problem.

EXAMPLE 3. Suppose that $H = \mathbb{R}^n$, $g(x) : \mathbb{R}^n \to \mathbb{R}$ is a real function and X is any non-empty subset in H. We define $f : H \otimes H \to \mathbb{R} \cup \{\infty\}$ and $\star$, respectively, as follows:

$$f(x) = \begin{cases} g(x), & x \in \mathbb{R}^n, \\ \infty, & \text{otherwise}, \end{cases} \tag{3}$$

$$x \star y = x$$

Clearly, OM can be reduced to the usual following constrained optimization problem.

$$\min_{x \in X \subseteq \mathbb{R}^n} g(x). \tag{4}$$

EXAMPLE 4. Let H be a finite set. Suppose that $f : H \otimes H \to \mathbb{R}$ is defined by $f(A) = CardA$, for all $A \in H \otimes H$ and X is any non-empty subset of H. Now, the OM gives the minimal elements of $P^*(X)$. In addition, if $P^*(X)$ is totally ordered, so the OM gives the minimum element. Let $x^* \star y^* = Argmin\{f(x \star y) | x,\ y \in X\}$. $(x^* \star y^*)^{-1}$ is the minimum element of X.

EXAMPLE 5. Let H be a finite set, $B \in H \otimes H$ be an arbitrary element and X be any non-empty subset in H. We define $f_B : P^*(X) \to \mathbb{R}$ as $f_B(A) = \max\{Card(A - B), Card(B - A)\}$ for all $A \in P^*(X)$. Clearly we have

$$B \in P^*(X) \Leftrightarrow 0 = \min_{x,y \in X} f_B(x \star y) \tag{5}$$

3.2 Optimization theory

Optimization theory is an important subject in almost all branches of sciences. We give some definitions and theorems in this respect.

Definition 3.1. Let $f : H \otimes H \to \mathbb{R}$ and consider the problem OM. If $\hat{x} \star \hat{y} \in P^*(X)$ and $f(\hat{x} \star \hat{y}) \leq f(x \star y)$ for each $x \star y \in P^*(X)$, then $\hat{x} \star \hat{y}$ is called a *global optimal solution*. If $\bar{x} \star \bar{y} \in P^*(X)$ and there exists an $\epsilon-$neighborhood $N^\star_\epsilon(\bar{x}, \bar{y}) = \{x \star y \in H \otimes H \mid x \in N_\epsilon(\bar{x}),\ y \in N_\epsilon(\bar{y})\}$ such that $f(\bar{x} \star \bar{y}) \leq f(x \star y)$ for each $x \star y \in N^\star_\epsilon(\bar{x}, \bar{y}) \cap P^*(X)$, then $\bar{x} \star \bar{y}$ is called a *local optimal solution.*

If we suppose that in (OM) (2), the function f and the set X are convex, then we have a convex programming.

The following theorem shows that each local minimum of convex programming is also a global minimum.

Theorem 3.2. *Let $f : P^*(X) \to \mathbb{R}$ where X is a non-empty convex subset in H and $\star(A, B) = \{x \star y \in H \otimes H \mid x \in A,\ y \in B\}$ be a convex set in $H \otimes H$, for all convex subsets A and B in H. Consider the OM problem and $\bar{x} \star \bar{y} \in P^*(X)$ is a local minimum, so we have*

(1) If f is convex, then $\bar{x} \star \bar{y}$ is a global minimum.

(2) If f is strictly convex, then $\bar{x} \star \bar{y}$ is the unique global minimum.

Proof. Firstly, we suppose that the function f is convex and $\bar{x} \star \bar{y}$ is not a global minimum. It means that there exists $\hat{x} \star \hat{y} \in P^*(X)$, such that $f(\hat{x} \star \hat{y}) < f(\bar{x} \star \bar{y})$. By convexity of f, we have

$$f([\lambda\hat{x} + (1-\lambda)\bar{x}] \star [\lambda\hat{y} + (1-\lambda)\bar{y}]) \leq \lambda f(\hat{x} \star \hat{y}) + (1-\lambda) f(\bar{x} \star \bar{y}) < f(\bar{x} \star \bar{y}).$$

