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Quantum Integrals And The Affineness Criterion For Quantum Yetter-Drinfeld π-Modules

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Abstract

In the paper, the quantum integrals associated to quantum Yetter-Drinfeld π-modules are defined. We shall prove the following affineness criterion: if there exists $\theta = \{\theta_\beta : H_\beta \to \mathrm{Hom}(H_{\beta^{-1}}, A)\}_{\beta\in\pi}$ a total quantum integral and the canonical map $\chi : A \otimes_B A \to \bigoplus_{\gamma\in\pi} H_\gamma \otimes A$, $\chi(a \otimes_B b) = \bigoplus_{\gamma\in\pi} S_\gamma^{-1}\phi_\alpha(b_{[1,\alpha^{-1}\gamma^{-1}\alpha]})b_{[0,0]<-1,\gamma>} \otimes ab_{[0,0]<0,0>}$ is surjective. Then the induction functor $-\otimes_B A : \mathcal{U}_B \to^H \mathcal{YD}_A^\alpha$ is an equivalence of categories. The affineness criterion proven by Menini and Militaru is recovered as special cases.

1 Introduction

The integrals for Hopf algebras were introduced in two fundamental paper: by Larson and Sweedler in [1] for the finite cases, and by Sweedler in [2] for the infinite cases. Then Doi ([3]) introduced the more general integral (called total integral) for H-comodule algebra A, where H is an ordinary Hopf algebra. In 2002, Menini and Militaru ([4]) defined the more general concept of an integral of a threetuple (H, A, C), where H is a Hopf algebra coacting on an algebra A and acting on a coalgebra C. Recently, the first author defined the more general concept of integrals for Doi-Hopf π-datums in Hopf group-coalgebra setting.

Let us note that there exists a symmetric monoidal category, the so-called Turaev category, constructed by Caenepeel and De Lombaerde ([5]) the Hopf algebras which are the same as Hopf π-coalgebras which appeared in the work of Turaev ([6]) on homotopy quantum field theories as a generalization of ordinary Hopf algebras. A purely algebraic study of Hopf π-coalgebras can be found in the references

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Virelizier [7-8], Wang [9-12], and Zunino [13-14].

In the paper, we try to introduce quantum Yetter-Drinfeld π-modules and the concept of quantum integrals to quantum Yetter-Drinfeld π-modules as a generalization of the concept of quantum integrals invented by Menini and Militaru. Then we prove the affineness criterion for quantum Yetter-Drinfeld π-modules.

The paper is organized as follows:

In Section 2, we recall some definitions and basic results related to Hopf π-coalgebras. In Section 3, we introduce the concept of quantum Yetter-Drinfeld π-modules, which can be interpreted as a special Doi-Hopf π-module.

In Section 4, quantum integrals to quantum Yetter-Drinfeld π-modules are introduced [See definition 4.1]. Then we prove the affineness criterion for quantum Yetter-Drinfeld π-modules [See Theorem 4.7].

2 Preliminaries

In this section, we recall some definitions and discuss properties of Hopf π-coalgebras. Most of the materials presented here can be found in [6]-[11].

Throughout this paper, we always let π be a finite discrete group with a neutral element e and k a commutative ring with a unit. If a tensor product is written without index, then it is assumed to be taken over k, that is, $\otimes = \otimes_k$. If U and V are k-modules, $T_{U,V} : U \otimes V \to V \otimes U$ will denote the flip map defined by $T_{U,V}(u \otimes v) \to v \otimes u$, for all $u \in U$ and $v \in V$.

The π-Coalgebras. A *π-coalgebra* is a family of k-module $C = \{C_\alpha\}_{\alpha\in\pi}$ together with a family of k-linear maps $\Delta = \{\Delta_{\alpha,\beta} : \Delta_{\alpha\beta} \to \Delta_\alpha \otimes \Delta_\beta\}_{\alpha,\beta\in\pi}$ (called a *comultiplication*) and a k-linear map $\varepsilon : C_e \to k$ (called a *counit*) such that Δ is coassociative in the sense that

$$(\Delta_{\alpha,\beta} \otimes id_{C\gamma}) \circ \Delta_{\alpha\beta,\gamma} = (id_{C_\alpha} \otimes \Delta_{\beta,\gamma}) \circ \Delta_{\alpha,\beta\gamma}, \tag{2. 1}$$

for any $\alpha, \beta, \gamma \in \pi$ and

$$(id_{C_\alpha} \otimes \varepsilon) \circ \Delta_{\alpha,e} = id_{C_\alpha} = (\varepsilon \otimes id_{C_\alpha}) \circ \Delta_{e,\alpha}, \tag{2. 2}$$

for all $\alpha \in \pi$.

Remark. $(C_e, \Delta_{e,e}, \varepsilon)$ is an ordinary coalgebra in the sense of Sweedler. Following the Sweedler$'$s notation for π-coalgebras, for any $\alpha, \beta \in \pi$ and $c \in C_{\alpha\beta}$, one write

$$\Delta_{\alpha,\beta}(c) = c_{(1,\alpha)} \otimes c_{(2,\beta)}. \tag{2. 3}$$

The coassociativity axiom(2.1) gives that, for any $\alpha,\beta,\gamma\in\pi$ and $c\in C_{\alpha\beta\gamma}$,

$$c_{(1,\alpha\beta)(1,\alpha)}\otimes c_{(1,\alpha\beta)(2,\beta)}\otimes c_{(2,\gamma)}=c_{(1,\alpha)}\otimes c_{(1,\beta\gamma)(1,\beta)}\otimes c_{(1,\beta\gamma)(2,\gamma)}, \qquad (2.\ 4)$$

which is written as $c_{(1,\alpha)}\otimes c_{(2,\beta)}\otimes c_{(3,\gamma)}$. Inductively, we can define $c_{(1,\alpha_1)}\otimes c_{(2,\alpha_2)}\otimes\cdots c_{(n,\alpha_n)}$, for any $c\in C_{\alpha_1\alpha_2\cdots\alpha_n}$. The axiom (2.2) gives that, for any $\alpha\in\pi$ and $c\in C_\alpha$,

$$\varepsilon(c_{(1,e)})c_{(2,\alpha)}=c=c_{(1,\alpha)}\varepsilon(c_{(2,e)}). \qquad (2.\ 5)$$

The Hopf π-Coalgebras. A *Hopf π-coalgebra* is a π-coalgebra $H=(\{H_\alpha\}_{\alpha\in\pi},\Delta,\varepsilon)$ together with a family of k-linear maps $S=\{S_\alpha:H_\alpha\to H_{\alpha^{-1}}\}_{\alpha\in\pi}$ (called an *antipode*) such that the following datas hold:

♦ Each H_α is an algebra with multiplication m_α and unit $1_\alpha\in H_\alpha$,

♦ For all $\alpha,\beta\in\pi$, $\Delta_{\alpha,\beta}$ and $\varepsilon:H_e\to k$ are algebra maps, i.e., for all $c,c^{'}\in H_{\alpha\beta}$,

$$(cc^{'})_{(1,\alpha)}\otimes(cc^{'})_{(2,\beta)}=c_{(1,\alpha)}c^{'}_{(1,\alpha)}\otimes c_{(2,\beta)}c^{'}_{(2,\beta)}, \qquad (2.\ 6)$$

♦ For all $a,a^{'}\in H_e$,

$$\varepsilon(aa^{'})=\varepsilon(a)\varepsilon(a^{'}), \qquad (2.\ 7)$$

♦ For all $\alpha\in\pi$,

$$m_\alpha\circ(id_{H_\alpha}\otimes S_{\alpha^{-1}})\circ\Delta_{\alpha,\alpha^{-1}}=\varepsilon 1_\alpha=m_\alpha\circ(S_{\alpha^{-1}}\otimes id_{H_\alpha})\circ\Delta_{\alpha^{-1},\alpha}. \qquad (2.\ 8)$$

Note that the notion of a Hopf π-coalgebra is not self-dual and that $(H_e,m_e,\Delta_{e,e},\varepsilon,S_e)$ is an ordinary Hopf algebra. A Hopf π-coalgebra H is of finite type, if H_α is finite-dimensional as k-vector space, for all $\alpha\in\pi$.

The π-C-Comodules. Let $C=\{C_\alpha\}_{\alpha\in\pi}$ be a π-coalgebra and V a k-module. A left *π-C-comodule* is a couple $(V,\rho^V=\{\rho^V_\alpha\}_{\alpha\in\pi})$, where for any $\alpha\in\pi$, $\rho^V_\alpha: V\to C_\alpha\otimes V$ is a k-linear morphism, which will be called a comodule structure and denoted by $\rho^V_\alpha(v)=v_{<-1,\alpha>}\otimes v_{<0,0>}$, satisfying the following conditions:

♦ ρ^V is coassociative in the sense that, for any $\alpha,\beta\in\pi$, we have

$$(id_{C_\alpha}\otimes\rho^V_\beta)\circ\rho^V_\alpha=(\Delta_{\alpha,\beta}\otimes id_V)\circ\rho^V_{\alpha\beta},$$

i.e.,

$$v_{<-1,\alpha>}\otimes v_{<0,0><-1,\beta>}\otimes v_{<0,0><0,0>}=v_{<-1,\alpha\beta>(1,\alpha)}\otimes v_{<-1,\alpha\beta>(2,\beta)}\otimes v_{<0,0>}, \qquad (2.\ 9)$$

for any $v\in V$.

