

Strong Whitney convergence

Agata Caserta

Abstract

The notion of strong uniform convergence on bornologies introduced in 2009 by Beer-Levi turns to give the classical convergence introduced by Arzelà in 1883. Evert in 2003 introduced the notion of Arzelà-Whitney or simply AW-convergence for a net of functions. We define a new type of convergence, a "strong" form of Whitney convergence on bornologies, and we prove that on some families it coincides with that AW-convergence. Furthermore, we study the countability properties of this new function space.

1 Introduction and notations

In 1883 Arzelà [1] gave a necessary and sufficient condition for the pointwise limit of a sequence of real valued continuous functions on a compact interval to be continuous. Arzelà's work paved the way for several outstanding papers. In 1905, the condition for which Arzelà introduced the term "uniform convergence by segments" was called "quasi uniform convergence" by Borel in [10], and Bartle in [3], extended Arzelà's theorem to nets of real valued continuous functions on a topological space. For a comprehensive guide to the literature on the preservation of continuity, the reader may consult [12].

Definition 1. A net $(f_\alpha)_{\alpha \in \Lambda}$ of functions $f_\alpha : X \rightarrow Y$ is said to be *quasi-uniformly convergent* to a function $f : X \rightarrow Y$ (or Arzelà convergent) on X , provided it pointwise converges to f , and for every $\epsilon > 0$ and α_0 there exists a finite number of indices $\alpha_1, \alpha_2, \dots, \alpha_n \geq \alpha_0$ such that for each $x \in X$ at least one of the following inequalities holds:

$$\rho(f_{\alpha_i}(x) - f(x)) < \epsilon \quad i = 1, \dots, n.$$

In 2009, in the realm of metric spaces, Beer and Levi [7] found a new theoretical approach giving another necessary and sufficient condition through the notion of strong uniform convergence on bornologies, when this bornology reduces to the that

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of all finite subsets of X . In [12] Caserta et al. offer a direct proof of the equivalence of Arzelà and Beer-Levi conditions.

We first recall the notion of bornology. A bornology \mathcal{B} on a metric space (X, d) is a family of subsets of X that is closed under taking finite unions, is hereditary and forms a cover of X (see [22]). For metric bornologies the interested reader may consult [4], [6], [8], [20]. By a base \mathcal{B}_0 for a bornology \mathcal{B} , we mean a subfamily of \mathcal{B} that is cofinal with respect to inclusion.

Given a bornology \mathcal{B} with a closed base on X , as announced, Beer and Levi present a new uniformizable topology on the set Y^X of all functions from X to Y .

Definition 2. [7] Let (X, d) and (Y, ρ) be metric spaces and let \mathcal{B} be a bornology with closed base on X . Then the topology of strong uniform convergence $\tau_{\mathcal{B}}^s$ is determined by a uniformity on Y^X having as a base all sets of the form

$$[B; \epsilon]^s := \{(f, g) : \exists \delta > 0 \text{ for every } x \in B^\delta \ \rho(f(x), g(x)) < \epsilon\}, \quad (B \in \mathcal{B}, \epsilon > 0).$$

where B^δ denotes the set $\{x : d(x, B) < \delta\} = \bigcup_{x \in B} S_\delta(x)$.

On $C(X, Y)$, the set of all continuous functions from X to Y , this topology is in general finer than the classical topology of uniform convergence on \mathcal{B} . This new function spaces has been intensively studied in [7], [12], [13], [14]. The notion of strong uniform convergence on bornologies is fundamentally variational in nature: we insist not only on uniform convergence on members of \mathcal{B} but convergence around the edge of elements of \mathcal{B} in some uniform sense. Notice also that since each bornology contains the singletons, we automatically have pointwise convergence, whatever the bornology might be. In the special case of the bornology of all finite subset of X , convergence in this sense of a net of continuous functions forces continuity of the limit, and conversely, if the limit is continuous, then this sort of convergence must ensue.

In 1936 Whitney in [27] introduced a new topology on $C(X, Y)$, named Whitney topology, widely used in Differential Geometry. An application of this topology in function spaces is in [26], in which the Baire space property of the function space is used to obtain embeddings into infinite-dimensional manifolds. This topology has been intensively investigated in the last decades in [21], [15], [19], [24].

We denote with $C^+(X)$ the set of all positive real valued functions defined on X .

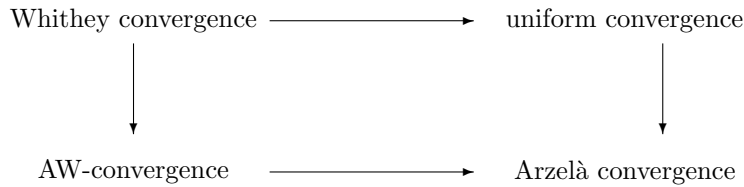
Definition 3. A net $(f_\alpha)_{\alpha \in \Lambda}$ of functions $f_\alpha : X \rightarrow Y$ is said to be convergent to a function $f : X \rightarrow Y$ in the sense of Whitney if for every $\epsilon \in C^+(X)$ there exists α_0 such that $\rho(f_{\alpha_i}(x) - f(x)) < \epsilon(x)$ for each $x \in X$ and for every $\alpha \geq \alpha_0$.