Now, for $\lambda > 0$ and sufficiently small, $[\lambda\hat{x} + (1-\lambda)\bar{x}] \star [\lambda\hat{y} + (1-\lambda)\bar{y}] \in N^\star_\epsilon(\bar{x}, \bar{y}) \cap P^*(X)$. Clearly the above inequality contradicts to the definition. Now, suppose that f is strictly convex. According to part one, we know that $\bar{x} \star \bar{y}$ is a global minimum. Suppose that $\bar{x} \star \bar{y}$ is not the unique global minimum, that is there exists $\tilde{x} \star \tilde{y} \in P^*(X)$, $\tilde{x} \star \tilde{y} \neq \bar{x} \star \bar{y}$, such that $f(\tilde{x} \star \tilde{y}) = f(\bar{x} \star \bar{y})$. By strictly convexity, we have

$$f([\lambda\tilde{x} + (1-\lambda)\bar{x}] \star [\lambda\tilde{y} + (1-\lambda)\bar{y}]) < \lambda f(\tilde{x} \star \tilde{y}) + (1-\lambda) f(\bar{x} \star \bar{y}) = f(\bar{x} \star \bar{y}),$$

for all $0 < \lambda < 1$. By the convexity of X, $[\lambda\tilde{x} + (1-\lambda)\bar{x}] \star [\lambda\tilde{y} + (1-\lambda)\bar{y}] \in P^*(X)$. This completes the proof.

$\square$

Example 6. Once again consider Example 2. Using Theorem 3.2, and by convexity of f, we conclude that the OM has a global minimum, where X is any non-empty convex subset in $\mathbb{R}^+ \times \mathbb{R}^+$. In order to find the global minimum, we use the below strategy. Since $f((x_1, y_1) \star (x_2, y_2)) = z_{\max} - z_{\min}$, for all $(x_1, y_1),\ (x_2, y_2) \in X$, we can use the following optimization problem equivalently:

$$\min_{(x_1,x_2) \in X \subseteq \mathbb{R}^+ \times \mathbb{R}^+} \quad x_1 + x_2. \tag{6}$$

Lemma 3.3. *Let $f : P^*(X) \to \mathbb{R}$ where X is a non-empty convex subset in H. The function f is convex if and only if*

$$f\left(\left[\sum_{i=1}^{n} \lambda_i x_i\right] \star \left[\sum_{i=1}^{n} \lambda_i y_i\right]\right) \leq \sum_{i=1}^{n} \lambda_i f(x_i \star y_i) \tag{7}$$

for each $x_i \star y_i$, $i = 1, \ldots, k$ in $P^\star(X)$ and $\sum_{i=1}^{n} \lambda_i = 1,\ \lambda_i \geq 0$.

Proof. If (7) is hold, then the convexity of f is obtained. Now, suppose that the function f is convex. By induction, we prove that the inequality (7) is true. Using convex function definition, the inequality (7) is hold for $n = 2$. We suppose that

(7) holds for n. we have

$$f\left(\left[\sum_{i=1}^{n+1}\lambda_i x_i\right]\star\left[\sum_{i=1}^{n+1}\lambda_i y_i\right]\right)$$
$$= f\left(\left[\sum_{i=1}^{n}\lambda_i x_i+\lambda_{n+1}x_{n+1}\right]\star\left[\sum_{i=1}^{n}\lambda_i y_i+\lambda_{n+1}y_{n+1}\right]\right)$$
$$= f\left(\left[(1-\lambda_{n+1})\sum_{i=1}^{n}\frac{\lambda_i}{(1-\lambda_{n+1})}x_i+\lambda_{n+1}x_{n+1}\right]\right.$$
$$\left.\star\left[(1-\lambda_{n+1})\sum_{i=1}^{n}\frac{\lambda_i}{(1-\lambda_{n+1})}y_i+\lambda_{n+1}y_{n+1}\right]\right)$$
$$\leq (1-\lambda_{n+1})f\left(\left[\sum_{i=1}^{n}\frac{\lambda_i}{(1-\lambda_{n+1})}x_i\right]\star\left[\sum_{i=1}^{n}\frac{\lambda_i}{(1-\lambda_{n+1})}y_i\right]\right)+\lambda_{n+1}f(x_{n+1}\star y_{n+1})$$
$$\leq \sum_{i=1}^{n+1}\lambda_i f(x_i\star y_i),$$

where $0<\lambda_{n+1}=1-\sum_{i=1}^{n}\lambda_i<1$. For the cases $\lambda_{n=1}=0$ and $\lambda_{n+1}=1$, the result is obvious. The proof is completed. $\square$

Note that the similar proof can be used for concave functions. The two following theorems give a method for solving many OM problems.