♦ V is counitary in the sense that

$$(\varepsilon\otimes id_V)\circ\rho^V_e=id_V,$$

i.e.,

$$\varepsilon(v_{<-1,e>})v_{<0,0>} = v. \tag{2. 10}$$

The π-H-Comodule Algebras. Let $H = (\{H_\alpha\}_{\alpha\in\pi}, m_\alpha, 1_\alpha, \Delta, \varepsilon, S)$ be a Hopf π-coalgebra and A an algebra with the unit 1_A. A left π-H*-comodule algebra* is a left π-H-comodule $(A, \rho^A = \{\rho^A_\alpha\}_{\alpha\in\pi})$ such that the following conditions are satisfied:

$$\rho^A_\alpha(ab) = a_{<-1,\alpha>}b_{<-1,\alpha>} \otimes a_{<0,0>}b_{<0,0>}, \tag{2. 11}$$

for all $\alpha \in \pi$ and $a, b \in A$ and

$$\rho^A_\alpha(1_\alpha) = 1_\alpha \otimes 1_A, \tag{2. 12}$$

for any $\alpha \in \pi$.

Notice that A endowed with the ρ^A_e is an ordinary left H_e−comodule algebra.

The π-H-Module Coalgebras. Let $H = (\{H_\alpha\}_{\alpha\in\pi}, m_\alpha, 1_\alpha, \Delta, \varepsilon, S)$ be a Hopf π-coalgebra and $C = (\{C_\alpha\}_{\alpha\in\pi}, \Delta, \varepsilon)$ a π-coalgebra. C is called a right π-H*-module coalgebra*, if there is a family of k-linear maps $\cdot := \{\cdot : C_\alpha \otimes H_\alpha \to C_\alpha\}$ such that the following conditions are satisfied:

- ♦ For all $\alpha \in \pi$, C_α is a right H_α−module,
- ♦ For all $\alpha, \beta \in \pi, c \in C_{\alpha\beta}, h \in H_{\alpha\beta}$,

$$\Delta_{\alpha,\beta}(c \cdot h) = c_{(1,\alpha)} \cdot h_{(1,\alpha)} \otimes c_{(2,\beta)} \cdot h_{(2,\beta)}, \tag{2. 13}$$

- ♦ For all $c \in C_e$ and $h \in H_e$, $\varepsilon(c \cdot h) = \varepsilon(c)\varepsilon(h)$.

The T-Coalgebras. A Hopf π-coalgebra $H = (\{H_\alpha\}_{\alpha\in\pi}, m_\alpha, 1_\alpha, \Delta, \varepsilon, S)$ is said to be a T*-coalgebra* , if H is endowed with a family of algebra isomorphisms $\phi = \{\phi_\beta : H_\alpha \to H_{\beta\alpha\beta^{-1}}\}_{\alpha,\beta\in\pi}$ (the crossing) such that each ϕ_β preserves the comultiplication and the counit, i.e., for all $\alpha, \beta, \gamma \in \pi$,

$$(\phi_\beta \otimes \phi_\beta) \circ \Delta_{\alpha,\gamma} = \Delta_{\beta\alpha\beta^{-1},\beta\gamma\beta^{-1}} \circ \phi_\beta,\ \varepsilon \circ \phi_\beta = \varepsilon$$

and ϕ is multiplicative in the sense that $\phi_{\alpha\beta} = \phi_\alpha \circ \phi_\beta$.

Let H be a T-coalgebra. Then one has that $\phi_e|_{H_\alpha} = id_{H_\alpha}$, $\phi_\alpha^{-1} = \phi_{\alpha^{-1}}$ for any $\alpha \in \pi$ and that ϕ preserves the antipode, i.e., $\phi_\alpha \circ S_\alpha = S_{\beta\alpha\beta^{-1}} \circ \phi_\beta$, for all $\alpha, \beta \in \pi$.

In the paper, let $H = (\{H_\alpha\}_{\alpha\in\pi}, m_\alpha, 1_\alpha, \Delta, \varepsilon, S)$ be a T-coalgebras. Suppose that the antipode $S = \{S_\alpha\}_{\alpha\in\pi}$ of H is bijective, which means each S_α is bijective.

3 Quantum Yetter-Drinfeld π-Modules

Let H be a Hopf π-coalgebra. A *Doi-Hopf π-datum* is a triple (H, A, C), where A is a left π-H-comodule algebra and C a right π-H-module coalgebra.

A *Doi-Hopf π-module* M is a right A-module which is also a left π-C-comodule with the coaction structure $\rho^M = \{\rho^M_{\alpha,\beta} : M_{\alpha\beta} \to C_\alpha \otimes M_\beta\}_{\alpha,\beta\in\pi}$ such that the following compatible condition holds:

$$\rho^M_{\alpha,\beta}(m \cdot a) = m_{<-1,\alpha>} \cdot a_{<-1,\alpha>} \otimes m_{<0,\beta>} \cdot a_{<0,0>},$$

for all $\alpha \in \pi$ and $m \in M_{\alpha\beta}, a \in A$.

The set of Doi-Hopf π-modules together with both a right A-module maps and a left π-C-comodule maps will form a category of Doi-Hopf π-modules and will be denoted by ${}^{\pi-C}\mathcal{U}(H)_A$ (called a *Doi-Hopf π-modules category*).

Definition 3.1. Let H be a Hopf π-coalgebra and A a k-algebra. The algebra A is called a *π-H-bicomodule algebra*, if A is not only a right π-H-comodule algebra $(A, {}^r\rho^A = \{{}^r\rho^A_\alpha\}_{\alpha\in\pi})$, but also a left π-H-comodule algebra $(A, {}^l\rho^A = \{{}^l\rho^A_\beta\}_{\beta\in\pi})$ such that the following condition:

$$a_{<-1,\alpha>} \otimes a_{<0,0>[0,0]} \otimes a_{<0,0>[1,\beta]} = a_{[0,0]<-1,\alpha>} \otimes a_{[0,0]<0,0>} \otimes a_{[1,\beta]}, \qquad (3.\ 1)$$

for any $\alpha, \beta \in \pi$ and $a \in A$, where we use the standard notation ${}^r\rho^A_\beta(a) = a_{[0,0]} \otimes a_{[1,\beta]}$.

Definition 3.2. Let H be a T-coalgebra and $(A, {}^r\rho^A, {}^l\rho^A)$ a π-H-bicomodule algebra. Let us fix $\alpha \in \pi$. A quantum Yetter-Drinfeld π-module M is a right A-module which is also a left π-H-comodule with a comodule structure $\rho^M = \{\rho^M_\beta : M \to H_\beta \otimes M\}_{\beta\in\pi}$ such that the following compatible condition holds:

$$m_{<-1,\beta>}a_{<-1,\beta>} \otimes m_{<0,0>} \cdot a_{<0,0>} = \phi_\alpha(a_{[1,\alpha^{-1}\beta\alpha]})(m \cdot a_{[0,0]})_{<-1,\beta>} \otimes (m \cdot a_{[0,0]})_{<0,0>}, \qquad (3.\ 2)$$

for any $\beta \in \pi, a \in A$ and $m \in M$.

Now, we can form the category ${}^H\mathcal{YD}^\alpha_A$ of quantum Yetter-Drinfeld π-modules for a fixed $\alpha \in \pi$ in which the composition of morphism of quantum Yetter-Drinfeld π-modules is the standard composition of the underlying linear maps.

Proposition 3.3. E.q (3.2) is equivalent to the following:

$$\rho^M_\beta(m \cdot a) = S^{-1}_\beta \phi_\alpha(a_{[1,\alpha^{-1}\beta^{-1}\alpha]})m_{<-1,\beta>}a_{[0,0]<-1,\beta>} \otimes m_{<0,0>} \cdot a_{[0,0]<0,0>}, \qquad (3.\ 3)$$

for all $m \in M$, $\beta \in \pi$ and $a \in A$.

Proof. A routine check can finish the proof. For example, in fact, for any $m \in M$, $\beta \in \pi$ and $a \in A$, we have

$$
\begin{aligned}
&\phi_\alpha(a_{[1,\alpha^{-1}\beta\alpha]})(m \cdot a_{[0,0]})_{<-1,\beta>} \otimes (m \cdot a_{[0,0]})_{<0,0>} \\
\overset{(3.3)}{=}\ & \phi_\alpha(a_{[1,\alpha^{-1}\beta\alpha]})S_\beta^{-1}\phi_\alpha(a_{[0,0][1,\alpha^{-1}\beta^{-1}\alpha]}) \\
& \qquad m_{<-1,\beta>}a_{[0,0][0,0]<-1,\beta>} \otimes m_{<0,0>} \cdot a_{[0,0][0,0]<0,0>} \\
=\ & \phi_\alpha(a_{[1,e](2,\alpha^{-1}\beta\alpha)})S_\beta^{-1}\phi_\alpha(a_{[1,e](1,\alpha^{-1}\beta^{-1}\alpha)}) \\
& \qquad m_{<-1,\beta>}a_{[0,0]<-1,\beta>} \otimes m_{<0,0>} \cdot a_{[0,0]<0,0>} \\
=\ & \phi_\alpha(a_{[1,e]})_{(2,\beta)}S_\beta^{-1}\phi_\alpha(a_{[1,e]})_{(1,\beta^{-1})}m_{<-1,\beta>}a_{[0,0]<-1,\beta>} \otimes m_{<0,0>} \cdot a_{[0,0]<0,0>} \\
\overset{(2.8)}{=}\ & m_{<-1,\beta>}a_{<-1,\beta>} \otimes m_{<0,0>} \cdot a_{<0,0>}.
\end{aligned}
$$

So we finish the proof. ■

Example 3.4. Let H be a T-coalgebra and $(A, {}^r\rho^A, {}^l\rho^A)$ a π-H-bicomodule algebra. Let us fix $\alpha \in \pi$. Then $(A, \cdot, \{\tilde{\rho}_\gamma^A\}_{\gamma\in\pi})$ is an object of ${}^H\mathcal{YD}_A^\alpha$, where the action $\cdot$ is the multiplication on A and the left comodule structure $\tilde{\rho}^A = \{\tilde{\rho}_\gamma^A\}_{\gamma\in\pi}$ is given by

$$\tilde{\rho}_\gamma^A(a) = S_\gamma^{-1}\phi_\alpha(a_{[1,\alpha^{-1}\gamma^{-1}\alpha]})a_{[0,0]<-1,\gamma>} \otimes a_{[0,0]<0,0>}, \tag{3. 4}$$

for all $\gamma \in \pi$, $a \in A$.