In 2003 Ewert and Jedrzejewshi [17] defined a stronger form of Arzelà convergence placed between Arzelà and Whitney convergence.

Definition 4. [17] A net $(f_\alpha)_{\alpha \in \Lambda}$ of functions $f_\alpha : X \rightarrow Y$ is said to be convergent to a function $f : X \rightarrow Y$ in the sense of Arzelà-Whitney, or simply *AW-convergent* on X , if $(f_\alpha)_{\alpha \in \Lambda}$ pointwise converges to f , and if, for every $\epsilon \in C^+(X)$ and α_0 there exists a finite number of indices $\alpha_1, \alpha_2, \dots, \alpha_n \geq \alpha_0$ such that for each $x \in X$

$$\min\{\rho(f_{\alpha_i}(x) - f(x)) : i = 1, \dots, n\} < \epsilon(x).$$

The following diagram visualizes the relations between the AW-convergence and the above mentioned types of convergence.



Observe that none of these implications is reversible. The notion of strong uniform convergence on bornology is equivalent to the Arzelà convergence on compacta, when the bornology reduces to that of all finite subsets of X (see Theorem 2.9 in [12]). We define a new type of convergence that is a strong version, in the sense of Beer-Levi, of Whitney convergence on bornologies. This new convergence, named strong Whitney convergence, lies between Whitney and strong uniform convergence. A goal of this paper is to compare this new type of convergence with the AW-convergence, and in the main theorem we prove, in analogy with the strong uniform convergence, that this new topology is equivalent to the one given by AW-convergence, when the bornology reduces to that of all finite subsets of X . Therefore it fits in the right place in the above diagram when we look at the corresponding topologies.

In the last section we work in the more general context of Hausdorff uniform spaces rather than metric spaces, developing in the process the rudiments of the theory of strong Whitney convergence in this setting in analogy with the recent studies of Beer [5] about strong uniform convergence and McCoy [24] about Whitney convergence. In this section we characterize the countability properties of this new function space in terms of properties of the involved space X and the bornology.

We denote the power set of X by $\mathcal{P}(X)$. The set $\mathbb{K}(X)$ denotes the family of all compact subsets of X . Let X and Y be topological spaces, $C(X, Y)$ (resp. $C(X)$ when $Y = \mathbb{R}$) denotes the set of all continuous functions from X to Y , and $C^+(X)$ the set of all positive real valued functions defined on X . The commonly used topologies on $C(X, Y)$ are the compact-open topology τ_k and the topology of pointwise convergence τ_p (see [25]). We denote the corresponding space by $(C(X, Y), \tau_k)$ (resp. $C_k(X)$ when $Y = \mathbb{R}$) and $(C(X, Y), \tau_p)$ (resp. $C_p(X)$ when $Y = \mathbb{R}$). The reader is referred to [16], [23] and [25] for standard notions and definitions.

2 Strong Whitney convergence on bornologies

Let \mathcal{B} be a family of subset of (X, d) , and $\epsilon(x) \in C(X)^+$. The classical uniformity for the topology $\tau_{\mathcal{B}}^w$ of Whitney convergence on \mathcal{B} for $C(X, Y)$ has as a base for its

entourages all sets of the form

$$[B; \epsilon]^w := \{(f, g) : \forall x \in B \rho(f(x), g(x)) < \epsilon(x)\}, \quad (B \in \mathcal{B}).$$

When $\mathcal{B} = \mathcal{P}_0(X)$, we get the standard uniformity for the topology of Whitney convergence on X . These uniformities make sense on Y^X as well.

Definition 5. Let (X, d) and (Y, ρ) be metric spaces and let \mathcal{B} be a bornology with closed base on X . Then the topology of strong Whitney convergence $\tau_{\mathcal{B}}^{sw}$ is determined by a uniformity on Y^X having as a base all sets of the form

$$[B; \epsilon]^{sw} := \{(f, g) : \exists \delta > 0 : \forall x \in B^\delta : \rho(f(x), g(x)) < \epsilon(x)\}$$

with $B \in \mathcal{B}, \epsilon \in C(X, \mathbb{R}^+)$.

If $\epsilon_1 \leq \epsilon_2$, then $[B; \epsilon_1]^{sw} \subseteq [B; \epsilon_2]^{sw}$. We clearly have that $\tau_{\mathcal{B}}^w \leq \tau_{\mathcal{B}}^{sw}$ and $\tau_{\mathcal{B}}^s \leq \tau_{\mathcal{B}}^{sw}$. Furthermore, for $\mathcal{B} = \mathcal{K}(X)$ we have that $\tau^w = \tau_{\mathcal{K}(X)}^{sw}$ (since ϵ restricted on $K \in \mathcal{K}(X)$ is uniformly continuous).