Theorem 3.4. *Let $f:P^*(X)\to\mathbb{R}$ be concave function, X be a polytope and $0\in X$. the optimal solution of OM will be obtained by*

$$f(\bar\theta x_l\star\bar\gamma x_s)=\min_{\substack{1\leq i\leq k\\1\leq j\leq k\\0\leq\theta_i\leq 1\\0\leq\gamma_j\leq 1}} f(\theta_i x_i\star\gamma_j x_j). \tag{8}$$

where $x_i(\ i=1,\dots,k)$ are extreme points of X.

Proof. Since X is a polytope, then using representation theorem, we can write every element of X as a convex combination of extreme points of X, that is, for all $x,\ y\in X,\ x=\sum_{i=1}^{n}\alpha_i x_i,\ \sum_{i=1}^{n}\alpha_i=1,\ \alpha_i\geq 0$ and $y=\sum_{i=1}^{n}\beta_i x_i,\ \sum_{i=1}^{n}\beta_i=1,\ \beta_i\geq 0$. let $I=\{i:\alpha_i>0\}$ and $J=\{i:\beta_i>0\}$. We know that $I\cup J=(I\cap J)\cup(I-J)\cup(J-I)$. We define λ_i, u_i and v_i, $i=1,...,k$, where $k\leq n$. Suppose that $i\in I\cap J$.

$$(\Lambda,U,V)_{I\cap J}=\begin{cases}\lambda_i=\alpha_i,\ u_i=x_i,\ v_i=\frac{\beta_i}{\alpha_i}x_i, & \text{if } \beta_i\leq\alpha_i,\\ \lambda_i=\beta_i,\ u_i=\frac{\alpha_i}{\beta_i}x_i,\ v_i=x_i, & \text{if } \alpha_i<\beta_i,\end{cases} \tag{9}$$

and $i \in J - I$,

$$(\Lambda, U, V)_{J-I} = \{(\beta_i, 0, x_i);\ i \in J - I\},$$

and $i \in I - J$,

$$(\Lambda, U, V)_{I-J} = \{(\alpha_i, x_i, 0);\ i \in I - J\},$$

where $(\Lambda, U, V) = \{(\lambda_i, u_i, v_i) \mid i \in I \cup J\}$. Let $A = \sum_{i=1}^{k} \lambda_i = \sum_{i \in I \cup J} \lambda_i$. Clearly, $1 \leq A \leq 2$, $x = \sum_{i=1}^{k} \lambda_i u_i$ and $y = \sum_{i=1}^{k} \lambda_i v_i$. However, since $x,\ y \in X$, then $\frac{x}{A} = \sum_{i=1}^{k} \frac{\lambda_i}{A} u_i$ and $\frac{y}{A} = \sum_{i=1}^{k} \frac{\lambda_i}{A} v_i$ belong to X. Because of concavity of f, Lemma 3.3 and this fact that $u_i = \theta_i x_i + (1-\theta_i)0,\ 0 \leq \theta_i \leq 1$ and $v_i = \gamma_i x_i + (1-\gamma_i)0,\ 0 \leq \gamma_i \leq 1$, we have

$$\begin{aligned} f\left(\tfrac{x}{A} \star \tfrac{y}{A}\right) &= f\left(\left[\sum_{i=1}^{k} \frac{\lambda_i}{A} u_i\right] \star \left[\sum_{i=1}^{k} \frac{\lambda_i}{A} v_i\right]\right) \\ &= f\left(\sum_{i=1}^{k}\sum_{j=1}^{k} \frac{\lambda_i \lambda_j}{A^2} u_i \star v_j\right) \\ &\geq \sum_{i=1}^{k}\sum_{j=1}^{k} \frac{\lambda_i \lambda_j}{A^2} f(u_i \star v_j) \\ &\geq f(\bar{\theta} x_l \star \bar{\gamma} x_s), \end{aligned}$$

where $\sum_{i=1}^{k}\sum_{j=1}^{k} \frac{\lambda_i \lambda_j}{A^2} = 1,\ \frac{\lambda_i \lambda_j}{A^2} \geq 0$ and

$$f(\bar{\theta} x_l \star \bar{\gamma} x_s) = \min_{\substack{1 \leq i \leq k \\ 1 \leq j \leq k \\ 0 \leq \theta_i \leq 1 \\ 0 \leq \gamma_j \leq 1}} f(\theta_i x_i \star \gamma_j x_j).$$