Example 3.5. Let H be a T-coalgebra with $H_\alpha = H_{\gamma\alpha}$ for a fixed $\alpha \in \pi$ and for all $\gamma \in \pi$. Set ${}^l\rho_\gamma^{H_\alpha} = \Delta_{\gamma,\alpha} : H_\alpha \to H_\gamma \otimes H_\alpha$ and ${}^r\rho_\gamma^{H_\alpha} = \Delta_{\alpha,\alpha^{-1}\gamma^{-1}\alpha} : H_\alpha \to H_\alpha \otimes H_{\alpha^{-1}\gamma^{-1}\alpha}$. Then $(H_\alpha, {}^l\rho_\gamma^{H_\alpha}, {}^r\rho_\gamma^{H_\alpha})$ is a π-H-bicomodule algebra, and $(H_\alpha, \cdot, \{\tilde{\rho}_\gamma^{H_\alpha}\}_{\gamma\in\pi})$ is an object of ${}^H\mathcal{YD}_A^\alpha$, where the action $\cdot$ is the multiplication on H_α and the left comodule structure $\tilde{\rho}^{H_\alpha} = \{\tilde{\rho}_\gamma^{H_\alpha}\}_{\gamma\in\pi}$ is given by

$$\tilde{\rho}_\gamma^{H_\alpha}(a) = S_\gamma^{-1}\phi_\alpha(a_{(2,\alpha^{-1}\gamma^{-1}\alpha)})a_{(1,\alpha)(1,\gamma)} \otimes a_{(1,\alpha)(2,\alpha)},$$

for all $\gamma \in \pi$, $a \in H_\alpha$.

Theorem 3.6. Let H be a T-coalgebra. Let us fix $\alpha \in \pi$. Then

(1) A can be made into a left π-$H \otimes H^{op}$-comodule algebra. The comodule structure $\rho^A = \{\rho_\gamma^A : A \to (H \otimes H^{op})_\gamma \otimes A\}$ is given by the following formula

$$\rho_\gamma^A(a) = a_{[0,0]<-1,\gamma>} \otimes S_\gamma^{-1}\phi_\alpha(a_{[1,\alpha^{-1}\gamma^{-1}\alpha]}) \otimes a_{[0,0]<0,0>}, \tag{3. 5}$$

for all $\gamma \in \pi$, $a \in A$.

(2) H can be turned into a right π-$H \otimes H^{op}$-module coalgebra. The action of $H \otimes H^{op}$ on H is given by the following formula

$$g \cdot (h \otimes k) = kgh \tag{3. 6}$$

for all $g, h \in H_\gamma$, $k \in H_\gamma^{op}$.

Proof. (1) We shall check that ρ_γ^A is an algebra morphism from $A \to (H \otimes H^{op})_\gamma \otimes A$. In fact, it is easy to see that $\rho_\gamma^A(1_A) = 1_{H_\gamma} \otimes 1_{H_\gamma^{op}} \otimes 1_A$. We also have

$$\begin{aligned}
\rho_\gamma^A(ab) &= a_{[0,0]<-1,\gamma>}b_{[0,0]<-1,\gamma>} \otimes S_\gamma^{-1}\phi_\alpha(a_{[1,\alpha^{-1}\gamma^{-1}\alpha]}b_{[1,\alpha^{-1}\gamma^{-1}\alpha]}) \\
&\quad \otimes a_{[0,0]<0,0>}b_{[0,0]<0,0>} \\
&= a_{[0,0]<-1,\gamma>}b_{[0,0]<-1,\gamma>} \otimes S_\gamma^{-1}\phi_\alpha(b_{[1,\alpha^{-1}\gamma^{-1}\alpha]})S_\gamma^{-1}\phi_\alpha(a_{[1,\alpha^{-1}\gamma^{-1}\alpha]}) \\
&\quad \otimes a_{[0,0]<0,0>}b_{[0,0]<0,0>} \\
&= \rho_\gamma^A(a)\rho_\gamma^A(b),
\end{aligned}$$

for all $a, b \in A$. In what follows, we need to prove that A is a left π-$H \otimes H^{op}$-comodule. It is sufficient to check that Eq.(2.9) and (2.10) hold. In fact, it is easy to see that Eq.(2.10) holds. For all $\gamma_1, \gamma_2 \in \pi$, $a \in A$, we also have

$$\begin{aligned}
&(\Delta_{H\otimes H^{op}} \otimes id) \circ \rho_{\gamma_1\gamma_2}^A(a) \\
= \;& a_{[0,0]<-1,\gamma_1\gamma_2>(1,\gamma_1)} \otimes S_{\gamma_1\gamma_2}^{-1}\phi_\alpha(a_{[1,\alpha^{-1}\gamma_2^{-1}\gamma_1^{-1}\alpha]})_{(1,\gamma_1)} \otimes a_{[0,0]<-1,\gamma_1\gamma_2>(2,\gamma_2)} \\
& \otimes S_{\gamma_1\gamma_2}^{-1}\phi_\alpha(a_{[1,\alpha^{-1}\gamma_2^{-1}\gamma_1^{-1}\alpha]})_{(2,\gamma_2)} \otimes a_{[0,0]<0,0>} \\
= \;& a_{[0,0]<-1,\gamma_1\gamma_2>(1,\gamma_1)} \otimes S_{\gamma_1}^{-1}(\phi_\alpha(a_{[1,\alpha^{-1}\gamma_2^{-1}\gamma_1^{-1}\alpha]})_{(2,\gamma_1^{-1})}) \otimes a_{[0,0]<-1,\gamma_1\gamma_2>(2,\gamma_2)} \\
& \otimes S_{\gamma_2}^{-1}(\phi_\alpha(a_{[1,\alpha^{-1}\gamma_2^{-1}\gamma_1^{-1}\alpha]})_{(1,\gamma_2^{-1})}) \otimes a_{[0,0]<0,0>} \\
= \;& a_{[0,0]<-1,\gamma_1\gamma_2>(1,\gamma_1)} \otimes S_{\gamma_1}^{-1}\phi_\alpha(a_{[1,\alpha^{-1}\gamma_2^{-1}\gamma_1^{-1}\alpha](2,\alpha^{-1}\gamma_1^{-1}\alpha)}) \otimes a_{[0,0]<-1,\gamma_1\gamma_2>(2,\gamma_2)} \\
& \otimes S_{\gamma_2}^{-1}\phi_\alpha(a_{[1,\alpha^{-1}\gamma_2^{-1}\gamma_1^{-1}\alpha](1,\alpha^{-1}\gamma_1^{-1}\alpha)}) \otimes a_{[0,0]<0,0>}
\end{aligned}$$

and

$$\begin{aligned}
&(id \otimes \rho_{\gamma_2}^A) \circ \rho_{\gamma_1}^A(a) \\
= \;& a_{[0,0]<-1,\gamma_1>} \otimes S_{\gamma_1}^{-1}\phi_\alpha(a_{[1,\alpha^{-1}\gamma_1^{-1}\alpha]}) \otimes a_{[0,0]<0,0>[0,0]<-1,\gamma_2>} \\
& \otimes S_{\gamma_2}^{-1}\phi_\alpha(a_{[0,0]<0,0>[1,\alpha^{-1}\gamma_2^{-1}\alpha]}) \otimes a_{[0,0]<0,0>[0,0]<0,0>} \\
= \;& a_{[0,0][0,0]<-1,\gamma_1>} \otimes S_{\gamma_1}^{-1}\phi_\alpha(a_{[1,\alpha^{-1}\gamma_1^{-1}\alpha]}) \otimes a_{[0,0][0,0]<0,0><-1,\gamma_2>} \\
& \otimes S_{\gamma_2}^{-1}\phi_\alpha(a_{[0,0][1,\alpha^{-1}\gamma_2^{-1}\alpha]}) \otimes a_{[0,0][0,0]<0,0><0,0>} \\
= \;& a_{[0,0][0,0]<-1,\gamma_1\gamma_2>(1,\gamma_1)} \otimes S_{\gamma_1}^{-1}\phi_\alpha(a_{[1,\alpha^{-1}\gamma_1^{-1}\alpha]}) \otimes a_{[0,0][0,0]<-1,\gamma_1\gamma_2>(2,\gamma_2)} \\
& \otimes S_{\gamma_2}^{-1}\phi_\alpha(a_{[0,0][1,\alpha^{-1}\gamma_2^{-1}\alpha]}) \otimes a_{[0,0][0,0]<0,0>} \\
= \;& a_{[0,0]<-1,\gamma_1\gamma_2>(1,\gamma_1)} \otimes S_{\gamma_1}^{-1}\phi_\alpha(a_{[1,\alpha^{-1}\gamma_2^{-1}\gamma_1^{-1}\alpha](2,\alpha^{-1}\gamma_1^{-1}\alpha)}) \otimes a_{[0,0]<-1,\gamma_1\gamma_2>(2,\gamma_2)} \\
& \otimes S_{\gamma_2}^{-1}\phi_\alpha(a_{[1,\alpha^{-1}\gamma_2^{-1}\gamma_1^{-1}\alpha](1,\alpha^{-1}\gamma_2^{-1}\alpha)}) \otimes a_{[0,0]<0,0>}.
\end{aligned}$$

So we prove that $(\Delta_{H\otimes H^{op}} \otimes id) \circ \rho_{\gamma_1\gamma_2}^A = (id \otimes \rho_{\gamma_2}^A) \circ \rho_{\gamma_1}^A$, for all $\gamma_1, \gamma_2 \in \pi$.