Remark 1. If (X, d) is locally pseudocompact, then $\tau_{\mathcal{F}(X)}^w = \tau_{\mathcal{F}(X)}^{sw}$. It suffices to prove that $\tau_{\mathcal{F}(X)}^{sw} \leq \tau_{\mathcal{F}(X)}^w$. Let $F = \{x_1, \dots, x_n\} \in \mathcal{F}(X)$ and $\epsilon \in C(X)^+$. Since X is locally pseudocompact, for every $x_i \in F$ there is a closed pseudocompact neighborhood I_{x_i} . Thus there is $\delta_{x_i} > 0$ such that $S(x_i, \delta_i) \subseteq I_{x_i}$ for all $i \leq n$. Set $\delta = \min_{i \leq n} \delta_{x_i}$, for every i , let $r_i = \inf\{\epsilon(x) : x \in S(x_i, \delta)\} > 0$, and $r = \min_{i \leq n} r_i$. Define $\phi \in C(X, \mathbb{R}^+) = \inf\{r, \epsilon\}$. Clearly $[F; \phi]^w(f) \subset [F; \epsilon]^{sw}(f)$.

It is clear that if X is compact the topology, then $\tau_{\mathcal{B}}^{sw}$ reduces to the topology of uniform convergence and therefore is metrizable. If, however, X is not compact, the strong Whitney topology is not in general metrizable.

Proposition 1. *If (X, d) is non compact then $(Y^X, \tau_{\mathcal{F}(X)}^{sw})$ is not first countable.*

PROOF. Assume by contradiction, that there is countable base at $f = 0$ given by $\{[F_n; \epsilon_n]^{sw}(0) : n \in \omega\}$. Since X is not compact, there is a sequence $(x_n)_{n \in \omega} \subset X$ with no cluster points. Let $\epsilon : (x_n)_{n \in \omega} \rightarrow (0, 1)$ continuous defined by $\epsilon(x_n) = \frac{1}{4}\epsilon_n(x_n)$. Since $(x_n)_{n \in \omega}$ is closed in X , by Tietze theorem, there is a continuous extension $\epsilon : X \rightarrow (0, 1)$. For every n , let $\hat{\epsilon}_n = \inf\{\epsilon, \epsilon_n\} \in C(X, \mathbb{R}^+)$. We have that $\frac{1}{2}\hat{\epsilon}_n \in [F_n; \epsilon_n]^{sw}(0)$. Now, $(\frac{1}{2}\hat{\epsilon}_n) \upharpoonright F_n$ is uniformly continuous, hence for $\epsilon' < \frac{1}{4} \min\{\hat{\epsilon}_n(x) : x \in F_n\}$, there is $\delta > 0$ such that for every $x, b \in F_n$ with $d(x, b) < \delta$ we have that $|\frac{1}{2}\hat{\epsilon}_n(x) - \frac{1}{2}\hat{\epsilon}_n(b)| < \epsilon'$, and so $|\frac{1}{2}\hat{\epsilon}_n(x)| \leq \hat{\epsilon}_n(b) < \epsilon_n(b)$. But $\frac{1}{2}\hat{\epsilon}_n$ does not belong to $[F_n; \epsilon']^{sw}(0)$ for every $n \in \omega$. Therefore we cannot have that $[F_n; \epsilon_n]^{sw}(0) \subset [F_n; \epsilon']^{sw}(0)$ for every n , a contradiction. ■

Corollary 1. *If (X, d) is non compact then $(Y^X, \tau_{\mathcal{F}(X)}^{sw})$ is not metrizable.*

In [7] Beer and Levi introduced the notion of strong uniform continuity on a bornology \mathcal{B} .

Definition 6. [7] Let (X, d) and (Y, ρ) be metric spaces and let B be a subset of X . A function $f : X \rightarrow Y$ is strongly uniformly continuous on B if for every $\epsilon > 0$ there is $\delta > 0$ such that if $d(x, w) < \delta$ and $\{x, w\} \cap B \neq \emptyset$, then $\rho(f(x), f(w)) < \epsilon$.

If \mathcal{B} is a family of nonempty subsets of X and (Y, ρ) a metric space, a function $f \in Y^X$ is called uniformly continuous (resp. strongly uniformly continuous) on \mathcal{B} if for each $B \in \mathcal{B}$, $f \upharpoonright B$ is uniformly continuous (resp. strongly uniformly continuous) on B . For a general bornology \mathcal{B} on X , strong uniform convergence is characterized by the preservation of the variational notion of strong uniform continuity of functions on members of \mathcal{B} [7] (Theorem 6.7), that reduces to ordinary pointwise continuity when \mathcal{B} is a bornology of relatively compact subsets.