This completes the proof. $\square$

Notice that a similar proof can be used for convex function such that instead of minimization of OM problem, we must maximize the objective function, that is,

$$\max_{x,y \in X} f(x \star y) \tag{10}$$

The following two examples show that how we can use Theorem 3.4, to find optimal solutions. In the first example, $(0,0)$ is an extreme point, but in the second example $(0,0)$ is not an extreme point.

Example 7. Consider the concave function $-f = z_{\min} - z_{\max}$ was given in Example 2 and the OM, where

$$X = \{(x_1, x_2):\ \ -x_1 + x_2 \leq 1,\ x_1 + x_2 \leq 2,\ x_1,\ x_2 \geq 0\}.$$

Since the extreme points of X are $(0,0)$, $(2,0)$, $(0,1)$, $(\frac{1}{2},\frac{3}{2})$, by using Theorem 3.4, we can find optimal solutions as follows:

$$f(\bar{\theta}x_l \star \bar{\gamma}x_s) = \min_{\substack{1\leq i\leq 4\\ 1\leq j\leq 4\\ 0\leq\theta_i\leq 1\\ 0\leq\gamma_j\leq 1}} f(\theta_i x_i \star \gamma_j x_j)$$

where $f(\bar{\theta}x_l \star \bar{\gamma}x_s) = \{(2,0)\star(0,0),\ (2,0)\star(0,1),\ (2,0)\star(2,0), (2,0)\star(\frac{1}{2},\frac{3}{2})\}$ and $f(\bar{\theta}x_l \star \bar{\gamma}x_s) = -2$.

Example 8. Consider the concave function $-f = z_{\min} - z_{\max}$ was given in Example 2 and the optimization model in (2), where $H = \mathbb{R}\times\mathbb{R}$,

$$X = \{(x_1,x_2):\ -x_1+x_2 \leq 1,\ x_1+x_2\leq 2,\ x_2\geq 0\}.$$

Since the extreme points of X are $(2,0)$, $(-1,0)$, $(\frac{1}{2},\frac{3}{2})$, by using Theorem 3.4, we can find optimal solutions as follows:

$$f(\bar{\theta}x_l \star \bar{\gamma}x_s) = \min_{\substack{1\leq i\leq 3\\ 1\leq j\leq 3\\ 0\leq\theta_i\leq 1\\ 0\leq\gamma_j\leq 1}} f(\theta_i x_i \star \gamma_j x_j)$$

where $f(\bar{\theta}x_l \star \bar{\gamma}x_s) = \{(2,0)\star(-1,0)\}$ and $f(\bar{\theta}x_l \star \bar{\gamma}x_s) = -3$.

Definition 3.5. A map $f : P^*(H) \to \mathbb{R}$ is called a *linear function* if

$$f([\lambda x_1 + (1-\lambda)x_2]\star[\lambda y_1 + (1-\lambda)y_2]) = \lambda f(x_1\star y_1) + (1-\lambda)f(x_2\star y_2), \quad (11)$$

where $x_1\star y_1,\ x_2\star y_2 \in P^*(H)$ and $\lambda\in R$.

The next theorem gives a condition for finiteness optimal solution in linear function and polyhedron.