(2) It is not hard to verify that H is a family of right $H \otimes H^{op}$-module. In order to check Eq.(2.13), we do a calculation as follows:

$$\begin{aligned}
\Delta_{\alpha,\beta}(g \cdot (h \otimes k)) &= k_{(1,\alpha)}g_{(1,\alpha)}h_{(1,\alpha)} \otimes k_{(2,\beta)}g_{(2,\beta)}h_{(2,\beta)} \\
&= g_{(1,\alpha)} \cdot (h_{(1,\alpha)} \otimes k_{(1,\alpha)}) \otimes g_{(2,\beta)} \cdot (h_{(2,\beta)} \otimes k_{(2,\beta)})
\end{aligned}$$

for all $g, h \in H_{\alpha\beta}$, $k \in H_{\alpha\beta}^{op}$. ■

From Theorem 3.6, we can view ${}^H\mathcal{YD}_A^\alpha$ (given a fixed $\alpha \in \pi$) as the category of Doi-Hopf π-modules associated to the Doi-Hopf π-datum $(H \otimes H^{op}, A, H)$. Then ${}^{\pi-H}\mathcal{U}(H \otimes H^{op})_A = {}^H\mathcal{YD}_A^\alpha$. Moreover, $\bigoplus_{\gamma\in\pi} H_\gamma \otimes A \in {}^H\mathcal{YD}_A^\alpha$ via the following structures

$$(h \otimes a) \cdot b = S_\gamma^{-1}\phi_\alpha(b_{[1,\alpha^{-1}\gamma^{-1}\alpha]})hb_{[0,0]<-1,\gamma>} \otimes ab_{[0,0]<0,0>}, \tag{3. 7}$$

$$\rho_\beta^{\bigoplus_{\gamma\in\pi} H_\gamma\otimes A}(h \otimes a) = h_{(1,\beta)} \otimes h_{(2,\beta^{-1}\gamma)} \otimes a, \tag{3. 8}$$

for any $\beta, \gamma \in \pi$, $h \in H_\gamma$ and $a, b \in A$.

4 The Affineness Criterion For Quantum Yetter-Drinfeld π-Modules

In the section, quantum integrals to quantum Yetter-Drinfeld π-modules are introduced. Then we prove the the affineness criterion for quantum Yetter-Drinfeld π-modules.

Definition 4.1. Let H be a T-coalgebra and $(A, {}^r\rho^A, {}^l\rho^A)$ a π-H-bicomodule algebra. Let us fix α in π. A family of k-linear map $\theta = \{\theta_\beta : C_\beta \to \mathrm{Hom}(C_{\beta^{-1}}, A)\}_{\beta\in\pi}$ is called a quantum integral of (H, A, C), if

$$\begin{aligned} c_{(1,\gamma)} \otimes \theta_\beta(c_{(2,\beta)})(d) \;=\; & S_\gamma^{-1}\phi_\alpha(\theta_{\gamma\beta}(c)(d_{(1,(\gamma\beta)^{-1})})_{[1,\alpha^{-1}\gamma^{-1}\alpha]})d_{(2,\gamma)} \\ & \theta_{\gamma\beta}(c)(d_{(1,(\gamma\beta)^{-1})})_{[0,0]<-1,\gamma>} \\ & \qquad \otimes\theta_{\gamma\beta}(c)(d_{(1,(\gamma\beta)^{-1})})_{[0,0]<0,0>} \end{aligned}$$

for all $\gamma, \beta \in \pi$ and $c \in C_{\gamma\beta}, d \in C_{\beta^{-1}}$. A quantum integral $\theta = \{\theta_\beta : C_\beta \to \mathrm{Hom}(C_{\beta^{-1}}, A)\}_{\beta\in\pi}$ is called total, if

$$\sum_{\beta\in\pi} \theta_\beta(c_{(1,\beta)})(c_{(2,\beta^{-1})}) = \varepsilon(c)1_A, \tag{4. 1}$$

for all $\beta \in \pi, c \in C_e$.

Proposition 4.2. Let $H = (\{H_\beta\}, \Delta, \varepsilon, S)$ be a T-coalgebra and $(A, {}^r\rho^A, {}^l\rho^A)$ a π-H-bicomodule algebra. Let us fix α in π. Assume that there exists $\theta = \{\theta_\beta : C_\beta \to \mathrm{Hom}(C_{\beta^{-1}}, A)\}_{\beta\in\pi}$ a total quantum integral. Then

$$\hat{\rho}^A = \bigoplus_{\gamma\in\pi} \tilde{\rho}_\gamma : A \to \bigoplus_{\beta\in\pi} H_\beta \otimes A$$

splits in ${}^H\mathcal{YD}_A^\alpha$.

Proof. We define the map

$$\tau_A : \bigoplus_{\beta\in\pi} H_\beta \otimes A \to A$$

$$\tau_A(\bigoplus_{\beta\in\pi} h_\beta \otimes a_\beta) = \sum_{\beta\in\pi} a_{\beta[0,0]<0,0>}\theta_\beta(h_\beta)(S^{-1}_{\beta^{-1}}\phi_\alpha(a_{\beta[1,\alpha^{-1}\beta\alpha]})a_{\beta[0,0]<-1,\beta^{-1}>}). \tag{4. 2}$$

Then the τ_A is a left π-H-colinear retraction of $\widetilde{\rho}^A$. In particular, $\tau_A(\bigoplus_{\beta\in\pi} 1_\beta \otimes 1_A) = 1_A$ and

$$\begin{aligned}
&\bigoplus_{\beta\in\pi} h_{\beta(1,\gamma)} \otimes \tau(h_{\beta(2,\gamma^{-1}\beta)} \otimes a_\beta)\\
=\ & S^{-1}_\gamma\phi_\alpha(\tau(\bigoplus_{\beta\in\pi} h_\beta \otimes a_\beta)_{[1,\alpha^{-1}\gamma^{-1}\alpha]})\\
& \tau(\bigoplus_{\beta\in\pi} h_\beta \otimes a_\beta)_{[0,0]<-1,\gamma>} \otimes \tau(\bigoplus_{\beta\in\pi} h_\beta \otimes a_\beta)_{[0,0]<0,0>},
\end{aligned} \tag{4. 3}$$

for all $\gamma \in \pi$. We define now

$$\Lambda : \bigoplus_{\beta\in\pi} H_\beta \otimes A \to A,$$

$$\Lambda(\bigoplus_{\gamma\in\pi} h_\gamma \otimes a_\gamma) = \tau(\bigoplus_{\gamma\in\pi} S^{-1}_\gamma(S^{-1}_{\gamma^{-1}}\phi_\alpha(a_{\gamma[1,\alpha^{-1}\gamma\alpha]}))h_\gamma S_{\gamma^{-1}}(a_{\gamma[0,0]<-1,\gamma^{-1}>}) \otimes 1_A)a_{\gamma[0,0]<0,0>}. \tag{4. 4}$$

Then, for $a \in A$, we have

$$\begin{aligned}
& (\Lambda \circ \hat{\rho}^A)(a)\\
=\ & \Lambda(\bigoplus_{\gamma\in\pi} S^{-1}_\gamma\phi_\alpha(a_{[1,\alpha^{-1}\gamma^{-1}\alpha]})a_{[0,0]<-1,\gamma>} \otimes a_{[0,0]<0,0>})\\
\overset{(4.4)}{=}\ & \tau(\bigoplus_{\gamma\in\pi} S^{-1}_\gamma(S^{-1}_{\gamma^{-1}}\phi_\alpha(a_{[0,0]<0,0>[1,\alpha^{-1}\gamma\alpha]}))S^{-1}_\gamma\phi_\alpha(a_{[1,\alpha^{-1}\gamma^{-1}\alpha]})a_{[0,0]<-1,\gamma>}\\
& \qquad S_{\gamma^{-1}}(a_{[0,0]<0,0>[0,0]<-1,\gamma^{-1}>}) \otimes 1_A)a_{[0,0]<0,0>[0,0]<0,0>}\\
=\ & \tau(\bigoplus_{\gamma\in\pi} S^{-1}_\gamma(S^{-1}_{\gamma^{-1}}\phi_\alpha(a_{[0,0][1,\alpha^{-1}\gamma\alpha]}))S^{-1}_\gamma\phi_\alpha(a_{[1,\alpha^{-1}\gamma^{-1}\alpha]})a_{[0,0][0,0]<-1,\gamma>}\\
& \qquad S_{\gamma^{-1}}(a_{[0,0][0,0]<0,0><-1,\gamma^{-1}>}) \otimes 1_A)a_{[0,0][0,0]<0,0><0,0>}\\
=\ & \tau(\bigoplus_{\gamma\in\pi} S^{-1}_\gamma(S^{-1}_{\gamma^{-1}}\phi_\alpha(a_{[0,0][1,\alpha^{-1}\gamma\alpha]}))S^{-1}_\gamma\phi_\alpha(a_{[1,\alpha^{-1}\gamma^{-1}\alpha]})a_{[0,0][0,0]<-1,e>(1,\gamma)}\\
& \qquad S_{\gamma^{-1}}(a_{[0,0][0,0]<-1,e>(2,\gamma^{-1})}) \otimes 1_A)a_{[0,0][0,0]<0,0>}\\
=\ & \tau(\bigoplus_{\gamma\in\pi} S^{-1}_\gamma(S^{-1}_{\gamma^{-1}}\phi_\alpha(a_{[1,e](1,\alpha^{-1}\gamma\alpha)}))S^{-1}_\gamma\phi_\alpha(a_{[1,e](2,\alpha^{-1}\gamma^{-1}\alpha)}) \otimes 1_A)a_{[0,0]}
\end{aligned}$$