Now we pause to place two new forms of continuity related to Whitney topology.

Definition 7. Let (X, d) and (Y, ρ) be metric spaces. A function $f : X \rightarrow Y$ is Whitney continuous at $B \subset X$, if for every $\epsilon \in C^+(X)$ there is $\delta > 0$ such that for all $x \in B$ and $y \in S(x, \delta) \cap B$ we have $\rho(f(x), f(y)) < \epsilon(x)$.

Definition 8. Let (X, d) and (Y, ρ) be metric spaces. A function $f : X \rightarrow Y$ is strongly Whitney continuous at $B \in \mathcal{B}$, if for every $\epsilon \in C^+(X)$ there is $\delta > 0$ such that for all $x \in B$ and $y \in S(x, \delta)$ we have $\rho(f(x), f(y)) < \epsilon(x)$.

We denote the set of all functions strongly Whitney continuous at every element of the family \mathcal{B} by $\mathfrak{F}_{\mathcal{B}}^{sw}(X, Y)$.

If f is strongly Whitney continuous at B , then f is strongly uniformly continuous at B .

Remark 2. Observe that if the bornology \mathcal{B} is such that every closed element is compact, then $\tau_{\mathcal{B}}^{sw} \leq \tau_w$. Indeed, given a $\tau_{\mathcal{B}}^{sw}$ - neighborhood of f_0 , $[B, \epsilon]^{sw}(f_0)$, we have that $[cl(B), \epsilon]^{sw}(f_0) \subseteq [B, \epsilon]^{sw}(f_0)$. By hypothesis $cl(B) = K \in \mathcal{K}$, thus $[K, \epsilon]^{sw}(f_0) \subseteq [B, \epsilon]^{sw}(f_0)$. Therefore $\tau_{\mathcal{B}}^{sw} \leq \tau_{\mathcal{K}}^{sw} = \tau_w$ since ϵ restricted on $K \in \mathcal{K}(X)$ is uniformly continuous.

Recall that a hereditary family of subsets of \mathcal{B} is called stable under small enlargements [9] if

$$\forall B \in \mathcal{B}, \exists \delta > 0 : B^\delta \in \mathcal{B}.$$

For example the family of d -bounded subsets is always stable under small enlargements; and the finite subsets are stable under small enlargements, if and only if, all points of X are isolated.

Next proposition describes when these topologies coincide under some conditions involving only the bornology on X .

Proposition 2. *Let (X, d) and (Y, ρ) be metric spaces, and let \mathcal{B} be a bornology stable under small enlargements and with a compact base on X . Then the following are satisfied*

- (i) $\mathfrak{F}_{\mathcal{B}}^{sw}(X, Y)$ is closed in Y^X equipped with the topology $\tau_{\mathcal{B}}^{sw}$,
- (ii) in $C(X, Y)$ we have $\tau_{\mathcal{B}}^{sw} = \tau_{\mathcal{B}}^s = \tau_{\mathcal{B}}$.

PROOF. (i) We show the complement is open. Suppose $f \in Y^X$ fails to be strongly Whitney continuous on some $B \in \mathcal{B}$. There is $\epsilon_f \in C^+(X)$ such that for every $n \in \omega$, there exist $\{x_n, w_n\} \subseteq B^{1/n}$ such that $d(x_n, w_n) < 1/n$ and $\rho(f(x_n), f(w_n)) \geq 3\epsilon_f(x_n)$. By assumption there is $\epsilon > 0$ such that $B^\epsilon \in \mathcal{B}$. Let B' be a basic element of \mathcal{B} such that $B^\epsilon \subset B'$. Set $\alpha \in C^+(X)$ defined by $\alpha(z) = \min \{\epsilon_f(x) : x \in B'\}$. Take $g \in [B'; \alpha]^{sw}(f)$, hence there is $\delta > 0$ such that for every $x \in (B')^\delta$ we have that $\rho(f(x), g(x)) < \alpha(x)$. Then for infinitely many $\{x_n, w_n\}$, we have that $\rho(f(x_n), g(x_n)) < \alpha(x_n)$ and $\rho(f(w_n), g(w_n)) < \alpha(w_n)$, therefore

$$\rho(g(x_n), g(w_n)) \geq \rho(f(x_n), f(w_n)) - \rho(f(x_n), g(x_n)) - \rho(f(w_n), g(w_n)) \geq \epsilon_f(x_n)$$

since $\rho(f(x_n), g(x_n)) < \epsilon_f(x_n)$ and $\rho(f(w_n), g(w_n)) < \epsilon_f(x_n)$. Thus g is not strongly Whitney continuous on \mathcal{B} .