Theorem 3.6. *Let $f : P^*(X)\to\mathbb{R}$ be a linear function, X be a polyhedron and $0\in X$. the optimal solution of OM will be finite, if*

$$\min\left\{\min_{\substack{1\leq i\leq p\\ 1\leq j\leq p\\ 0\leq\zeta_i\leq 1\\ 0\leq\xi_j\leq 1}} \{f(\zeta_i d_i\star\xi_j d_j)\},\quad \min_{\substack{1\leq i\leq k\\ 1\leq j\leq p\\ 0\leq\zeta_i\leq 1\\ 0\leq\xi_j\leq 1}} \{f(\zeta_i x_i\star\xi_j d_j)\}\right\} \geq 0, \quad (12)$$

then the optimal solution is

$$f(\bar{\theta}x_l \star \bar{\gamma}x_s) = \min_{\substack{1\leq i\leq k\\ 1\leq j\leq k\\ 0\leq\theta_i\leq 1\\ 0\leq\gamma_j\leq 1}} f(\theta_i x_i \star \gamma_j x_j),$$

if there exists $i,\ j$, such that at least one of these two inequalities $f(\zeta_i d_i\star\xi_j d_j) < 0$ or $f(\zeta_i x_i \star\xi_j d_j) < 0$ is held, then the optimal value is $-\infty$, where $x_1,\ldots,x_k$ and $d_1,\ldots,d_p$, are extreme points and extreme directions, respectively.

Proof. By representation theorem, we have for every $x, y \in X$,

$$x = \sum_{i=1}^{n} \alpha_i x_i + \sum_{j=1}^{m} \mu_j d_j \text{ and } y = \sum_{i=1}^{n} \beta_i x_i + \sum_{j=1}^{m} \eta_j d_j,$$

where $\sum_{i=1}^{n} \alpha_i = 1, \alpha_i \geq 0,\ \sum_{i=1}^{n} \beta_i = 1,\ \beta \geq 0,\ \mu_j \geq 0$ and $\eta_j \geq 0$. We consider λ_i as previous theorem and let $I = \{i : \mu_i > 0\}$ and $J = \{i : \eta_i > 0\}$. We know that $I \cup J = (I \cap J) \cup (I - J) \cup (J - I)$. We define π_i, z_i and w_i, $i = 1, \ldots, p$, where $p \leq m$. Suppose that $i \in I \cap J$.

$$(\Pi, Z, W)_{I \cap J} = \begin{cases} \pi_i = \mu_i,\ z_i = d_i,\ w_i = \frac{\eta_i}{\mu_i} d_i, & \text{if } \eta_i \leq \mu_i, \\ \pi_i = \eta_i,\ z_i = \frac{\mu_i}{\eta_i} d_i,\ w_i = d_i, & \text{if } \mu_i < \eta_i, \end{cases} \tag{13}$$

and $i \in J - I$,

$$(\Pi, Z, W)_{J-I} = \{(\eta_i, 0, d_i);\ i \in J - I\},$$

and $i \in I - J$,

$$(\Pi, Z, W)_{I-J} = \{(\mu_i, d_i, 0);\ i \in I - J\},$$

where $(\Pi, Z, W) = \{(\pi_i, z_i, w_i) \mid i \in I \cup J\}$. Because of linearity of f, we have

$$\begin{aligned} f(\tfrac{x}{A} \star \tfrac{y}{A}) &= f\left(\left[\sum_{i=1}^{k} \frac{\lambda_i}{A} u_i + \sum_{j=1}^{p} \frac{\pi_j}{A} z_j\right] \star \left[\sum_{i=1}^{k} \frac{\lambda_i}{A} v_i + \sum_{j=1}^{p} \frac{\pi_j}{A} w_j\right]\right) \\ &= \sum_{i=1}^{k}\sum_{j=1}^{k} \frac{\lambda_i \lambda_j}{A^2} f(u_i \star v_j) + \sum_{i=1}^{k}\sum_{j=1}^{p} \frac{\lambda_i \pi_j}{A^2} f(u_i \star w_j) \\ &\quad + \sum_{i=1}^{k}\sum_{j=1}^{p} \frac{\lambda_i \pi_j}{A^2} f(v_i \star z_j) + \sum_{i=1}^{p}\sum_{j=1}^{p} \frac{\pi_i \pi_j}{A^2} f(z_i \star w_j). \end{aligned}$$

Now, if (12) is held, we set $\mu_j = 0,\ \eta_j = 0$ and as previous theorem we obtain optimal solutions. Otherwise we can tend some μ_j or η_j to infinity and the value function tends to $-\infty$. This completes the proof. □

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