$$= \tau(\bigoplus_{\gamma\in\pi} S_\gamma^{-1}(\phi_\alpha(a_{[1,e](2,\alpha^{-1}\gamma^{-1}\alpha)})S_{\gamma^{-1}}^{-1}\phi_\alpha(a_{[1,e](1,\alpha^{-1}\gamma\alpha)}))\otimes 1_A)a_{[0,0]}$$
$$= \tau(\bigoplus_{\gamma\in\pi} S_\gamma^{-1}(\phi_\alpha(a_{[1,e]})_{(2,\gamma^{-1})}S_{\gamma^{-1}}^{-1}\phi_\alpha(a_{[1,e]})_{(1,\gamma)})\otimes 1_A)a_{[0,0]}$$
$$= \tau(\bigoplus_{\gamma\in\pi} 1_\gamma\otimes 1_A)a = a,$$

i.e., Λ is still a retraction of $\hat{\rho}^A$. Now, for all $b\in\pi$, $h_\gamma\in H_\gamma$ and $a_\gamma\in A$, we have

$$\Lambda((\bigoplus_{\gamma\in\pi} h_\gamma\otimes a_\gamma)\cdot b)$$
$$\overset{(3.7)}{=} \Lambda(\bigoplus_{\gamma\in\pi} S_\gamma^{-1}\phi_\alpha(b_{[1,\alpha^{-1}\gamma^{-1}\alpha]})h_\gamma b_{[0,0]<-1,\gamma>}\otimes a_\gamma b_{[0,0]<0,0>})$$
$$\overset{(4.4)}{=} \tau(\bigoplus_{\gamma\in\pi} S_\gamma^{-1}(S_{\gamma^{-1}}^{-1}\phi_\alpha((a_\gamma b_{[0,0]<0,0>})_{[1,\alpha^{-1}\gamma\alpha]}))S_\gamma^{-1}\phi_\alpha(b_{[1,\alpha^{-1}\gamma^{-1}\alpha]})h_\gamma b_{[0,0]<-1,\gamma>}$$
$$S_{\gamma^{-1}}((a_\gamma b_{[0,0]<0,0>})_{[0,0]<-1,\gamma^{-1}>})\otimes 1_A)(a_\gamma b_{[0,0]<0,0>})_{[0,0]<0,0>}$$
$$= \tau(\bigoplus_{\gamma\in\pi} S_\gamma^{-1}(S_{\gamma^{-1}}^{-1}\phi_\alpha(a_{\gamma[1,\alpha^{-1}\gamma\alpha]}b_{[0,0]<0,0>[1,\alpha^{-1}\gamma\alpha]}))$$
$$S_\gamma^{-1}\phi_\alpha(b_{[1,\alpha^{-1}\gamma^{-1}\alpha]})h_\gamma b_{[0,0]<-1,\gamma>}S_{\gamma^{-1}}(a_{\gamma[0,0]<-1,\gamma^{-1}>}$$
$$b_{[0,0]<0,0>[0,0]<-1,\gamma^{-1}>})\otimes 1_A)(a_{\gamma[0,0]<0,0>}b_{[0,0]<0,0>[0,0]<0,0>})$$
$$= \tau(\bigoplus_{\gamma\in\pi} S_\gamma^{-1}(S_{\gamma^{-1}}^{-1}\phi_\alpha(a_{\gamma[1,\alpha^{-1}\gamma\alpha]}b_{[0,0][1,\alpha^{-1}\gamma\alpha]}))$$
$$S_\gamma^{-1}\phi_\alpha(b_{[1,\alpha^{-1}\gamma^{-1}\alpha]})h_\gamma b_{[0,0][0,0]<-1,\gamma>}S_{\gamma^{-1}}(a_{\gamma[0,0]<-1,\gamma^{-1}>}$$
$$b_{[0,0][0,0]<0,0><-1,\gamma^{-1}>})\otimes 1_A)(a_{\gamma[0,0]<0,0>}b_{[0,0][0,0]<0,0><0,0>})$$
$$= \tau(\bigoplus_{\gamma\in\pi} S_\gamma^{-1}(S_{\gamma^{-1}}^{-1}\phi_\alpha(a_{\gamma[1,\alpha^{-1}\gamma\alpha]}b_{[0,0][1,\alpha^{-1}\gamma\alpha]}))$$
$$S_\gamma^{-1}\phi_\alpha(b_{[1,\alpha^{-1}\gamma^{-1}\alpha]})h_\gamma b_{[0,0][0,0]<-1,e>(1,\gamma)}S_{\gamma^{-1}}(a_{\gamma[0,0]<-1,\gamma^{-1}>}$$
$$b_{[0,0][0,0]<-1,e>(2,\gamma^{-1})})\otimes 1_A)(a_{\gamma[0,0]<0,0>}b_{[0,0][0,0]<0,0>})$$
$$= \tau(\bigoplus_{\gamma\in\pi} S_\gamma^{-1}(S_{\gamma^{-1}}^{-1}\phi_\alpha(a_{\gamma[1,\alpha^{-1}\gamma\alpha]}b_{[0,0][1,\alpha^{-1}\gamma\alpha]}))$$
$$S_\gamma^{-1}\phi_\alpha(b_{[1,\alpha^{-1}\gamma^{-1}\alpha]})h_\gamma S_{\gamma^{-1}}(a_{\gamma[0,0]<-1,\gamma^{-1}>})\otimes 1_A)(a_{\gamma[0,0]<0,0>}b_{[0,0][0,0]})$$
$$= \tau(\bigoplus_{\gamma\in\pi} S_\gamma^{-1}(S_{\gamma^{-1}}^{-1}\phi_\alpha(a_{\gamma[1,\alpha^{-1}\gamma\alpha]}b_{[1,e](1,\alpha^{-1}\gamma\alpha)})$$
$$S_\gamma^{-1}\phi_\alpha(b_{[1,e](2,\alpha^{-1}\gamma^{-1}\alpha)})h_\gamma S_{\gamma^{-1}}(a_{\gamma[0,0]<-1,\gamma^{-1}>})\otimes 1_A)(a_{\gamma[0,0]<0,0>}b_{[0,0]})$$
$$= \tau(\bigoplus_{\gamma\in\pi} S_\gamma^{-1}(S_{\gamma^{-1}}^{-1}\phi_\alpha(a_{\gamma[1,\alpha^{-1}\gamma\alpha]})S_\gamma^{-1}(S_{\gamma^{-1}}^{-1}\phi_\alpha(b_{[1,e](1,\alpha^{-1}\gamma\alpha)})$$
$$S_\gamma^{-1}\phi_\alpha(b_{[1,e](2,\alpha^{-1}\gamma^{-1}\alpha)})h_\gamma S_{\gamma^{-1}}(a_{\gamma[0,0]<-1,\gamma^{-1}>})\otimes 1_A)(a_{\gamma[0,0]<0,0>}b_{[0,0]})$$
$$= \tau(\bigoplus_{\gamma\in\pi} S_\gamma^{-1}(S_{\gamma^{-1}}^{-1}\phi_\alpha(a_{\gamma[1,\alpha^{-1}\gamma\alpha]})S_\gamma^{-1}(S_{\gamma^{-1}}^{-1}\phi_\alpha(b_{[1,e]})_{(1,\gamma)})$$

$$
\begin{aligned}
& S_\gamma^{-1}\phi_\alpha(b_{[1,e]})_{(2,\gamma^{-1})}h_\gamma S_\gamma(a_{\gamma[0,0]<-1,\gamma^{-1}>})\otimes 1_A)(a_{\gamma[0,0]<0,0>}b_{[0,0]}) \\
= \;& \tau(\bigoplus_{\gamma\in\pi} S_\gamma^{-1}(S_{\gamma^{-1}}^{-1}\phi_\alpha(a_{\gamma[1,\alpha^{-1}\gamma\alpha]}))h_\gamma S_{\gamma^{-1}}(a_{\gamma[0,0]<-1,\gamma^{-1}>})\otimes 1_A)a_{\gamma[0,0]<0,0>}b \\
= \;& \Lambda(\bigoplus_{\gamma\in\pi} h_\gamma\otimes a_\gamma)\cdot b.
\end{aligned}
$$

So we finish the proof. ■

We can define now the coinvariants of A as

$$
\begin{aligned}
B &= A^{co(H)} \\
&= \{a\in A|S_\gamma^{-1}\phi_\alpha(a_{[1,\alpha^{-1}\gamma^{-1}\alpha]})a_{[0,0]<-1,\gamma>}\otimes a_{[0,0]<0,0>} = 1_\gamma\otimes a,\ \text{for all } \gamma\in\pi\}.
\end{aligned}
$$

Then B is a subalgebra of A and will be called the subalgebra of quantum coinvariants.

Proposition 4.3. Let H be a T-coalgebra and $(A,^r\rho^A,^l\rho^A)$ a π-H-bicomodule algebra, and B the subalgebra of quantum coinvariants. Let us fix α in π. Assume that there exists $\theta = \{\theta_\beta : H_\beta \to \mathrm{Hom}(H_{\beta^{-1}}, A)\}_{\beta\in\pi}$ a total quantum integral. Then B is a direct summand of as a left B-submodule.