(ii) For the first equality it suffices to show that $\tau_{\mathcal{B}}^{sw} \leq \tau_{\mathcal{B}}^s$. Let $f \in C(X, Y)$ and $[B; \epsilon]^{sw}(f)$ be a neighborhood of f in $\tau_{\mathcal{B}}^{sw}$. By assumption there is a $\delta_0 > 0$ such that $B^{\delta_0} \in \mathcal{B}$ and there is $B' \in \mathcal{B}$ compact such that $B^{\delta_0} \subseteq B'$. Let $\alpha > 0$ defined by $\alpha = \min\{\epsilon(x) : x \in B'\}$. $[B'; \alpha]^s(f)$ is a $\tau_{\mathcal{B}}^s$ -neighborhood of f , such that $[B'; \alpha]^s(f) \subseteq [B, \epsilon]^{sw}(f)$. The second equality follows from Theorem 6.2 in [7]. \blacksquare

As announced above, the notion of strong uniform convergence on bornologies introduced by Beer-Levi turns to give the classical convergence introduced by Arzelà. We recall that the notion of Arzelà-Whitney convergence for a net of functions is independent from the uniform convergence and plays a key role in the mentioned types of convergence. In the main theorem we prove that the strong Whitney convergence on some family is equivalent to the AW-convergence.

Theorem 1. *Let (X, d) and (Y, ρ) be metric spaces and let $f \in Y^X$. Let $(f_\alpha)_{\alpha \in \Lambda}$ be a net in $C(X, Y)$ that is pointwise convergent to f . The following are equivalent:*

- (i) $(f_\alpha)_{\alpha \in \Lambda}$ is $\tau_{\mathcal{F}}^{sw}$ -convergent to f ,
- (ii) $(f_\alpha)_{\alpha \in \Lambda}$ is AW-convergent to f on compact sets.

PROOF. (i) \Rightarrow (ii) Let K be a compact subset of X . Let $\epsilon > 0$ and α_0 be fixed. Then $K \subseteq \bigcup_{i=1}^n S(x_i, \delta_i)$ with $x_i \in K$. Let $F = \{x_1, \dots, x_n\}$. Since $(f_\alpha)_{\alpha \in \Lambda}$ is $\tau_{\mathcal{F}}^{sw}$ -convergent to f at x_i , there exists an α_i such that for every $\alpha \geq \alpha_i$ we have $f_\alpha \in [\{x_i\}, \epsilon]^{sw}(f)$. Define

$$\Lambda_{\alpha_i} = \{\alpha \geq \alpha_i : \forall y \in S(x_i, \delta_i) \rho(f_\alpha(y), f(y)) < \epsilon(y)\} \quad i = 1, \dots, n.$$

We claim that for all $i = 1, \dots, n$ the set Λ_{α_i} are nonempty. Assume, by contradiction, that there is $i_0 \in \{i_1, \dots, i_n\}$ such that $\Lambda_{\alpha_{i_0}} = \emptyset$, therefore for all $\beta \geq \alpha_{i_0}$ there is $y_\beta \in S(x_{i_0}, \delta_{i_0})$ such that $\rho(f_\beta(y_\beta), f(y_\beta)) \geq \epsilon(y_\beta)$. Let $\epsilon_0 = \epsilon(y_\beta)$. Since $(f_\alpha)_{\alpha \in \Lambda}$ is pointwise convergent to f at y_β , there exists an α' such that for every $\hat{\alpha} \geq \alpha'$ we have that $\rho(f_{\hat{\alpha}}(y_\beta), f(y_\beta)) < \epsilon_0/4$. The continuity of $f_{\hat{\alpha}}$ and f_β at y_β implies that there exists $\delta' > 0$ such that for all $z \in S(y_\beta, \delta')$ we have that $\rho(f_{\hat{\alpha}}(z), f_{\hat{\alpha}}(y_\beta)) < \epsilon_0/8$ and $\rho(f_\beta(z), f_\beta(y_\beta)) < \epsilon_0/8$, therefore $\rho(f_\beta(y_\beta), f_{\hat{\alpha}}(y_\beta)) \leq \epsilon_0/4$. It follows that

$$\rho(f_\beta(y_\beta), f(y_\beta)) \leq \rho(f_\beta(y_\beta), f_{\hat{\alpha}}(y_\beta)) + \rho(f_{\hat{\alpha}}(y_\beta), f(y_\beta)) \leq \epsilon_0/2 < \epsilon(y_\beta)$$

a contradiction. Thus for every $x \in K$ at least one of the following is satisfied: $\rho(f_{\alpha_i}(x), f(x)) < \epsilon(x)$ for $i = 1, \dots, n$.