Proof. We shall prove that there exist a well defined left trace given by the formula

$$t^l : A \to B,$$

$$t^l(a) = \tau_A(\bigoplus_{\beta\in\pi} 1_\beta\otimes a) = \sum_{\beta\in\pi} a_{[0,0]<0,0>}\theta_\beta(1_\beta)(S_{\beta^{-1}}^{-1}\phi_\alpha(a_{[1,\alpha^{-1}\beta\alpha]})a_{[0,0]<-1,\beta^{-1}>})$$

for all $a\in A$. Notice that $t^l(a)\in A^{co(H)}$, for all $a\in A$, in fact, taking $h_\beta = 1_\beta$ in Eq. (4.3), for all $\beta\in\pi$, we have

$$
\begin{aligned}
1_\gamma\otimes\tau(\bigoplus_{\beta\in\pi} 1_{\gamma^{-1}\beta}\otimes a) \;=\;& S_\gamma^{-1}\phi_\alpha(\tau(\bigoplus_{\beta\in\pi} 1_\beta\otimes a)_{[1,\alpha^{-1}\gamma^{-1}\alpha]}) \\
& \tau(\bigoplus_{\beta\in\pi} 1_\beta\otimes a)_{[0,0]<-1,\gamma>}\otimes\tau(\bigoplus_{\beta\in\pi} 1_\beta\otimes a)_{[0,0]<0,0>}
\end{aligned}
$$

for all $\gamma\in\pi$, i.e.,

$$1_\gamma\otimes t^l(a) = S_\gamma^{-1}\phi_\alpha(t^l(a)_{[1,\alpha^{-1}\gamma^{-1}\alpha]})t^l(a)_{[0,0]<-1,\gamma>}\otimes t^l(a)_{[0,0]<0,0>}.$$

Now, for $b\in B$ and $a\in A$,

$$
\begin{aligned}
t^l(ba) &= \sum_{\beta\in\pi}(ba)_{[0,0]<0,0>}\theta_\beta(1_\beta)(S_{\beta^{-1}}^{-1}\phi_\alpha((ba)_{[1,\alpha^{-1}\beta\alpha]})(ba)_{[0,0]<-1,\beta^{-1}>}) \\
&= \sum_{\beta\in\pi} b_{[0,0]<0,0>}a_{[0,0]<0,0>}\theta_\beta(1_\beta)(S_{\beta^{-1}}^{-1}\phi_\alpha(a_{[1,\alpha^{-1}\beta\alpha]})S_{\beta^{-1}}^{-1}\phi_\alpha(b_{[1,\alpha^{-1}\beta\alpha]})
\end{aligned}
$$

$$
\begin{aligned}
& b_{[0,0]<-1,\beta^{-1}>}a_{[0,0]<-1,\beta^{-1}>}) \\
= & \sum_{\beta\in\pi} ba_{[0,0]<0,0>}\theta_\beta(1_\beta)(S^{-1}_{\beta^{-1}}\phi_\alpha(a_{[1,\alpha^{-1}\beta\alpha]})a_{[0,0]<-1,\beta^{-1}>}) \\
= & bt^l(a).
\end{aligned}
$$

Hence t^l is a left B-module map and finally

$$t^l(1_A) = \tau_A(\bigoplus_{\beta\in\pi} 1_\beta \otimes a) = \sum_{\beta\in\pi}\theta_\beta(1_\beta)(1_{\beta^{-1}}) = 1_A.$$

Hence t^l is a left B-module retraction of the inclusion $B \subset A$.

We finish the proof. ■

Definition 4.4. Let H be a T-coalgebra and $(A, {}^r\rho^A, {}^l\rho^A)$ a π-H-bicomodule algebra, and B the subalgebra of quantum coinvariants. Let us fix α in π. Assume that there exists $\theta = \{\theta_\beta : H_\beta \to \mathrm{Hom}(H_{\beta^{-1}}, A)\}_{\beta\in\pi}$ a total quantum integral. The map

$$t^l : A \to B,$$

$$t^l(a) = \tau_A(\bigoplus_{\beta\in\pi} 1_\beta \otimes a) = \sum_{\beta\in\pi} a_{[0,0]<0,0>}\theta_\beta(1_\beta)(S^{-1}_{\beta^{-1}}\phi_\alpha(a_{[1,\alpha^{-1}\beta\alpha]})a_{[0,0]<-1,\beta^{-1}>})$$

for all $a \in A$ is called the (left) quantum trace associated to θ.

Now, fix an $\alpha \in \pi$, we shall construct functors connecting ${}^H\mathcal{YD}^\alpha_A$ and $\mathcal{U}_B$. First, if $M \in {}^H\mathcal{YD}^\alpha_A$, then

$$M^{co(H)} = \{m \in M |\ \rho^M_\gamma(m) = 1_\gamma \otimes m,\ \text{for all } \gamma \in \pi\}$$

is the right B-module of the coinvariants of M, in fact, for all $\gamma \in \pi, m \in M^{co(H)}$ and $a \in B$, we have

$$
\begin{aligned}
\rho^M_\gamma(m\cdot a) &= S^{-1}_\lambda\phi_\alpha(a_{[1,\alpha^{-1}\lambda^{-1}\alpha]})m_{<-1,\lambda>}a_{[0,0]<-1,\lambda>} \otimes m_{<0,0>}\cdot a_{[0,0]<0,0>} \\
&= S^{-1}_\lambda\phi_\alpha(a_{[1,\alpha^{-1}\lambda^{-1}\alpha]})a_{[0,0]<-1,\lambda>} \otimes m\cdot a_{[0,0]<0,0>} \\
&= 1_\gamma \otimes m\cdot a = \rho^M_\gamma(m)a.
\end{aligned}
$$

Furthermore, we have a covariant functor

$$(-)^{co(H)} : {}^H\mathcal{YD}^\alpha_A \to \mathcal{U}_B.$$

Now, for $N \in \mathcal{U}_B,\ N\otimes_B A \in {}^H\mathcal{YD}^\alpha_A$ via the structures

$$(n\otimes_B a)\cdot a' = n\otimes_B aa',$$

$$\rho^{N\otimes_B A}_\gamma(n\otimes_B a) = S^{-1}_\gamma\phi_\alpha(a_{[1,\alpha^{-1}\gamma^{-1}\alpha]})a_{[0,0]<-1,\gamma>} \otimes n\otimes_B a_{[0,0]<0,0>}$$

for all $n \in N, a, a^{'} \in A$ and $\gamma \in \pi$. In this way, we have constructed a covariant functor called the induction functor

$$- \otimes_B A : \mathcal{U}_B \rightarrow^H \mathcal{YD}_A^{\alpha}.$$

We shall prove now that the above functors are an adjoint pair.

Proposition 4.5. Let H be a T-coalgebra and (A, ρ^r, ρ^l) a π-H-bicomodule algebra. Then the induction functor $- \otimes_B A : \mathcal{U}_B \rightarrow^H \mathcal{YD}_A^{\alpha}$ is a left adjoint of the coinvariant functor $(-)^{co(H)} :^H \mathcal{YD}_A^{\alpha} \rightarrow \mathcal{U}_B$.

Proof. the unit and the counit of the adjointness are given by

$$\eta_N : N \rightarrow (N \otimes_B A)^{co(H)},\ \eta_N(n) = n \otimes_B 1_A$$

for all $N \in \mathcal{U}_B, n \in N$, and

$$\beta_M : M^{co(H)} \otimes_B A \rightarrow M,\ \beta_M(m \otimes_B a) = ma$$

for all $M \in^H \mathcal{YD}_A^{\alpha},\ m \in M^{co(H)}$ and $a \in A$. So The proof is finished. ■

We are going to prove now an affineness condition for quantum Yetter-Drinfeld π-modules. First, we need the following

Theorem 4.6. Let $H = (\{H_\beta\}, \Delta, \varepsilon, S)$ be a T-coalgebra and $(A,^r \rho^A,^l \rho^A)$ a π-H-bicomodule algebra, and $B = A^{co(H)}$. Let us fix α in π. Assume that there exists $\theta = \{\theta_\beta : H_\beta \rightarrow \mathrm{Hom}(H_{\beta^{-1}}, A)\}_{\beta \in \pi}$ a total quantum integral. Then

$$\eta_N : N \rightarrow (N \otimes_B A)^{co(H)},\ \eta_N(n) = n \otimes_B 1_A$$

is an isomorphism of right B-modules for all $N \in \mathcal{U}_B$.

Proof. Using the left quantum trace $t^l : A \rightarrow B$, we shall construct an inverse of η_N. We define

$$\chi_N : (N \otimes_B A)^{co(H)} \rightarrow N, \chi_N(\sum_i n_i \otimes a_i) = \sum_i n_i t^l(a_i)$$

for all $\sum_i n_i \otimes_B a_i \in (N \otimes_B A)^{co(H)}$. It is easy to see that $\chi_N \circ \eta_N = id$. Let $\sum_i n_i \otimes_B a_i \in (N \otimes_B A)^{co(H)}$. Then, for all $\gamma \in \pi$,

$$S_\gamma^{-1} \phi_\alpha(a_{i[1,\alpha^{-1}\gamma^{-1}\alpha]}) a_{i[0,0]<-1,\gamma>} \otimes n_i \otimes_B a_{i[0,0]<0,0>} = 1_\gamma \otimes n_i \otimes_B a_i.$$

It follows that

$$n_i \otimes_B a_{i[0,0]<0,0>} \otimes \theta_{\gamma^{-1}}(1_{\gamma^{-1}})(S_\gamma^{-1} \phi_\alpha(a_{i[1,\alpha^{-1}\gamma^{-1}\alpha]}) a_{i[0,0]<-1,\gamma>}) = n_i \otimes_B a_i \otimes 1_A.$$

Now, if we multiply the last factors, we get

$$n_i \otimes_B t^l(a_i) = n_i \otimes_B a_i.$$

Hence we obtain

$$(\eta_N \circ \chi_N)(\sum_i n_i \otimes_B a_i) = \sum_i n_i t^l(a_i) \otimes_B 1_A = \sum_i n_i \otimes_B t^l(a_i) = \sum_i n_i \otimes_B a_i,$$

i.e., χ_N is an inverse of η_N. ■

Theorem 4.7. Let H be a T-coalgebra and $(A,{}^r\rho^A,{}^l\rho^A)$ a π-H-bicomodule algebra, and $B = A^{co(H)}$. Let us fix α in π. Assume that there exists $\theta = \{\theta_\beta : H_\beta \to \mathrm{Hom}(H_{\beta^{-1}}, A)\}_{\beta\in\pi}$ a total quantum integral, and the canonical map

$$\chi : A \otimes_B A \to \bigoplus_{\gamma\in\pi} H_\gamma \otimes A,$$

$$\chi(a \otimes_B b) = \bigoplus_{\gamma\in\pi} S_\gamma^{-1}\phi_\alpha(b_{[1,\alpha^{-1}\gamma^{-1}\alpha]})b_{[0,0]<-1,\gamma>} \otimes ab_{[0,0]<0,0>},$$

for all $a, b \in A$, is surjective. Then the induction functor $- \otimes_B A : \mathcal{U}_B \to {}^H\mathcal{YD}_A^\alpha$ is an equivalence of categories.

Proof. In Theorem 4.6, we have shown that, under the assumption of the existence of a total quantum integral, the adjunction map $\eta_N : N \to (N \otimes_B A)^{co(H)}$ is an isomorphism for all $N \in \mathcal{U}_B$. It remains to prove that the other adjunction map, namely $\beta_M : M^{co(H)} \otimes_B A \to M$, $\beta_M(m \otimes_B a) = ma$ is an isomorphism for all $M \in {}^H\mathcal{YD}_A^\alpha$.

Let V be a k-module. Then $A \otimes V \in {}^H\mathcal{YD}_A^\alpha$ via the structures induced by A, i.e.,

$$(a \otimes v)b = ab \otimes v,$$

$$\rho_\gamma^{A\otimes V}(a \otimes v) = S_\gamma^{-1}\phi_\alpha(a_{[1,\alpha^{-1}\gamma^{-1}\alpha]})a_{[0,0]<-1,\gamma>} \otimes a_{[0,0]<0,0>} \otimes v,$$

for all $a, b \in A$, $v \in V$ and $\gamma \in \pi$. In particular, for $V = A$, $A \otimes A \in {}^H\mathcal{YD}_A^\alpha$ via

$$(a \otimes a')b = ab \otimes a',$$

$$\rho_\gamma^{A\otimes V}(a \otimes a') = S_\gamma^{-1}\phi_\alpha(a_{[1,\alpha^{-1}\gamma^{-1}\alpha]})a_{[0,0]<-1,\gamma>} \otimes a_{[0,0]<0,0>} \otimes a'$$

for all $a, a', b \in A$ and $\gamma \in \pi$. We will prove first that the adjunction map $\beta_{A\otimes V} : A \otimes V^{co(H)} \otimes_B A \to A \otimes V$ is an isomorphism for any k-module V.

First, $V \otimes B$ and $B \otimes V \in \mathcal{U}_B$ via

$$(v \otimes b)b' = v \otimes bb', \quad (b \otimes v)b' = bb' \otimes v$$

for all $b, b' \in B$, $v \in V$. The flip map $\tau : V \otimes B \to B \otimes V$, $\tau(v \otimes b) = b \otimes v$ is an isomorphism in $\mathcal{U}_B$. On the other hand, $V \otimes A \in {}^H\mathcal{YD}_A^\alpha$ via

$$(v \otimes a)b = v \otimes ab,$$

$$\rho_{\gamma}^{V\otimes A}(v\otimes a)=S_{\gamma}^{-1}\phi_{\alpha}(a_{[1,\alpha^{-1}\gamma^{-1}\alpha]})a_{[0,0]<-1,\gamma>}\otimes v\otimes a_{[0,0]<0,0>},$$

for all $a,b\in A$, $v\in V$ and $\gamma\in\pi$. The flip map $\tau:A\otimes V\to V\otimes A$, $\tau(a\otimes v)=v\otimes a$ is an isomorphism in ${}^{H}\mathcal{YD}_{A}^{\alpha}$. Applying Theorem 4.6 to $N=V\otimes B\cong B\otimes V$, we obtain the following isomorphisms in $\mathcal{U}_B$

$$B\otimes V\cong V\otimes B\cong (V\otimes B\otimes_B A)^{co(H)}\cong (V\otimes A)^{co(H)}\cong (A\otimes V)^{co(H)}.$$

Considering the composition of the canonical isomorphisms

$$(A\otimes V)^{co(H)}\otimes_B A\cong (V\otimes A)^{co(H)}\otimes_B A\cong V\otimes B\otimes_B A\cong V\otimes A\cong A\otimes V,$$

we have that the adjunction map $\beta_{A\otimes V}$ for $A\otimes V$ is an isomorphism. We consider $\widetilde{\beta}$ be the composition

$$A\otimes A\stackrel{can}{\to} A\otimes_B A\stackrel{\beta}{\to}\bigoplus_{\gamma\in\pi}H_{\gamma}\otimes A,$$

where *can* is the canonical surjection. As β is surjective, $\tilde{\beta}$ is surjective. Let us define now

$$\xi:A\otimes A\to\bigoplus_{\gamma\in\pi}H_{\gamma}\otimes A,$$

$$\begin{aligned}\xi(a\otimes b)=(\widetilde{\beta}\circ\tau)(a\otimes b)&=(\beta\circ can\circ\tau)(a\otimes b)\\&=\bigoplus_{\gamma\in\pi}S_{\gamma}^{-1}\phi_{\alpha}(a_{[1,\alpha^{-1}\gamma^{-1}\alpha]})a_{[0,0]<-1,\gamma>}\otimes ba_{[0,0]<0,0>}\end{aligned}$$

for all $a,b\in A$. Notice that ξ is a surjective. We will prove that ξ is a morphism in ${}^{H}\mathcal{YD}_{A}^{\alpha}$. Indeed,

$$\begin{aligned}&\xi((a\otimes b)a^{'})=\xi(aa^{'}\otimes b)\\=&\bigoplus_{\gamma\in\pi}S_{\gamma}^{-1}\phi_{\alpha}(a^{'}_{[1,\alpha^{-1}\gamma^{-1}\alpha]})S_{\gamma}^{-1}\phi_{\alpha}(a_{[1,\alpha^{-1}\gamma^{-1}\alpha]})a_{[0,0]<-1,\gamma>}a^{'}_{[0,0]<-1,\gamma>}\\&\otimes ba_{[0,0]<0,0>}a^{'}_{[0,0]<0,0>}\\=&(\bigoplus_{\gamma\in\pi}S_{\gamma}^{-1}\phi_{\alpha}(a_{[1,\alpha^{-1}\gamma^{-1}\alpha]})a_{[0,0]<-1,\gamma>}\otimes ba_{[0,0]<0,0>})a^{'}\\=&\xi(a\otimes b)a^{'}\end{aligned}$$

and on one hand,

$$\begin{aligned}&\rho_{\lambda}^{\bigoplus_{\gamma\in\pi}H_{\gamma}\o A}(\xi(a\otimes b))\\=&\rho_{\lambda}^{\bigoplus_{\gamma\in\pi}H_{\gamma}\o A}(\bigoplus_{\gamma\in\pi}S_{\gamma}^{-1}\phi_{\alpha}(a_{[1,\alpha^{-1}\gamma^{-1}\alpha]})a_{[0,0]<-1,\gamma>}\o ba_{[0,0]<0,0>})\\=&\bigoplus_{\gamma\in\pi}S_{\gamma}^{-1}\phi_{\alpha}(a_{[1,\alpha^{-1}\gamma^{-1}\alpha]})_{(1,\lambda)}a_{[0,0]<-1,\gamma>(1,\lambda)}\\&\otimes S_{\gamma}^{-1}\phi_{\alpha}(a_{[1,\alpha^{-1}\gamma^{-1}\alpha]})_{(2,\lambda^{-1}\gamma)}a_{[0,0]<-1,\gamma>(2,\lambda^{-1}\gamma)}\otimes ba_{[0,0]<0,0>}\\\stackrel{\lambda^{-1}\gamma=\omega}{=}&\bigoplus_{\omega\in\pi}(S_{\lambda\omega}^{-1}\phi_{\alpha}(a_{[1,\alpha^{-1}\omega^{-1}\lambda^{-1}\alpha]}))_{(1,\lambda)}a_{[0,0]<-1,\lambda\omega>(1,\lambda)}\end{aligned}$$

$$
\begin{aligned}
& \otimes(S_{\lambda\omega}^{-1}\phi_\alpha(a_{[1,\alpha^{-1}\omega^{-1}\lambda^{-1}\alpha]}))_{(2,\omega)}a_{[0,0]<-1,\lambda\omega>(2,\omega)}\otimes ba_{[0,0]<0,0>}\\
= & \bigoplus_{\omega\in\pi} S_\lambda^{-1}(\phi_\alpha(a_{[1,\alpha^{-1}\omega^{-1}\lambda^{-1}\alpha]})_{(2,\lambda^{-1})})a_{[0,0]<-1,\lambda\omega>(1,\lambda)}\\
& \otimes S_\omega^{-1}(\phi_\alpha(a_{[1,\alpha^{-1}\omega^{-1}\lambda^{-1}\alpha]})_{(1,\omega^{-1})})a_{[0,0]<-1,\lambda\omega>(2,\omega)}\otimes ba_{[0,0]<0,0>},
\end{aligned}
$$