(ii) \Rightarrow (i) Given $\epsilon \in C^+(X)$ and $x \in X$ we prove that there exists an α_0 such that for every $\alpha \geq \alpha_0$ we have $f_\alpha \in [\{x\}, \epsilon]^{sw}(f)$. Let $\epsilon_0 = \epsilon(x)/4$. Since $(f_\alpha)_{\alpha \in \Lambda}$ is pointwise convergent to f at x , there exists an α_0 such that for every $\alpha \geq \alpha_0$ we have $\rho(f_\alpha(x), f(x)) < \epsilon_0$. We claim that for every $\alpha \geq \alpha_0$ there is $\delta > 0$ such that for all $y \in S(x, \delta)$ we have $\rho(f_\alpha(y), f(y)) < \epsilon(y)$. Assume not, so there exists $\alpha \geq \alpha_0$ and a sequence $(x_n)_{n \in \omega}$ converging to x such that $\rho(f_\alpha(x_n), f(x_n)) \geq \epsilon(x_n)$ for all $n \in \omega$. Set $B = \{x_n\}_{n \in \omega} \cup \{x\}$. Since B is a compact subset of X and $(f_\alpha)_{\alpha \in \Lambda}$ is AW-convergent to f on B , there are $\alpha_1, \dots, \alpha_n \geq \alpha_0$ such that for each $z \in B$, $\min\{\rho(f_{\alpha_i}(z), f(z)) < \epsilon/4 : i = 1, \dots, n\} < \epsilon(z)/4$. Thus there is $i \in \{1, \dots, n\}$ and an infinite set $B^* \subset B$ such that

$$\rho(f_{\alpha_i}(r), f(r)) < \epsilon(r)/4 \quad \forall r \in B^*.$$

Since $\alpha_i \geq \alpha_0$ and x is fixed, we have that $\rho(f_{\alpha_i}(x), f_\alpha(x)) < \epsilon(x)/2$. Since $\text{cl}(B^*)$ is a compact subset of B , set $k = \min\{\epsilon(x) : x \in \text{cl}(B^*)\}/8$. Since f_{α_i} and f_α are continuous at x it follows that there exists $\delta > 0$ such that $\forall r \in B^* \cap B_\delta(x)$ we have that $\rho(f_{\alpha_i}(r), f_{\alpha_i}(x)) < k$ and $\rho(f_\alpha(x), f_\alpha(r)) < k$, therefore

$$\rho(f_{\alpha_i}(r), f_\alpha(r)) \leq \rho(f_{\alpha_i}(r), f_{\alpha_i}(x)) + \rho(f_{\alpha_i}(x), f_\alpha(x)) + \rho(f_\alpha(x), f_\alpha(r)) \leq \frac{3}{4}\epsilon(x).$$

Let $n \in \omega$ be such that $x_n \in B^* \cap B_\delta(x)$. By continuity of ϵ and since $(x_n)_{n \in \omega}$ converges to x the set $\{x_n : \epsilon(x_n) > \frac{3}{5}\epsilon_0\}$ is a non empty open set, hence there are infinitely many x_n such that $\epsilon(x_n) > \frac{3}{5}\epsilon_0$. Then

$$\rho(f_\alpha(x_n), f(x_n)) \leq \rho(f_\alpha(x_n), f_{\alpha_i}(x_n)) + \rho(f_{\alpha_i}(x_n), f(x_n)) \leq \epsilon(x_n)$$

a contradiction. ■

3 Countability properties

We first give a straightforward extension of the definition of strong Whitney topology to the uniform setting.

In what follows, letters in bold caps will denote diagonal uniformities. For results and terminology about diagonal uniformities we will rely on the textbook by Willard [25]. If (X, \mathbf{D}) is a Hausdorff uniform space and $x_0 \in X$, and $D \in \mathbf{D}$, we write $D(x_0)$ for $\{x \in X : (x_0, x) \in D\}$. Of course, $\{D(x_0) : D \in \mathbf{D}\}$ forms a local base at x_0 for the induced topology. If $A \in \mathcal{P}(X)$, we call the uniform neighborhood $D(A) = \cup_{a \in A} D(a)$ an enlargements of A . A bornology \mathcal{B} on a Hausdorff uniform space is said to be stable under small enlargements if it contains an enlargement of each of its members. Evidently the relatively compact sets are stable under small enlargements, if and only if, X is locally compact.

Definition 9. Let (X, \mathbf{D}) be a Hausdorff uniform space and (Y, ρ) be metric spaces. Let \mathcal{B} be a bornology with closed base on X . Then the topology of strong Whitney convergence $\tau_{\mathcal{B}}^{sw}$ is determined by a uniformity on Y^X having as a base all sets of the form

$$[B; \epsilon]^{sw} := \{(f, g) : \exists D \in \mathbf{D} : \forall x \in D(B) : \rho(f(x), g(x)) < \epsilon(x)\}$$

with $B \in \mathcal{B}, \epsilon \in C(X, \mathbb{R}^+)$.

In the following we extend to a more general setting the results obtained in [11] (Theorem 4) and [17] (Corollary 1) when the bornology reduces to that of all finite subsets of X .