on the other hand,

$$
\begin{aligned}
& (id\otimes\xi)\circ\rho_\lambda^{A\otimes A}(a\otimes b)\\
= & (id\otimes\xi)(S_\lambda^{-1}\phi_\alpha(a_{[1,\alpha^{-1}\lambda^{-1}\alpha]})a_{[0,0]<-1,\lambda>}\otimes a_{[0,0]<0,0>}\otimes b)\\
= & S_\lambda^{-1}\phi_\alpha(a_{[1,\alpha^{-1}\lambda^{-1}\alpha]})a_{[0,0]<-1,\lambda>}\otimes\bigoplus_{\gamma\in\pi}S_\gamma^{-1}\phi_\alpha(a_{[0,0]<0,0>[1,\alpha^{-1}\gamma^{-1}\alpha]})\\
& a_{[0,0]<0,0>[0,0]<-1,\gamma>}\otimes ba_{[0,0]<0,0>[0,0]<0,0>}\\
= & S_\lambda^{-1}\phi_\alpha(a_{[1,\alpha^{-1}\lambda^{-1}\alpha]})a_{[0,0][0,0]<-1,\lambda>}\otimes\bigoplus_{\gamma\in\pi}S_\gamma^{-1}\phi_\alpha(a_{[0,0][1,\alpha^{-1}\gamma^{-1}\alpha]})\\
& a_{[0,0][0,0]<0,0><-1,\gamma>}\otimes ba_{[0,0][0,0]<0,0><0,0>}\\
= & S_\lambda^{-1}\phi_\alpha(a_{[1,\alpha^{-1}\lambda^{-1}\alpha]})a_{[0,0][0,0]<-1,\lambda\gamma>(1,\lambda)}\otimes\bigoplus_{\gamma\in\pi}S_\gamma^{-1}\phi_\alpha(a_{[0,0][1,\alpha^{-1}\gamma^{-1}\alpha]})\\
& a_{[0,0][0,0]<-1,\lambda\gamma>(2,\gamma)}\otimes ba_{[0,0][0,0]<0,0>}\\
= & \bigoplus_{\gamma\in\pi}S_\lambda^{-1}\phi_\alpha(a_{[1,\alpha^{-1}\gamma^{-1}\lambda^{-1}\alpha](2,\alpha^{-1}\lambda^{-1}\alpha)})a_{[0,0]<-1,\lambda\gamma>(1,\lambda)}\\
& \otimes S_\gamma^{-1}\phi_\alpha(a_{[1,\alpha^{-1}\gamma^{-1}\lambda^{-1}\alpha]((1,\alpha^{-1}\gamma^{-1}\alpha))})a_{[0,0]<-1,\lambda\gamma>(2,\gamma)}\otimes ba_{[0,0]<0,0>}\\
= & \bigoplus_{\gamma\in\pi}S_\lambda^{-1}(\phi_\alpha(a_{[1,\alpha^{-1}\gamma^{-1}\lambda^{-1}\alpha]})_{(2,\lambda^{-1})})a_{[0,0]<-1,\lambda\gamma>(1,\lambda)}\\
& \otimes S_\gamma^{-1}\phi_\alpha(a_{[1,\alpha^{-1}\gamma^{-1}\lambda^{-1}\alpha]})_{(1,\gamma^{-1})})a_{[0,0]<-1,\lambda\gamma>(2,\gamma)}\otimes ba_{[0,0]<0,0>},
\end{aligned}
$$

for all $a, a^{'}, b\in A$ and $\lambda\in\pi$. Hence ξ is a surjective morphism in ${}^H\mathcal{YD}_A^\alpha$.

Since $\bigoplus_{\gamma\in\pi}H_\gamma\otimes A$ is projective as a usual right A-module, where $\bigoplus_{\gamma\in\pi}H_\gamma\otimes A$ is a usual right A-module via

$$(\bigoplus_{\gamma\in\pi}h_\gamma\otimes a_\gamma)b=\bigoplus_{\gamma\in\pi}h_\gamma\otimes a_\gamma b,$$

for all $b\in A$. On the other hand, the map $u:\bigoplus_{\gamma\in\pi}H_\gamma\otimes A\to\bigoplus_{\gamma\in\pi}H_\gamma\otimes A$ which is defined by

$$u(\bigoplus_{\gamma\in\pi}h_\gamma\otimes a_\gamma)=\bigoplus_{\gamma\in\pi}S_\gamma^{-1}\phi_\alpha(a_{\gamma[1,\alpha^{-1}\gamma^{-1}\alpha]})h_\gamma a_{\gamma[0,0]<-1,\gamma>}\otimes a_{\gamma[0,0]<0,0>}$$

is an isomorphism of right A-modules, where the first $\bigoplus_{\gamma\in\pi}H_\gamma\otimes A$ has the usual right A-module structure and the second $\bigoplus_{\gamma\in\pi}H_\gamma\otimes A$ has the right A-module structure given by Eq. (3.7). The A-linear inverse of u is given by $u^{-1}:\bigoplus_{\gamma\in\pi}H_\gamma\otimes A\to\bigoplus_{\gamma\in\pi}H_\gamma\otimes A$,

$$u^{-1}(\bigoplus_{\gamma\in\pi}h_\gamma\otimes a_\gamma)=\bigoplus_{\gamma\in\pi}S_\gamma^{-1}(S_{\gamma^{-1}}^{-1}\phi_\alpha(a_{\gamma[1,\alpha^{-1}\gamma\alpha]}))h_\gamma S_{\gamma^{-1}}(a_{\gamma[0,0]<-1,\gamma^{-1}>})\otimes a_{\gamma[0,0]<0,0>}.$$

So we obtain that $\bigoplus_{\gamma\in\pi} H_\gamma \otimes A$, with the A-module structure given by Eq.(3.7) is still projective as a right A-module. It follows that the surjective morphism $\xi : A\otimes A \to \bigoplus_{\gamma\in\pi} H_\gamma \otimes A$ splits in the category of right A-modules. In particular, ξ is a k-split epimorphism in ${}^H\mathcal{YD}^\alpha_A$.

Let now $M \in^H \mathcal{YD}^\alpha_A$. Then $A\otimes A\otimes M \in^H \mathcal{YD}^\alpha_A$ via the structures:

$$(a\otimes b\otimes m)a^{'} = aa^{'} \otimes b\otimes m,$$

$$\rho^{A\otimes A\otimes M}_\gamma (a\otimes b\otimes m) = S^{-1}_\gamma \phi_\alpha(a_{[1,\alpha^{-1}\gamma^{-1}\alpha]})a_{[0,0]<-1,\gamma>} \otimes a_{[0,0]<0,0>} \otimes b\otimes m$$

for all $\gamma\in\pi$, $a, b \in A$ and $m \in M$. On the other hand, $\bigoplus_{\gamma\in\pi} H_\gamma \otimes A\otimes M \in^H \mathcal{YD}^\alpha_A$ via

$$(\bigoplus_{\gamma\in\pi} h_\gamma \otimes a_\gamma \otimes m_\gamma)b = \bigoplus_{\gamma\in\pi} S^{-1}_\gamma \phi_\alpha(b_{[1,\alpha^{-1}\gamma^{-1}\alpha]})h_\gamma b_{[0,0]<-1,\gamma>} \otimes a_\gamma b_{[0,0]<0,0>} \otimes m_\gamma,$$

$$\rho^{\bigoplus_{\gamma\in\pi} H_\gamma\otimes A\otimes M}_\beta (\bigoplus_{\gamma\in\pi} h_\gamma \otimes a_\gamma \otimes m_\gamma) = \bigoplus_{\gamma\in\pi} h_{\gamma(1,\beta)} \otimes h_{\gamma(2,\beta^{-1}\gamma)} \otimes a_\gamma \otimes m_\gamma, \quad (4.\ 5)$$

for any $\beta\in\pi, b\in A$. We obtain that

$$\xi\otimes id : A\otimes A\otimes M \to \bigoplus_{\gamma\in\pi} H_\gamma \otimes A\otimes M$$

is a k-split epimorphism in ${}^H\mathcal{YD}^\alpha_A$. For $\bigoplus_{\gamma\in\pi} H_\gamma \otimes A\otimes M$, we obtain

$$f : \bigoplus_{\gamma\in\pi} H_\gamma \otimes A\otimes M \to M,$$

$$\begin{aligned} f(\bigoplus_{\gamma\in\pi} h_\gamma \otimes a_\gamma \otimes m_\gamma) \quad &= \quad \sum_{\gamma\in\pi} m_{\gamma<0,0>} \theta_\gamma (S^{-1}_\gamma S^{-1}_{\gamma^{-1}} \phi_\gamma(a_{\gamma[1,\alpha^{-1}\gamma\alpha]}) \\ &\quad\quad h_\gamma S_{\gamma^{-1}}(a_{\gamma<-1,\gamma^{-1}>}))(m_{\gamma<-1,\gamma^{-1}>})a_{\gamma[0,0]<0,0>} \end{aligned}$$

is a k-split epimorphism in ${}^H\mathcal{YD}^\alpha_A$. Hence, the composition

$$g = f\circ(\xi\otimes id) : A\otimes A\otimes M \to M,$$

$$\begin{aligned} g(a\otimes b\otimes m) \quad &= \quad \sum_{\gamma\in\pi} m_{<0,0>} \theta_\gamma (S^{-1}_\gamma S^{-1}_{\gamma^{-1}} \phi_\gamma(b_{[1,\alpha^{-1}\gamma\alpha]}) \\ &\quad\quad S_{\gamma^{-1}}(b_{<-1,\gamma^{-1}>}))(m_{<-1,\gamma^{-1}>})b_{[0,0]<0,0>} a, \end{aligned}$$

for all $a, b \in A$, $m \in M$, is a k-split epimorphism in ${}^H\mathcal{YD}^\alpha_A$. The rest of proof is similar to [4]. ■

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