Proposition 3. *The space (X, \mathbf{D}) is pseudocompact, if and only if, $\tau_{\mathcal{B}}^w = \tau_{\mathcal{B}}^{sw}$. In this case it coincides with the uniform convergence on the family \mathcal{B} .*

PROOF. (\Rightarrow) It suffices to prove that $\tau_{\mathcal{B}}^{sw} \leq \tau^w$. We recall that from pseudocompactness of the space X we have that $\inf\{\epsilon(x) : x \in X\} = r > 0$. Therefore, for all $B \in \mathcal{B}, \epsilon \in C(X, \mathbb{R}^+)$ we have $[B; r]^w(f) \subset [B; \epsilon]^{sw}(f)$. By a similar argument we can prove that $\tau_{\mathcal{B}}^{sw}$ coincide with the uniform convergence on the family \mathcal{B} .

(\Leftarrow) Assume that X is not pseudocompact, hence there is $f \in C^+(X)$ unbounded. Thus there is $(x_n)_{n \in \omega} \subset X$ such that $(\frac{1}{f(x_n)})_n$ is strictly decreasing, unbounded with distinct terms. Let $\epsilon = 1/f \in C(X, \mathbb{R}^+)$. For each $\phi > 0$, we have that $[B; \phi]^w(f) \not\subset [B; \epsilon]^{sw}(f)$. ■

In this section we also study two cardinal invariants for this new function space, namely the character and the weight, which correspond to well-known countability properties.

Define a subset F of $C(X)$ to be dominating provided that for each $g \in C(X)$, there exists $f \in F$ such that $g \leq f$ (i.e., $g(x) \leq f(x)$ for every $x \in X$). We recall that the dominating number of X is defined by

$$d_n(X) = \min\{|F| : F \text{ is dominating subset of } C(X)\}$$

and a space X is pseudocompact, if and only if, $d_n(X) = \aleph_0$.

The space $(C(X), \tau_{\mathcal{B}}^{sw})$ is homogeneous, so the topology is determined by the family of the neighborhoods of $f = 0$

Proposition 4. $\chi(C(X), \tau_{\mathcal{B}}^{sw}) = d_n(X) \cdot \omega(\mathcal{B})$.

PROOF. Let \mathcal{B}_0 be a base for the bornology \mathcal{B} . First we show that $\chi(C(X), \tau_{\mathcal{B}}^{sw}) \leq d_n(X) \cdot \omega(\mathcal{B})$. Let $F \subset C^+(X)$ with $|F| = d_n(X)$. Let $\{1/\varphi : \varphi \in F\}$, and denote with $\mathcal{B}(0) = \{[B; 1/\varphi]^{sw}(0) : B \in \mathcal{B}_0, \varphi \in F\}$. We show that $\mathcal{B}(0)$ is a base at 0. Let $[B_1; \psi]^{sw}(0)$ such that $B_1 \in \mathcal{B}$ and $\psi \in C^+(X)$. Thus there is $\tilde{B}_1 \in \mathcal{B}_0$ such that $B_1 \subseteq \tilde{B}_1$ and there is $\phi \in F$ such that $1/\psi \leq \phi$, that is $1/\phi \leq \psi$. Hence we have that $[B_1; 1/\phi]^{sw}(0) \subset [B_1; \psi]^{sw}(0)$.

We show the reverse inequality. Let $\mathcal{B}(0)$ be a base at 0 of size $\chi(C(X), \tau_{\mathcal{B}}^{sw})$. We can assume that $\mathcal{B}(0) = \{[B; \phi_B]^{sw}(0) : B \in \mathcal{B}_0\}$ for some $\phi_B \in C^+(X)$. Thus $|\mathcal{B}(0)| =$

$\omega(\mathcal{B})$ and $|\mathcal{B}(0)| \cdot \omega(\mathcal{B}) = \chi(C(X), \tau_{\mathcal{B}}^{sw})$, therefore $|\mathcal{B}(0)| = \chi(C(X), \tau_{\mathcal{B}}^{sw})/\omega(\mathcal{B})$. Define $F = \{1/\phi_B : [B; \phi_B]^{sw}(0) \in \mathcal{B}(0)\}$. We claim that F is dominating in $C(X)$. Let $f \in C(X)$, then there is $\psi \in C^+(X)$ with $f \leq \psi$ and there is $B_1 \in \mathcal{B}_0$ such that $[B_1; \phi_{B_1}]^{sw}(0) \subset [B_1; 1/\psi]^{sw}(0)$. We need to prove that $\phi_{B_1} \leq 1/\psi \leq 1/f$, hence $f \leq \psi \leq 1/\phi_{B_1}$. Assume not. Hence, for some $x \in X$ we have that $\phi_{B_1}(x) > 1/\psi(x)$. Let $0 < \kappa = 1/(\phi_{B_1}(x) \cdot \psi(x)) < 1$ and $\kappa\phi_{B_1} = 1/\psi$. It follows that $\kappa\phi_{B_1} \in [B_1; \phi_{B_1}]^{sw}(0)$, but $1/\psi$ does not belong to $[B_1; 1/\psi]^{sw}(0)$ a contradiction. We have that $d_n(X) \leq |F| \leq |\mathcal{B}(0)|$, therefore $d_n(X) \cdot \omega(\mathcal{B}) \leq \chi(C(X), \tau_{\mathcal{B}}^{sw})$. ■

Proposition 5. $\omega(C(X), \tau_{\mathcal{B}}^{sw}) = \chi(C(X), \tau_{\mathcal{B}}^{sw}) \cdot d(X)$.

PROOF. Since $\omega(Z) \geq \chi(Z) \cdot d(Z)$ for every space Z we need to prove the reverse inequality. Let \mathcal{B}_0 be a base at 0. By the previous proposition we can assume that a base at 0 is of the form $\mathcal{B}_0 = \{[B; 1/\varphi]^{sw}(0) : B \in \mathcal{B}_0, \varphi \in F\}$, where F is dominating in $C(X)$, with $|F| = d_n(X)$ and \mathcal{B}_0 be a base for the bornology \mathcal{B} . Let D be a dense subset of $(C(X), \tau_{\mathcal{B}}^{sw})$ such that $|D| = d(X)$. Define $\mathfrak{B} = \{[B; 1/\varphi]^{sw}(f) : B \in \mathcal{B}_0, f \in D, \varphi \in F\}$. We claim that \mathfrak{B} is a base of $(C(X), \tau_{\mathcal{B}}^{sw})$ of size $\chi(C(X), \tau_{\mathcal{B}}^{sw}) \cdot d(X)$. Let $f \in C(X)$, and $[B; \phi]^{sw}(f)$ open. Hence there is $g \in D \cap [B; \phi]^{sw}(f)$. Then by definition of open set there is $\psi \in C^+(X)$ and $B_1 \in \mathcal{B}$ such that $[B_1; \psi]^{sw}(g) \subseteq [B; \phi]^{sw}(f)$. Moreover, since $\mathcal{B}(0)$ is a local base we have that there is $\varphi \in F$ and $B^* \in \mathcal{B}_0$ such that $[B^*; 1/\varphi]^{sw}(0) \subseteq [B_1; \psi]^{sw}(0)$. Therefore, we have that $[B^*; 1/\varphi]^{sw}(g) \subseteq [B; \phi]^{sw}(f)$. Indeed let $h \in [B^*; 1/\varphi]^{sw}(g)$, we have that $h - g \in [B^*; 1/\varphi]^{sw}(0) \subseteq [B_1; \psi]^{sw}(0)$. Thus $h \in [B_1; \psi]^{sw}(g)$. ■

Corollary 2. Let (X, \mathcal{D}) be a Hausdorff uniform space and let \mathcal{B} be a bornology on X with closed base. The following conditions are equivalent:

- (i) $(C(X, R), \tau_{\mathcal{B}}^{sw})$ is first countable;
- (ii) $(C(X, R), \tau_{\mathcal{B}}^{sw})$ is metrizable;
- (iii) $(C(X, R), \tau_{\mathcal{B}}^{sw})$ is completely metrizable;
- (iv) $(C(X, R), \tau_{\mathcal{B}}^{sw})$ is Čech complete;
- (v) X pseudocompact and \mathcal{B} has a countable base.

PROOF. The equivalence between (i), (ii), (iii), (v) it follows from Proposition 4 and Proposition 3. The fact that (iv) implies (i) holds because τ_u on \mathcal{B} is coarser than $\tau_{\mathcal{B}}^{sw}$, hence submetrizable, i.e., each point is a G_δ -set. A Čech complete space in which points are G_δ -set is first countable. ■

It is easy to verify as in Theorem 3.5 in [12] that the following hold.

Lemma 1. Let (X, \mathcal{D}) be a Hausdorff uniform space and let \mathcal{B} be a bornology on X with closed base. If $(C(X, R), \tau_{\mathcal{B}}^{sw})$ has ccc, then every closed element of \mathcal{B} is compact.

Corollary 3. *Let (X, \mathcal{D}) be a Hausdorff uniform space and let \mathcal{B} be a bornology on X with closed base. The following conditions are equivalent:*

- (i) $(C(X, R), \tau_{\mathcal{B}}^{sw})$ is second countable;
- (ii) X pseudocompact separable and \mathcal{B} has a countable base of compacta.

PROOF. (ii) implies (i) follows directly from Proposition 5. For the other implication by cardinality equality in Proposition 5 we have that X is pseudocompact and separable, and \mathcal{B} has a countable base. Since $(C(X, R), \tau_{\mathcal{B}}^{sw})$ is ccc it follows that \mathcal{B} has a countable base of compacta. ■

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Agata Caserta

Department of Mathematics, Seconda Università degli Studi di Napoli, Caserta
81100, Italy
E-mail: agata.caserta@unina2.it